The Friedman rule in a model with endogenous growth and cash-in-advance constraint

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Abstract

This paper introduces money into an overlapping generations model with endogenous growth. The main message of the paper is that, as long as the modified golden rule is attained, the Friedman rule is optimal. The result holds regardless of the ability of the government to internalize the externality and control the level of human capital. Other results include: (i) violation of the Friedman rule for a different second-best environment wherein human capital accumulation is controlled but not physical capital accumulation and (ii) existence of a negative relationship between money growth rate and the economy's endogenous growth rate.

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1 Introduction

For over four decades, the optimal money supply literature has studied the environments under which the Friedman rule may or may not hold. One interesting result, in the context of overlapping-generations models à la Diamond (1965), is that if the (modified) golden rule is satisfied the Friedman rule will hold. Weiss (1980) had famously shown that the Friedman rule does not hold in overlapping generations models. The subsequent literature showed that this result was due to the generic failure of the laissez-faire equilibrium of overlapping generations models to deliver the (modified) golden rule; see, e.g., Able (1987) and Gahvari (1988). Introducing generational lump-sum tax and transfers, or a debt policy, to control capital accumulation allows the economy to attain the (modified) golden rule and restores the optimality of the Friedman rule.[1]

The primary aim of this paper is to examine if the above result is robust to the introduction of some form of market failure. Put differently, I ask if the control of physical capital and offsetting intergenerational wealth transfers due to money creation is sufficient for the application of the Friedman rule in overlapping generations model; or it is also required that the model should contain no other intrinsic sources of market failures, such as externalities. For example, van der Ploeg and Alogoskoufis (1994) demonstrate that the Friedman rule is violated in an overlapping-generations model that exhibits endogenous growth.[2] Given that van der Ploeg and Alogoskoufis allow for lump-sum taxation and debt policy, capital can be controlled in their model. It might then appear that it is the market friction underlying the endogenous growth that is behind the violation of the Friedman rule in their setup. This deduction is unwarranted, however, as van der Ploeg and Alogoskoufis do not set the available fiscal instruments to fully control the capital stock of the economy. Indeed, in their model, it is the non-neutrality of monetary policy that leads to the breakdown of the Friedman rule.[3]

I set out to address this question sequentially. First, I consider an environment wherein the government has enough fiscal instruments to offset the other sources of distortion in the economy. In this setting, as in Diamond, one can attain a first-best equilibrium; albeit through a Pigouvian tax as well as the control of capital. In the second environment, the government lacks enough fiscal instruments to control both sources of frictions that impinge on the proper working of a market economy (capital accumulation and the externality). I will do so once allowing for the control of capital and once for the correction of the externality.

The distortion I consider is an externality associated with investment in education. I use a model due to Docquier *et al.* (2007) who have recently extended Diamond's model to allow for endogenous growth that emanates from building up one's human capital. Two aspects of this model are particularly interesting. One is that it allows for (per capita) growth that does not exist in Diamond (1965); secondly, its laissez-faire equilibrium exhibits an additional source of market failure (an intergenerational spillover). Specifically, the externality is due to the positive effect of investment in education on the human capital of not just the investor, but his children as well. It arises when one's human capital is determined partly through education and partly through the human capital one inherits from his parents. However, in deciding how much to spend on education, individuals ignore the effect that their decision has on the human capital of their children.[4]

To address these questions, the paper introduces money in Docquier *et al.*'s

(2007) model and rationalizes it through a cash-in-advance constraint. It derives the dynamics of the model and characterize its balanced growth path under laissez faire. Subsequently, it derives the first-best allocations of this economy and shows that they are not affected by the introduction of money.[5] It proves that the implementation of the first-best, i.e. when both physical and human capital are fully controlled, requires the Friedman rule to be satisfied. This result generalizes the earlier result of Able (1987) and Gahvari (1988) to overlapping generations models that incorporate an intrinsic source of externality. It also puts van der Ploeg and Alogoskoufis' (1994) contrary result in the right perspective.

Secondly, the paper proves that the Friedman rule holds even in a second-best environment without an instrument to internalize the externality of education. As such, it contributes to the literature on the suboptimality of the Friedman rule in second-best environments that followed Phelps' (1973) original argument to this effect. The lesson of that literature is that the validity of the Friedman rule depends, to a great extent, on what tax instruments the government has at its disposal. See, e.g., Chari *et al.* (1991, 1996) who, studying this issue in the context of the optimal tax literature in public finance, also emphasize the structure of consumer preferences for this result. Most recent contribution in this genre is Petrucchi (2011) who shows that restrictions on tax instruments that violate Diamond and Mirrlees (1971) production efficiency result also violate the Friedman rule (as well as Chamley (1986) and Judd (1985) zero capital income taxation result).[6]

As a contribution to the second-best literature on the Friedman rule, the current paper's novelty is its juxtaposition of an externality to another market failure problem; namely, that of optimal capital accumulation. That the Friedman rule holds even in this environment, may at first appear counter intuitive. The key to understanding it is that introducing another distortion in the economy via the violation of the Friedman rule does nothing to alleviate the existing distortion in human capital accumulation, as long as one fully controls the economy's stock of physical capital. The paper highlights this point. Third, the paper studies a different second-best environment; one in which human capital accumulation can be controlled but not physical capital accumulation. The paper proves that this setting calls for the violation of the Friedman rule. In this case, the distortion due to the violation of Friedman rule does alleviate the distortion due to the lack of physical capital accumulation. However, in this case, it is also possible for the Friedman to be satisfied as a boundary condition.

Other grounds covered, and results obtained, include a comparison between the values that the variables of the model assume in the first- and second-best environments studied. The second-best with control of physical capital leads to a lower money growth rate, a higher physical to human capital ratio, and a lower endogenous rate of growth than the first best. In the second-best with control of human capital, physical to human capital ratio and the endogenous growth rate are lower when the monetary growth rate is set optimally and greater when the Friedman rule is satisfied as a boundary condition. A second result is the existence of a negative relationship between money growth rate and the endogenous growth rate of the economy in the cash-in-advance-constraint models à la Hahn and Solow (1995).

Finally, the paper is related to the vast literature on the Friedman rule, particularly those written in the context of endogenous growth and/or within the overlapping generations framework. van der Ploeg and Alogoskoufis (1994) referred to earlier is one. Paal and Smith (2000) discuss the suboptimality of the Friedman rule in a monetary growth model where spatial separation and limited communication rationalize money holding for transaction purposes. There are no (offsetting) fiscal instruments in their model and the suboptimality of the Friedman rule is caused by the impact of bank portfolio reallocations on the real economy. Most recently, Lai and Chin (2010) show that the Friedman rule is valid if global capital markets are perfect; in this case distortions cannot be remedied through monetary policy. On the other hand, with imperfect capital markets becomes an effective tool for correcting market distortions.[7]

2 The model and its laissez-faire equilibrium

Consider Diamond's (1965) two-period overlapping generations model wherein individuals work in the first period supplying one unit of labor and derive utility from consuming a composite consumption good in the first- and second-period of their lives. There is no bequest motive, and population grows at a constant rate. Append to this model (i) human capital accumulation as modeled by Docquier *et al.* (2007) and (ii) money holdings. Output of each period can be used for consumption in the same period, or retained with no depreciation, to be used next period as an input either in the educational process of the young or the production process.[8] Endowed with an initial level of human capital, the young decide how much to invest in their own education to increase their human capital, how much to save in real assets to finance their future consumption, and how much money to carry forward into the future. The old, who own all the economy's physical and monetary assets, sell their assets and use the proceeds, along with the return on their real assets, to finance their consumption. The firms hire capital and labor to produce the output (used as consumption goods and real savings). All decisions on human capital accumulation, production, consumption and savings are undertaken at the beginning of period one and in the order stated.[9] These decisions by individuals and firms determine, as explained below, the temporal laissez faire equilibrium of the economy under the perfect-foresight assumption on the part of the young.

2.1 Education

The first decision the young makes is on education. At the beginning of period t, the young start life with a given amount of human capital, or "effective labor," h_{t-1}, that they have inherited from their parents. This level of inherited human capital may be increased to h_t by one's investment in education, e_t. Assume that the human capital formation technology is characterized by a linear homogeneous function $\Phi\left(e_t, h_{t-1}\right)$ so that

$$h_t = \Phi\left(e_t, h_{t-1}\right) = h_{t-1}\varphi\left(\frac{e_t}{h_{t-1}}\right), \tag{1}$$

where $\varphi\left(\cdot\right) \equiv \Phi\left(\cdot, 1\right)$. I assume that $\varphi\left(\cdot\right)$ is positive, increasing, and strictly concave with $\varphi\left(0\right) = 1$; it also satisfies the Inada conditions: $\varphi'\left(0\right) = \infty$ and $\varphi'\left(\infty\right) = 0$.[10]

When deciding on how much to invest in their education, the young have no resources. They thus borrow e_t each for their own education from the old of the previous generation. Let w_t denote the *real* wage at time t (measured in units of composite consumption good), and r_t the *real* interest rate. The individual chooses e_t to maximize $w_t h_t - e_t(1 + r_t)$ subject to (1). This yields a solution for

e_t characterized by

$$\varphi'\left(\frac{e_t}{h_{t-1}}\right) = \frac{1+r_t}{w_t}. \tag{2}$$

2.2 Production

Firms decide on production after the young's decision on education. The production technology, which also exhibits constant returns to scale, uses capital, K_t, and *effective* labor, H_t, to produce a composite output, $Y_t = F(K_t, H_t)$. Let N_t denote the number of young persons—equivalently workers—at time t, and define output and capital according to $y_t = Y_t/N_t$, $k_t = K_t/N_t$. From the definitions of h_t and H_t, one also has $h_t = H_t/N_t$. The production function can then be presented by $y_t = F(k_t, h_t)$. Assuming a competitive setting w_t and r_t are determined according to

$$w_t = F_h(k_t, h_t), \tag{3}$$

$$r_t = F_k(k_t, h_t). \tag{4}$$

At the beginning of period t, prior to the educational and production decisions, the *sum* of aggregate capital to be used for educational investment of the young and for production, $N_t e_t + K_t$, is pre-determined from the savings decisions of the current old at time $t-1$ when they were young. Specifically, denoting these savings, on an individual basis, by s_{t-1}, it must be the case that $N_t e_t + K_t = N_{t-1}s_{t-1}$. Dividing this equation by N_t, it follows that

$$e_t + k_t = \frac{s_{t-1}}{1+n}, \tag{5}$$

where n denotes the constant population growth rate. The system of equations (1)–(5) determine the equilibrium values for k_t, e_t, h_t, w_t, and r_t (as functions of s_{t-1} and h_{t-1}).

2.3 Monetary policy

The monetary authority injects money into (or retires money from) the economy at the constant rate of θ per period.[11] This occurs before consumption takes place. Denote the aggregate stock of money at the *end* of period t by M_t. With money stock changing at the rate of θ in every period, $M_t = (1+\theta)\, M_{t-1}$. The transfers are given to (or taken from) the old—who hold all the stock of money—via lump-sum monetary transfers at the *beginning* of period t. Denote the lump-sum money transfer to each old person, his *share* of the change in money supply, by b_t. With N_{t-1} old individuals, it must be the case that $b_t = \theta\,(M_{t-1}/N_{t-1})$. Denoting the cash holdings of an old person at the beginning of period t by m_{t-1}, one can write this relationship as

$$b_t = \theta m_{t-1}, \tag{6}$$

where $m_{t-1} \equiv M_{t-1}/N_{t-1}$. Observe that the relationship (6) holds in equilibrium; otherwise the individual treats b_t as lump-sum in his optimization problem.

The old's money holdings during period t thus consists of two components: their own cash savings from the previous period, m_{t-1}, and the lump-sum money transfer from (or to) the government, b_t. Holdings of money is rationalized through a Clower cash-in-advance constraint. I specify this constraint through the assumption that all agents must finance a fraction of their expenditures on second-period consumption, d_t, through cash balances. That is,

$$m_{t-1} + b_t \geqq \alpha p_t d_t. \tag{7}$$

where p_t is the price level at time t and $\alpha < 1$ is the proportion of $p_t d_t$ that has to be financed through cash. This specification is due to Hahn and Solow (1995) and has been used extensively in overlapping-generations models; see, e.g., Crettez *et al.* (1999, 2002) and Michel and Wigniolle (2003, 2005). It may at first appear restrictive in that it does not apply to first-period consumption expenditures. However, this is not the case. Extending it to the young will not change the main results of the paper concerning the Friedman rule; see Appendix B for details.[12]

Finally, for future reference, denote the inflation rate during period $t+1$ by π_{t+1}. This is defined by

$$1 + \pi_{t+1} \equiv \frac{p_{t+1}}{p_t}. \tag{8}$$

Also denote the nominal interest rate at $t+1$ by i_{t+1} and observe that the nominal and real interest rates are related according to the Fisher equation,

$$1 + i_{t+1} = (1 + r_{t+1})(1 + \pi_{t+1}). \tag{9}$$

2.4 Consumption and saving

The second decision the young makes concerns consumption and savings. This occurs after production and money injection. Preferences of the young are represented by

$$u = u(c_t, d_{t+1}) \tag{10}$$

where c_t denotes consumption in the first period, and $u(\cdot,\cdot)$ is strictly quasi-concave, twice differentiable, and homothetic. Denote a young person's savings in real assets by s_t, and his cash balances by m_t. His first- and second-period budget constraints are given by[13]

$$c_t + s_t + \frac{m_t}{p_t} = w_t h_t - e_t(1 + r_t), \tag{11}$$

$$d_{t+1} = s_t(1 + r_{t+1}) + \frac{m_t + b_{t+1}}{p_{t+1}}. \tag{12}$$

The young person, having already determined his educational investment e_t and human capital h_t, must choose his present consumption c_t, savings in real assets s_t, and real cash balances m_t/p_t. His decision on cash balances is tied to his future consumption via the Clower cash-in-advance constraint. Assume constraint (7) is binding. Rewrite it for time t, then divide it by p_{t+1}, and rearrange the terms to get

$$\frac{m_t}{p_{t+1}} = \alpha d_{t+1} - \frac{b_{t+1}}{p_{t+1}}. \tag{13}$$

Incorporate this constraint in the young's first- and second-period budget constraints. Simplification yields,[14]

$$\begin{aligned} c_t + \left[1 + \frac{\alpha\left(1+i_{t+1}\right)}{1-\alpha}\right] s_t &= w_t h_t - e_t(1+r_t) + \frac{b_{t+1}}{p_t}, &(14)\\ (1-\alpha)\, d_{t+1} &= s_t\left(1+r_{t+1}\right). &(15)\end{aligned}$$

Substituting for c_t and d_{t+1} from constraints (14)–(15) into $u(c_t, d_{t+1})$ and optimizing with respect to s_t yields the following first-order condition

$$\frac{\partial u\left(c_t, d_{t+1}\right)/\partial d_{t+1}}{\partial u\left(c_t, d_{t+1}\right)/\partial c_t} = \frac{1+\alpha i_{t+1}}{1+r_{t+1}}. \tag{16}$$

Equations (14), (15), and (16) determine the values for the current-period variables c_t and s_t under perfect-foresight assumption (as functions of r_{t+1}, i_{t+1} and b_{t+1}/p_t).[15]

2.5 Dynamics

First, to simplify the exposition, rewrite the variables of the model as a fraction of effective labor. Denote $\widehat{e}_t \equiv e_t/h_{t-1}$ and thus rewrite equations (1) and (2) as

$$\begin{aligned} h_t &= h_{t-1}\varphi\left(\widehat{e}_t\right), \\ \varphi'\left(\widehat{e}_t\right) &= \frac{1+r_t}{w_t}. \end{aligned}$$

Similarly, denote $\widehat{y}_t = Y_t/H_t = y_t/h_t$ and $\widehat{k}_t = K_t/H_t = k_t/h_t$. This allows the production function to be represented by $\widehat{y}_t = f(\widehat{k}_t)$, where $f(\cdot)$ is positive, increasing, and strictly concave. One can then rewrite equations (3) and (4) as

$$\begin{aligned} w_t &= f(\widehat{k}_t) - \widehat{k}_t f'(\widehat{k}_t), \\ r_t &= f'(\widehat{k}_t). \end{aligned}$$

Observe that the values of $\widehat{e}_t$ and $\widehat{k}_t$ will be determined as soon as e_t and k_t are determined.

Now, to examine the dynamic evolution of this economy, substitute for s_t from (5) into equation (15) to get

$$(1-\alpha)\, d_{t+1} = (1+n)\left(e_{t+1} + k_{t+1}\right)\left(1 + r_{t+1}\right).$$

Divide this relationship by h_t, substitute $\widehat{d}_{t+1}$ for d_{t+1}/h_t, $\widehat{e}_{t+1}$ for e_{t+1}/h_t, $\widehat{k}_{t+1}$ for k_{t+1}/h_{t+1}, $\varphi\left(\widehat{e}_{t+1}\right)$ for h_{t+1}/h_t, and rearrange the terms to arrive at

$$(1-\alpha)\, \widehat{d}_{t+1} = (1+n)\left(1+r_{t+1}\right)\left[\widehat{e}_{t+1} + \widehat{k}_{t+1}\varphi\left(\widehat{e}_{t+1}\right)\right]. \tag{17}$$

With r_{t+1} being determined by $\widehat{k}_{t+1}$, write the above expression as $\widehat{d}_{t+1} = A\left(\widehat{k}_{t+1}, \widehat{e}_{t+1}\right)$.

Second, "solve" equation (13) and $b_{t+1} = \theta m_t$ for m_t and b_{t+1} to get

$$m_t = \frac{\alpha}{1+\theta} p_{t+1} d_{t+1}, \tag{18}$$

$$b_{t+1} = \frac{\alpha\theta}{1+\theta} p_{t+1} d_{t+1}. \tag{19}$$

It follows from (18) that

$$\frac{p_{t+1}d_{t+1}}{p_t d_t} = \frac{m_t}{m_{t-1}} = \frac{M_t/N_t}{M_{t-1}/N_{t-1}} = \frac{1+\theta}{1+n},$$

or

$$\frac{d_{t+1}}{d_t} = \frac{1+\theta}{1+n}\frac{1}{1+\pi_{t+1}}.$$

Multiply the numerator by h_t/h_t and the denominator of this relationship by h_{t-1}/h_{t-1} to rewrite it as

$$\varphi\left(\widehat{e}_t\right)\frac{\widehat{d}_{t+1}}{\widehat{d}_t} = \frac{1+\theta}{1+n}\frac{1}{1+\pi_{t+1}}. \tag{20}$$

This expression leads to $\widehat{d}_{t+1} = B\left(\pi_{t+1}, \widehat{e}_t, \widehat{d}_t\right)$.

Third, delete s_t between equations (14)–(15) to get the young's intertemporal budget constraint

$$c_t + (1+\alpha i_{t+1})\frac{d_{t+1}}{1+r_{t+1}} = w_t h_t - e_t(1+r_t) + \frac{b_{t+1}}{p_t}. \tag{21}$$

Substitute for b_{t+1} from (19) in (21) and rearrange the terms to get

$$c_t + \left[1 + \frac{\alpha(i_{t+1}-\theta)}{1+\theta}\right]\frac{d_{t+1}}{1+r_{t+1}} = w_t h_t - (1+r_t)e_t.$$

Divide this equation by h_t, substitute $\widehat{c}_t$ for c_t/h_t, $\widehat{d}_{t+1}$ for d_{t+1}/h_t, $\widehat{e}_t$ for e_t/h_{t-1}, and $\varphi\left(\widehat{e}_t\right)$ for h_t/h_{t-1} to arrive at

$$\widehat{c}_t + \left[1 + \frac{\alpha(i_{t+1}-\theta)}{1+\theta}\right]\frac{\widehat{d}_{t+1}}{1+r_{t+1}} = w_t - (1+r_t)\frac{\widehat{e}_t}{\varphi\left(\widehat{e}_t\right)}. \tag{22}$$

Turning to the first-order condition (16) for utility maximization, the assumption that the utility function (10) is homothetic, implies that the marginal rate of substitution between $\widehat{c}_t$ and $\widehat{d}_{t+1}$ and between c_t and d_{t+1} are the same. One can rewrite (16) as

$$\frac{u_d\left(\widehat{c}_t, \widehat{d}_{t+1}\right)}{u_c\left(\widehat{c}_t, \widehat{d}_{t+1}\right)} = \frac{1 + \alpha\, i_{t+1}}{1 + r_{t+1}}. \tag{23}$$

Eliminating $\widehat{c}_t$ between (22) and (23), and using the Fisher equation $1 + i_{t+1} = (1 + r_{t+1})\,(1 + \pi_{t+1})$, one finds $\widehat{d}_{t+1} = C\left(\pi_{t+1}, r_{t+1}, w_t, r_t, \widehat{e}_t\right)$. Eliminating π_{t+1} between this relationship and $\widehat{d}_{t+1} = B\left(\pi_{t+1}, \widehat{e}_t, \widehat{d}_t\right)$, the expression for $\widehat{d}_{t+1}$ can be rewritten as $\widehat{d}_{t+1} = E(r_{t+1}, w_t, r_t, \widehat{d}_t, \widehat{e}_t)$. With r_{t+1} being determined by $\widehat{k}_{t+1}$, and w_t and r_t by $\widehat{k}_t$, one can further rewrite this expression as $\widehat{d}_{t+1} = F\left(\widehat{k}_{t+1}, \widehat{k}_t, \widehat{d}_t, \widehat{e}_t\right)$.

Finally, the system of equations $\widehat{d}_{t+1} = A\left(\widehat{k}_{t+1}, \widehat{e}_{t+1}\right)$, $\widehat{d}_{t+1} = E\left(\widehat{k}_{t+1}, \widehat{k}_t, \widehat{d}_t, \widehat{e}_t\right)$, and the first-order condition (2), which determines $\widehat{e}_t$ as a function of $\widehat{k}_t$, leads to a pair of first-order difference equations of the form

$$\widehat{k}_{t+1} = \Psi\left(\widehat{k}_t, \widehat{d}_t\right), \tag{24}$$

$$\widehat{d}_{t+1} = \Phi\left(\widehat{k}_t, \widehat{d}_t\right). \tag{25}$$

The system of difference equations (24)–(25) determines the dynamic path of the economy. Moreover, the economy tends to a balanced growth path if equations (24)–(25) have a steady-state solution and if it is stable. Under the latter two assumptions, the steady-state values of $\left(\widehat{k}_t, \widehat{d}_t\right)$ will be the rest point of equations (24)–(25).[16]

2.6 Balanced growth

At the steady state, equations (23), (22), (17), (20), (9), (2), (4), and (3) are simplified to[17]

$$\frac{u_d\left(\widehat{c},\widehat{d}\right)}{u_c\left(\widehat{c},\widehat{d}\right)}=\frac{1+\alpha\, i}{1+r}, \tag{26}$$

$$\widehat{c}+\left[\frac{1-\alpha}{1+r}+\frac{\alpha}{(1+n)\,\varphi\left(\widehat{e}\right)}\right]\widehat{d}=w-(1+r)\frac{\widehat{e}}{\varphi\left(\widehat{e}\right)}, \tag{27}$$

$$(1-\alpha)\,\widehat{d}=(1+n)\,(1+r)\left[\widehat{e}+\widehat{k}\varphi\left(\widehat{e}\right)\right], \tag{28}$$

$$1+\pi=\frac{1+\theta}{(1+n)\,\varphi\left(\widehat{e}\right)}, \tag{29}$$

$$1+i=(1+r)\,\frac{1+\theta}{(1+n)\,\varphi\left(\widehat{e}\right)}, \tag{30}$$

$$\varphi'\left(\widehat{e}\right)=\frac{1+r}{w}, \tag{31}$$

$$r=f'(\widehat{k}), \tag{32}$$

$$w=f(\widehat{k})-\widehat{k}f'(\widehat{k}). \tag{33}$$

These equations determine the steady-state values of real variables $\widehat{c},\widehat{d},\widehat{e},\widehat{k},w,r$, and monetary variables i, and π. Observe that while these values remain unchanged, the values of the other variables of the model grow over time at a constant rate.[18] This is the balanced growth path of the economy. Specifically, let

$$g\equiv\varphi\left(\widehat{e}\right)-1. \tag{34}$$

Then, $e_t=(1+g)e_{t-1}$, $h_t=(1+g)h_{t-1}$, $k_t=(1+g)k_{t-1}$, $c_t=(1+g)c_{t-1}$, $d_t=(1+g)d_{t-1}$, and $K_t=(1+n)(1+g)K_{t-1}$. Utility also grows over time. Assuming that the utility function is homogeneous of degree $\beta>0$, $u(c_t,d_{t+1})=h_t^{\beta}u\left(\widehat{c}_t,\widehat{d}_{t+1}\right)$ so that utility increases by a factor of $(1+g)^{\beta}$.[19]

The following proposition summarizes these results:

Proposition 1 *Consider a version of Diamond's (1965) overlapping-generations model wherein each generation's human capital is determined via the level they inherit from their parents and their own educational attainment. Assume further that money is an alternative asset to physical capital and required for second period transactions. On a balanced growth path, per capita educational expenditures, human capital, physical capital, consumption during working years, and consumption during retirement all grow at a constant rate g specified in (34). The monetary variables of the economy, the inflation rate and the nominal rate of interest, remain constant and vary with the rate of money growth, θ. Equations (26)–(33) characterize the laissez faire balanced growth path of the economy (where a "hat" on a variable denotes its value per "effective labor").*

3 First best

Let the social welfare function be presented by the discounted sum of the average of all generations' lifetime utilities. The first best is then characterized by maximizing this function subject to every generation's human capital formation and resource constraint. The human capital formation is given by equation (1); the resource constraint for the generation born at time t is given by

$$K_t + F(K_t, H_t) = N_t c_t + N_{t-1} d_t + K_{t+1} + N_{t+1} e_{t+1},$$

which one can alternatively write, by dividing it by N_t, as

$$k_t + h_t f\left(\frac{k_t}{h_t}\right) = c_t + \frac{1}{1+n} d_t + (1+n)\left(k_{t+1} + e_{t+1}\right).$$

The Lagrangian expression for this optimization problem can then be written as,

$$\begin{aligned}\pounds \quad \equiv \quad & \sum_{t=0}^{\infty}\frac{1}{(1+\rho)^t}\left\{u\left(c_t,d_{t+1}\right)+\mu_t\left[h_t\varphi\left(\frac{e_{t+1}}{h_t}\right)-h_{t+1}\right]\right. \\ & \left.+\lambda_t\left[k_t+h_tf\left(\frac{k_t}{h_t}\right)-c_t-\frac{1}{1+n}d_t-(1+n)\left(k_{t+1}+e_{t+1}\right)\right]\right\} \quad (35)\end{aligned}$$

where $\rho\in(0,1)$ is the discount rate reflecting the "planner's" social time preference, and $\mu_t/(1+\rho)^t$ and $\lambda_t/(1+\rho)^t$ are the multipliers associated with the resource constraint and the human capital equation at time t. The first-order conditions with respect to $c_t, d_{t+1}, h_{t+1}, k_{t+1}$, and e_{t+1}, are

$$u_c(c_t,d_{t+1})=\lambda_t, \quad (36)$$

$$u_d(c_t,d_{t+1})=\frac{\lambda_{t+1}}{(1+n)(1+\rho)}, \quad (37)$$

$$(1+\rho)\mu_t=\lambda_{t+1}\left[f(\widehat{k}_{t+1})-\widehat{k}_{t+1}f'(\widehat{k}_{t+1})\right]+\mu_{t+1}\left[\varphi(\widehat{e}_{t+2})-\widehat{e}_{t+2}\varphi'\left(\widehat{e}_{t+2}\right)\right], \quad (38)$$

$$\lambda_t(1+\rho)(1+n)=\lambda_{t+1}\left[1+f'(\widehat{k}_{t+1})\right], \quad (39)$$

$$\mu_t\varphi'\left(\widehat{e}_{t+1}\right)=\lambda_t(1+n). \quad (40)$$

Finally, the transversality condition is

$$\lim_{t\to\infty}\frac{\lambda_tk_{t+1}+\mu_th_{t+1}}{(1+\rho)^t}=0. \quad (41)$$

In characterizing the optimal values of the economic variables, when manipulating the first-order conditions (36)–(40), I pay close attention to which derivations use (39), the first-order condition with respect to k_{t+1}, and which use (40), the first-order condition with respect to e_{t+1}. This facilitates the characterization of second-best outcomes that I shall discuss later in which the planner cannot control one or the other of the variables k_{t+1} and e_{t+1}. Thus derive an expression for λ_{t+1}/λ_t once from (37) by writing it for $t+1$ and t and dividing one by the other, and once from dividing (37) by (36). One gets the following two equations

$$\frac{\lambda_{t+1}}{\lambda_t} = \frac{u_d(c_t, d_{t+1})}{u_d(c_{t-1}, d_t)} = \frac{h_t^{\beta-1} u_d\left(\widehat{c}_t, \widehat{d}_{t+1}\right)}{h_{t-1}^{\beta-1} u_d\left(\widehat{c}_{t-1}, \widehat{d}_t\right)} = \left[\varphi\left(\widehat{e}_t\right)\right]^{\beta-1} \frac{u_d\left(\widehat{c}_t, \widehat{d}_{t+1}\right)}{u_d\left(\widehat{c}_{t-1}, \widehat{d}_t\right)}, \tag{42}$$

$$\frac{\lambda_{t+1}}{\lambda_t} = (1+n)(1+\rho)\frac{u_d(c_t, d_{t+1})}{u_c(c_t, d_{t+1})} = (1+n)(1+\rho)\frac{u_d\left(\widehat{c}_t, \widehat{d}_{t+1}\right)}{u_c\left(\widehat{c}_t, \widehat{d}_{t+1}\right)}. \tag{43}$$

It then follows from (42)–(43) that

$$\frac{u_d\left(\widehat{c}_t, \widehat{d}_{t+1}\right)}{u_c\left(\widehat{c}_t, \widehat{d}_{t+1}\right)} = \frac{\left[\varphi\left(\widehat{e}_t\right)\right]^{\beta-1}}{(1+n)(1+\rho)} \frac{u_d\left(\widehat{c}_t, \widehat{d}_{t+1}\right)}{u_d\left(\widehat{c}_{t-1}, \widehat{d}_t\right)}. \tag{44}$$

Observe that neither (39) nor (40) has been used in deriving (44).

Next, substituting from (4) in (39) yields $\lambda_{t+1}/\lambda_t = (1+\rho)(1+n)/(1+r_{t+1})$. Equating this to the expression for λ_{t+1}/λ_t in (43) and using (44)

$$1 + r_{t+1} = \frac{(1+n)(1+\rho)}{\left[\varphi\left(\widehat{e}_t\right)\right]^{\beta-1}} \frac{u_d\left(\widehat{c}_{t-1}, \widehat{d}_t\right)}{u_d\left(\widehat{c}_t, \widehat{d}_{t+1}\right)}. \tag{45}$$

Observe that (45) uses (39) but not (40).

Finally, use equation (40) to substitute for μ_t and μ_{t+1} into (38). Divide the resulting equation by λ_t, substitute for λ_{t+1}/λ_t from (42), and manipulate to arrive at

$$\varphi'\left(\widehat{e}_{t+1}\right) = \frac{1+n}{w_{t+1}} \left\{ \left[\varphi\left(\widehat{e}_t\right)\right]^{1-\beta} (1+\rho) \frac{u_d\left(\widehat{c}_{t-1}, \widehat{d}_t\right)}{u_d\left(\widehat{c}_t, \widehat{d}_{t+1}\right)} - \frac{\varphi'\left(\widehat{e}_{t+1}\right)}{\varphi'\left(\widehat{e}_{t+2}\right)} \left[\varphi(\widehat{e}_{t+2}) - \widehat{e}_{t+2}\varphi'\left(\widehat{e}_{t+2}\right)\right] \right\}. \tag{46}$$

This equation uses (40) but not (39).

3.1 Balanced growth

On the balanced growth path, $\widehat{c}_t$ and $\widehat{d}_{t+1}$ remain constant over time so that equations (44), (45), and (46) simplify to

$$
\begin{aligned}
\frac{u_d\left(\widehat{c},\widehat{d}\right)}{u_c\left(\widehat{c},\widehat{d}\right)} &= \frac{1}{(1+n)(1+\rho)\left[\varphi\left(\widehat{e}\right)\right]^{1-\beta}}, && (47)\\
1+r &= (1+n)(1+\rho)\left[\varphi\left(\widehat{e}\right)\right]^{1-\beta}, && (48)\\
\varphi'\left(\widehat{e}\right) &= \frac{1+n}{w}\left\{(1+\rho)\left[\varphi\left(\widehat{e}\right)\right]^{1-\beta} - \left[\varphi(\widehat{e}) - \widehat{e}\varphi'\left(\widehat{e}\right)\right]\right\}, && (49)
\end{aligned}
$$

Observe also that on a balanced-growth path the transversality condition (41) reduces to[20]

$$\frac{\left[\varphi\left(\widehat{e}\right)\right]^{\beta}}{1+\rho} < 1. \tag{50}$$

In what follows I assume $\beta < 1$.[21]

Equations (47)–(49) constitute the three margins that should be determined "correctly"; none is assured at the laissez-faire equilibrium. Condition (47) shows the first-best marginal rate of intertemporal substitution in consumption. Its counterpart at the steady-state laissez-faire equilibrium is equation (26) indicating that the real and nominal interest rates govern its value in the market. Condition (48) is a version of the modified golden rule, adjusted for the endogenous growth rate. As is well known, this condition is not generally satisfied at the laissez faire equilibrium. The third condition, equation (49), characterizes the optimal educational investment rule. To see how this condition differs from its counterpart at the steady-state laissez faire equilibrium, substitute from (48) into (49) to get

$$\varphi'\left(\widehat{e}\right) = \frac{1+r}{w} - \frac{(1+n)\left[\varphi(\widehat{e}) - \widehat{e}\varphi'\left(\widehat{e}\right)\right]}{w}. \tag{51}$$

This equation differs from the corresponding laissez faire equation (31) in that the latter does not include the second expressions on the right-hand side. This reflects the externality that parents bestow on their children by educating themselves. Recall that education increases one's human capital which also enhances the human capital of one's children. Parents do not take this externality into account when deciding on their own educational attainment. Observe also that this is a positive externality. The above equation is in accordance with this in that with $\varphi(\cdot)$ being concave, $\varphi(\widehat{e}) - \widehat{e}\varphi'(\widehat{e}) > 0$ so that $\varphi'(\widehat{e}) < [1+r]/w$.

Now while the laissez-faire equilibrium, in and of itself, does not satisfy any of these three conditions, the government can muster enough instruments to ensure the satisfaction of all three. The most crucial element for this is ensuring that capital accumulation follows the modified golden rule. This can be achieved by levying of generation-specific lump-sum taxes. Given the satisfaction of this rule, i.e. equation (48), condition (47) will be satisfied by setting $i = 0$ so that the opportunity cost of holding money is zero (as required by the Friedman rule). Finally, subsidizing educational expenditures guaranteed that condition (49) is satisfied. I demonstrate this below.

3.2 Decentralization

Let z_t^y and z_t^o denote lump sum taxes imposed at time t on the living young and the old, and σ denote the rate of subsidy on educational expenditures. These taxes are related through the government's budget constraint so that

$$N_t z_t^y + N_{t-1} z_t^o - N_t \sigma (1 + r_t) e_t = 0.$$

Or, dividing by $N_{t-1}h_t$,

$$(1+n)\left[\widehat{z}_t^y - \sigma(1+r_t)\frac{\widehat{e}_t}{\varphi\left(\widehat{e}_t\right)}\right] + \frac{\widehat{z}_t^o}{\varphi\left(\widehat{e}_t\right)} = 0, \tag{52}$$

where $\widehat{z}_t^y = z_t^y/h_t$ and $\widehat{z}_t^o = z_t^o/h_{t-1}$. Observe also that $\widehat{z}_t^y$and $\widehat{z}_{t+1}^o$ remain invariant on the balanced growth path so that $z_t^y = (1+g)z_{t-1}^y$ and $z_t^o = (1+g)z_{t-1}^o$.

Allowing for the above tax rates, and eliminating $\widehat{z}_t^y$ through the government's budget constraint (52),[22] equations (27), (28), and (31) change to

$$\widehat{c} + \left[\frac{1-\alpha}{1+r} + \frac{\alpha}{(1+n)\,\varphi\left(\widehat{e}\right)}\right]\widehat{d} = w - (1+r)\frac{\widehat{e}}{\varphi\left(\widehat{e}\right)} + \left[\frac{1}{(1+n)\,\varphi\left(\widehat{e}\right)} - \frac{1}{1+r}\right]\widehat{z}^o, \tag{53}$$

$$(1-\alpha)\widehat{d} = (1+n)\,(1+r)\left[(1-\sigma)\,\widehat{e} + \widehat{k}\,\varphi\left(\widehat{e}\right)\right] - \widehat{z}^o, \tag{54}$$

$$\varphi'\left(\widehat{e}\right) = \frac{(1-\sigma)\,(1+r)}{w}. \tag{55}$$

The remaining steady-state equations in (26)–(33) do not change. Thus the market solutions for $\widehat{c}, \widehat{d}, \widehat{e}, \widehat{k}, w, r, i$, and π are now found from equations (26), (29), (30), (32), (33), and (53)–(55). To have these equations lead to the first-best balanced growth path characterized by (47)–(49), the policy instruments must be set as follows. First, lump-sum taxes z^y, z^o must be set such that the modified golden rule (48) is attained. This requirement is commonplace in overlapping generations models. Second, to have the first-best condition (49), or (51), satisfied, a subsidy is required on the purchase of e. The subsidy rate is found from (49) and (55) to be[23]

$$\sigma = \frac{1 + r - (1+n)(1+\rho)\left[\varphi\left(\widehat{e}\right)\right]^{1-\beta}}{1+r} + \frac{(1+n)\left[\varphi(\widehat{e}) - \widehat{e}\varphi'\left(\widehat{e}\right)\right]}{1+r},$$

where $\widehat{e}$ and r take their first-best values. Observe also that in the first best, with condition (48) satisfied, the above relationship is simplified to

$$\sigma = \frac{(1+n)\left[\varphi(\widehat{e}) - \widehat{e}\varphi'\left(\widehat{e}\right)\right]}{1+r}. \tag{56}$$

Finally, attaining (47), simultaneously with (48), is predicated on the well-known result in the money literature that at the optimum there should be no opportunity cost in holding money; that is that i must be equal to zero. To have (47) satisfied, in the face of (48), one must set $i = 0$ in equation (26).[24] The implication of this for the rate of money growth is straightforward. Substituting $i = 0$ in equation (9) yields $(1+r)(1+\pi) = 1$ so that from (48)

$$1+\pi = \frac{1}{(1+\rho)(1+n)\left[\varphi\left(\widehat{e}\right)\right]^{1-\beta}}$$

It then follows from (29) that

$$\theta = \frac{\left[\varphi\left(\widehat{e}\right)\right]^{\beta}}{1+\rho} - 1 = \frac{(1+g)^{\beta}}{1+\rho} - 1 < 0, \tag{57}$$

where $\widehat{e}$ and g are set at their first-best values. The sign of θ follows from the transversality condition (50).[25]

The following proposition summarizes the results of this section.

Proposition 2 *Consider the economy of Proposition 1. The first-best balanced-growth path of this economy is characterized by equations (47)–(49). It can be decentralized using generation- specific lump-sum taxes, a subsidy on educational expenditures given by equation (56), and the satisfaction of the Friedman rule that requires the opportunity cost of holding money to be zero.*

4 Second best with control of physical capital

Assume now that educational subsidies are unavailable so that the level of investment in education is not optimal. The question I want to address is if the Friedman rule continues to be optimal as long as the government can control the level of physical capital in the economy. To answer this question, I first characterize the second-best allocation when k can be controlled but not e. The formulation of this problem is exactly as that in the first best with the sole difference of not being able to optimize over e. The optimization problem is summarized by the Lagrangian (35) where the optimization is done with respect to c_t, d_{t+1}, h_{t+1}, and k_{t+1}. The corresponding first-order conditions are (36)–(39). Specifically, first-order condition (40) with respect to e_{t+1} does not hold which means equation (46), and thus (49), do not hold. The equations that characterize this second best are (44) and (45) and their corresponding balanced growth versions, equations (47) and (48).

The satisfaction of equation (47) in this second best means that the intertemporal consumption decision must remain undistorted (even though the choice of e is distorted). Interestingly, as long as one can control $\widehat{k}$ and thus $r = f'(\widehat{k})$, the condition for an undistorted intertemporal consumption decision remains $i = 0$. This follows because the other second-best optimality condition, the modified golden-rule condition (48), requires one to set $1 + r = (1 + n)(1 + \rho)\left[\varphi\left(\widehat{e}\right)\right]^{1-\beta}$. Setting $i = 0$ on top of this ensures that the intertemporal consumption decision remains undistorted. We have:

Proposition 3 *Consider the economy of Proposition 1. The Friedman rule holds in a second-best environment of this economy wherein the levels of education and human capital are suboptimal, as long as physical capital can be controlled and the modified golden rule is satisfied.*

4.1 Why the Friedman rule?

Given our second-best environment, one may wonder why introducing an additional distortion through the violation of the Friedman rule does not improve welfare. The answer lies in the manner that inflation affects the decentralized equilibrium solution of the economy in relation to the optimality conditions that characterize its first-best allocation. Specifically, observe that inflation has no *direct* effect on educational investment decisions. It can affect $\widehat{e}$ only through $\widehat{k}$ via equation (31). Now there is no benefit to using inflation to affect $\widehat{k}$ given that $\widehat{k}$ is fully controlled through generation-specific lump-sum taxes. Consequently, deviation from the Friedman rule, while having an adverse effect on intertemporal consumption decisions, has no countervailing effect on $\widehat{e}$.

Figure 1 illustrates this point diagrammatically. Given full control of $\widehat{k}$, the government can ensure that the equilibrium values of $\widehat{k}$ and $\widehat{e}$ satisfy the modified golden rule condition (48)

$$1 + r = (1 + n)(1 + \rho) \left[\varphi \left(\widehat{e}\right)\right]^{1-\beta}.$$

Differentiation establishes that the graph of this curve slopes downward as drawn in Figure 1. Observe that the government need not resort to inflation to satisfy this relationship.

Next consider the graph of (31) that determines $\widehat{e}$ as a function of $\widehat{k}$ in the laissez-faire (second-best equilibrium):

$$\varphi' \left(\widehat{e}\right) = \frac{1 + r}{w} = \frac{1 + f'(\widehat{k})}{f(\widehat{k}) - \widehat{k} f'(\widehat{k})}.$$

Differentiating this equation with respect to $\widehat{k}$ yields

$$\frac{d\widehat{e}}{d\widehat{k}} = \frac{w + (1+r)\,\widehat{k}}{w^2} \frac{f''\left(\widehat{k}\right)}{\varphi''\left(\widehat{e}\right)} > 0. \tag{58}$$

Intuitively, as $\widehat{k}$ increases, $(1+r)/w$ declines and this lowers the cost of borrowing to educate oneself relative to returns to education. This, in turn, increases the demand for education. It follows from equation (58) that the graph of (31) slopes upward. Inflation has no effect on (31); it does not shift it. The intersection of the graphs of (48) and (31) shows the second-best equilibrium.

4.2 Second-best versus first-best

To compare the second-best values for $\widehat{e}$ and $\widehat{k}$ versus their first-best counterparts, observe that whereas the modified golden rule condition (48) applies in both cases, the second-best laissez-faire condition (31) $\varphi'\left(\widehat{e}\right) = (1+r)/w$ is replaced by

$$\varphi'\left(\widehat{e}\right) = \frac{1+r}{w} - \frac{(1+n)\left[\varphi(\widehat{e}) - \widehat{e}\varphi'\left(\widehat{e}\right)\right]}{w} < \frac{1+r}{w} \tag{59}$$

in the first best. The inequality sign in (59) follows from the concavity of $\varphi(\cdot)$. It then follows from equations (31) and (59) that $\varphi'\left(\widehat{e}^{FB}\left(\widehat{k}\right)\right) < \varphi'\left(\widehat{e}^{SB}\left(\widehat{k}\right)\right)$ where superscripts FB and SB denote first best and second best. The concavity of $\varphi(\cdot)$ then implies $\widehat{e}^{FB}\left(\widehat{k}\right) > \widehat{e}^{SB}\left(\widehat{k}\right)$ so that the graph of $\widehat{e}^{FB}\left(\widehat{k}\right)$ is *above* graph $\widehat{e}^{SB}\left(\widehat{k}\right)$. This is depicted in Figure 1. Observe that the first-best value of $\widehat{e}$ exceeds its second-best value, while the first-best value of $\widehat{k}$ is smaller than its second-best value. We have $\widehat{e}^{SB} < \widehat{e}^{FB}$ and $\widehat{k}^{SB} > \widehat{k}^{FB}$.

The finding that $\widehat{e}^{SB} < \widehat{e}^{FB}$ and $\widehat{k}^{SB} > \widehat{k}^{FB}$ allows one to compare the first-best and second-best values of the other variables of interest. Specifically, with $\widehat{k}^{SB} > \widehat{k}^{FB}$, it follows that $r^{SB} < r^{FB}$ and $w^{SB} > w^{FB}$. However, at any given time t, it is not w that matters to the individual but wh_t. Now h_t grows at the rate of $g^{FB} = \varphi(\widehat{e}^{FB}) - 1$ in the first best and $g^{SB} = \varphi(\widehat{e}^{SB}) - 1$ in the second best, with $g^{SB} < g^{FB}$. Finally, observe that while $i = 0$ in both first and second best, the corresponding money growth rate that induces this differs across the two equilibria. Because condition (45) holds in the second best as well, the corresponding monetary growth rate continues to be given by equation (57)

$$\theta = \frac{[\varphi(\widehat{e})]^{\beta}}{1+\rho} - 1.$$

With $\widehat{e}^{SB} < \widehat{e}^{FB}$, the optimal monetary growth rate—the Friedman rule—calls for a smaller monetary growth rate. That is

$$\theta^{SB} < \theta^{FB} < 0.$$

These results are summarized as:

Proposition 4 *Consider the economy of Propositions 2 and 3. Let $\widehat{k}$ denote physical capital to human capital ratio, g denote the growth rate of the economy, θ denote the money growth rate, and superscripts FB and SB denote the corresponding first- and second-best solutions of this economy. Then, $\widehat{k}^{SB} > \widehat{k}^{FB}$, $\widehat{e}^{SB} < \widehat{e}^{FB}$, $g^{SB} < g^{FB}$, and $\theta^{SB} < \theta^{FB} < 0$.*

5 Second best without control of physical capital

Assume now that differential lump-sum tax and transfers are not feasible and the government cannot control the level of physical capital in the economy. However, educational subsidies are available and $\widehat{e}$ is set optimally. To characterize this second-best equilibrium, one can again formulate the problem as in the first best with the exception of not optimizing over k. Specifically, the optimization problem is summarized by the Lagrangian (35) where the optimization is done with respect to c_t, d_{t+1}, h_{t+1}, and e_{t+1}. The first-order condition that does not hold in this case is (39), and thus (45) and (48). The equations that characterize this second-best allocation are then (47) and (49). One continues to require an undistorted intertemporal consumption decision and an undistorted decision concerning educational expenditures.

Unlike the previous two cases, the Friedman rule of $i = 0$ no longer ensures that equation (47), requiring an undistorted intertemporal consumption decision, is satisfied. This arises because the modified golden rule condition (48) does not hold in this second best. Specifically, to bring about an undistorted intertemporal consumption decision, i.e. to satisfy (47), one must set i and θ, from (26) and (30), according to condition is satisfied if

$$i = \frac{(1+r)\left[\varphi\left(\widehat{e}\right)\right]^{\beta-1} - (1+n)(1+\rho)}{\alpha\left(1+n\right)\left(1+\rho\right)}, \tag{60}$$

$$\theta = -1 + \frac{(1+r)\left[\varphi\left(\widehat{e}\right)\right]^{\beta} - \left(1-\alpha\right)\left(1+n\right)(1+\rho)\varphi\left(\widehat{e}\right)}{\alpha\left(1+r\right)\left(1+\rho\right)}. \tag{61}$$

However, given that one does not control $\widehat{k}$ to ensure $1 + r = (1 + n)(1 + \rho)\left[\varphi\left(\widehat{e}\right)\right]^{1-\beta}$, and given that $i \geq 0$, equation (60) tells us that (47) is satisfied only if $1 + r > (1 + \rho)(1 + n)\left[\varphi\left(\widehat{e}\right)\right]^{1-\beta}$. That is, the equilibrium is characterized by a "low" physical-capital to human-capital ratio $\widehat{k}$. This makes sense. With $i \geq 0$, one can push $u_d\left(\widehat{c}, \widehat{d}\right)/u_c\left(\widehat{c}, \widehat{d}\right) = (1 + \alpha i)/(1 + r)$ only upwards. And this will be helpful if $u_d\left(\widehat{c}, \widehat{d}\right)/u_c\left(\widehat{c}, \widehat{d}\right)$ is, in the absence of inflation, "too small" (which arises if r is too high relative to its modified golden rule level). Figure 2 depicts this case wherein $\widehat{e}^{SB} < \widehat{e}^{FB}$ and $\widehat{k}^{SB} < \widehat{k}^{FB}$. It then also follows that in this case $g^{SB} < g^{FB}$.

On the other hand, if the laissez-faire solution entails too much capital, and a low r, then (47) is not satisfied. Under this circumstance, one wants to deflate the economy and the Friedman rule is satisfied as a boundary condition. Figure 3 depicts this case for which $\widehat{e}^{SB} > \widehat{e}^{FB}$ and $\widehat{k}^{SB} > \widehat{k}^{FB}$, so that $g^{SB} > g^{FB}$.

Proposition 5 *Consider the economy of Proposition 1. Assume that the government can control human capital fully but not physical capital.*

(i) The second-best balanced growth is characterized by equations (47) and (49).

(ii) There are two types of equilibrium. One is characterized by a "low" value of physical-capital to human-capital ratio for which the Friedman rule is not optimal. Under this circumstance, the optimal nominal interest rate is given by equation (60) and the optimal monetary growth rate by (61). In this case, $\widehat{e}^{SB} < \widehat{e}^{FB}, \widehat{k}^{SB} < \widehat{k}^{FB}$*, and* $g^{SB} < g^{FB}$.

(iii) The other type is characterized by a "high" value of physical-capital to human-capital ratio for which the Friedman rule is satisfied as a boundary condition. In this case, $\widehat{e}^{SB} > \widehat{e}^{FB}, \widehat{k}^{SB} > \widehat{k}^{FB}$*, and* $g^{SB} > g^{FB}$.

6 Third best and the Friedman rule

I now consider the desirability of the Friedman rule in an environment without generations-specific lump-sum taxes and education subsidies. The only available instrument to influence the economy is the monetary authority's control of the growth rate of money supply (without any coordination from the fiscal side). As a first step, I establish a result concerning the effect of money growth on the steady-state balanced-growth laissez-faire solution of the economy. Specifically, I prove in Appendix A that

$$\frac{d\widehat{k}}{d\theta} < 0.$$

This result is, in and of itself, an interesting one. It also implies that inflation lowers the growth rate of the economy. This follows from the laissez-faire condition (31) that governs the relationship between $\widehat{e}$ and $\widehat{k}$. Its differentiation with respect to θ yields

$$\frac{d\widehat{e}}{d\theta} = \frac{w + (1+r)\,\widehat{k}}{w^2} \frac{f''\left(\widehat{k}\right)}{\varphi''\left(\widehat{e}\right)} \frac{d\widehat{e}}{d\widehat{k}},$$

so that $d\widehat{e}/d\theta$ and $d\widehat{k}/d\theta$ are of the same sign. Consequently, $d\widehat{e}/d\theta < 0$. Observe also that with the growth rate of the economy being $g = \varphi\left(\widehat{e}\right) - 1$ and $\varphi'\left(\widehat{e}\right) > 0$, g moves positively with $\widehat{e}$ and

$$\frac{dg}{d\theta} < 0.$$

The dampening effect of money creation on growth appears at odds with the earlier results of van der Ploeg and Alogoskoufis (1994) and Mino and Shibata (1995) who find that a rise in money growth rate tends to boost the balanced growth rate. The main reason for this difference lies in the way money holdings are rationalized in those models as opposed to here. In those studies, real cash balances enter directly into the utility function. Given this formulation, an increase in θ increases the cost of holding money relative to consumption goods. Individuals respond by substituting away from consumption of money services into consumption of physical goods. Accommodating this increase in demand then generally requires a higher amount of real savings and capital accumulation.[26] In the present model, on the other hand, inflation works like a tax on future goods relative to present goods. A higher θ decreases the demand for future goods relative to today's. This results in lower real savings and capital accumulation.

It is important to point out that it is not just the cash-in-advance constraint modeling that is behind this result. The negative effect of money creation on growth depend also on the particular modeling of the cash-in-advance constraint. Unlike the rest of the results of the paper particularly those concerning the Friedman rule, this result may be reversed if the constraint applies to the expenditures of the young as well. In this latter formulation, the constraint works like a tax on consumption in both periods. If, as a percentage of their expenditures, the young will have to hold more cash than the old, the effective tax on present consumption will be higher than on future consumption leading to a substitution into future goods. See Appendix B.

Turning to the desirability of the Friedman rule in this case, consider two possible solutions that can emerge at $i = 0$. One is that $1 + r < (1 + n)(1 + \rho)\left[\varphi\left(\widehat{e}\right)\right]^{1-\beta}$ so that the equilibrium is characterized by "too much" capital. Under this circumstance, the $d\widehat{k}/d\theta < 0$ result suggests that increasing the money growth rate moves the economy in the right direction by lowering $\widehat{k}$. However, that $1+r < (1 + n)(1 + \rho)\left[\varphi\left(\widehat{e}\right)\right]^{1-\beta}$ also suggests $\widehat{u}_d/\widehat{u}_c$ exceeds its optimal value at $i = 0$; see (47). Increasing $\widehat{u}_d/\widehat{u}_c$ through inflation increases the intertemporal distortion further.

Second, assume that at $i = 0$, $1 + r < (1 + n)(1 + \rho)\left[\varphi\left(\widehat{e}\right)\right]^{1-\beta}$ so that the economy is characterized by "too little" capital. Under this circumstance, inflation further reduces $\widehat{k}$ and, on this score, worsens welfare. However, in this case, $\widehat{u}_d/\widehat{u}_c$ is less than optimal and inflation moves it in the right direction. Either way, the opposing effects on welfare renders the result ambiguous.

The results of this section are summarized as:

Proposition 6 *Consider the economy of Proposition 1.*

(i) An increase in the money growth rate leads to a reduction in the balanced growth path values of the physical capital per human capital ratio and the rate of growth of the economy.

(ii) If the only instrument through which the government can influence the economy is the growth rate of money supply, with no coordination from the fiscal side, the Friedman rule may or may not be optimal.

7 Conclusion

This paper has studied the implications of introducing money into an overlapping-generations model with endogenous growth where the rationalization for money holding comes from the Clower's cash-in-advance constraint. It has considered an economy populated with finitely-lived individuals whose human capital is determined partly through education and partly through their inherited human capital. Throughout the paper, to emphasize allocative efficiency, the paper has assumed no government tax requirements. With an external revenue requirement, one can always reject the Friedman rule by suitably ruling out certain fiscal instruments.[27]

The main message of the paper has been that, as long as the government can fully control the level of physical capital in the economy and sets it to satisfy the modified golden rule, the Friedman rule remains optimal. The result holds regardless of the ability of the government to control the level of human capital. With the control of human capital, controlling physical capital yields a first-best environment. Given this perspective, the result has generalized the earlier result of Abel (1987) and Gahvari (1988) derived for overlapping-generations models without endogenous growth. Without the control of human capital, the economy is in a second-best environment. The result thus contradicts one's intuition based on the second-best theory. The paper has clarified this apparent contradiction by demonstrating that introducing an additional distortion in the economy, via the violation of the Friedman rule, does nothing to alleviate the existing distortion in human capital accumulation as long as one can fully control physical capital accumulation. Furthermore, the paper has demonstrated that notwithstanding a zero nominal interest rate in both first- and second-best environments, the monetary growth rate and the economy's endogenous rate of growth are both smaller in the second best.

The paper has also studied another second-best environment; one in which human capital accumulation can be controlled but not physical capital. It has shown that in this setting the distortion due to the violation of the Friedman rule does alleviate the distortion due to the lack of physical capital accumulation. Here, second-best optimality calls for the violation of the Friedman rule. However, it is also possible for the Friedman rule to be satisfied in this case as a boundary condition.

A final interesting result is the possibility of a negative relationship between money growth rate on the one hand and the endogenous growth rate of the economy on the other. The reason for this is that, in a cash-in-advance constraint model, holding on to cash works like a tax on consumption goods. To the extent that the constraint applies more to the future goods as opposed to present, as in the cash-in-advance-constraint models à la Hahn and Solow (1995), future goods are taxed more heavily relative to present goods. This decreases the demand for future goods relative to today's resulting in lower real savings and capital accumulation. However, this particular result may be reversed if the young have to hold on to cash more than the old making the present goods relatively more expensive.

Appendix A

Satisfaction of the economy's resource constraint: Rewrite the old's budget constraint (12) for time t while substituting $(1+\theta)\,m_{t-1}$ for $m_{t-1}+b_t$ so that

$$\begin{aligned} d_t &= s_{t-1}\,(1+r_t) + \frac{(1+\theta)\,m_{t-1}}{p_t} \\ &= s_{t-1}\,(1+r_t) + \frac{(1+n)\,m_t}{p_t}. \end{aligned}$$

Eliminate m_t/p_t between this equation and equation (11) to get

$$\frac{d_t - s_{t-1}\,(1+r_t)}{1+n} = w_t h_t - e_t(1+r_t) - (c_t + s_t)$$

Rearranging the terms yields

$$c_t + \frac{d_t}{1+n} = w_t h_t - e_t(1+r_t) - s_t + \frac{s_{t-1}\,(1+r_t)}{1+n}$$

Substituting for s_t and s_{t-1} from (5) in above and simplifying results in

$$\begin{aligned} c_t + \frac{d_t}{1+n} + (1+n)\,(e_{t+1} + k_{t+1}) &= w_t h_t + k_t\,(1+r_t) \\ &= k_t + w_t h_t + r_t k_t \\ &= k_t + y_t, \end{aligned}$$

which is the economy's resource constraint.

Local stability of the steady-state solution: To examine the (local) stability properties of the steady-state solution, I linearize the system of difference equations (24)–(25) around the steady-state solution $\left(\widehat{k},\widehat{d}\right)$ according to

$$\begin{pmatrix} \widehat{k}_{t+1} - \widehat{k} \\ \widehat{d}_{t+1} - \widehat{d} \end{pmatrix} = \begin{pmatrix} \Psi_k\left(\widehat{k},\widehat{d}\right) & \Psi_d\left(\widehat{k},\widehat{d}\right) \\ \Phi_k\left(\widehat{k},\widehat{d}\right) & \Phi_d\left(\widehat{k},\widehat{d}\right) \end{pmatrix} \begin{pmatrix} \widehat{k}_t - \widehat{k} \\ \widehat{d}_t - \widehat{d} \end{pmatrix} \equiv \Omega \begin{pmatrix} \widehat{k}_t - \widehat{k} \\ \widehat{d}_t - \widehat{d} \end{pmatrix}, \quad \text{(A1)}$$

where $\Psi_k, \Psi_d, \Phi_k,$ and Φ_d denote the partial derivatives of $\Psi(\cdot)$ and $\Phi(\cdot)$ with respect to k and d. The dynamic path given by (A1) converges to a steady state as t increases (i.e., $\widehat{k}_t - \widehat{k}$ and $\widehat{d}_t - \widehat{d}$ tend to zero), if at time $t = 0$, the initial values of $\widehat{k}_0$ and $\widehat{d}_0$ are such that $\left(\widehat{k}_0 - \widehat{k}, \widehat{d}_0 - \widehat{d}\right)$ is in the space spanned by the eigenvectors of Ω that are associated with the eigenvalues of Ω with modulus smaller than one. Now, at $t = 0$, the value of $\widehat{k}_0$ is pre-determined. However, the value of $\widehat{d}_0$ depends on p_0. Consequently, the system will be stable for any value of p_0 if Ω possesses two eigenvalues with modulus less than one. If there is only one eigenvalue with a modulus less than one, then there will be one value for p_0 and a unique path which leads to the steady state.

Proof of $d\widehat{k}/d\theta < 0$: Substitute for i from equation (30) into equation (26) and simplify to get

$$\frac{u_d\left(\widehat{c}, \widehat{d}\right)}{u_c\left(\widehat{c}, \widehat{d}\right)} = \frac{1-\alpha}{1+r} + \frac{\alpha(1+\theta)}{(1+n)\varphi(\widehat{e})}. \tag{A2}$$

With equations (32)–(33) determining w and r as functions of $\widehat{k}$, equations (27) and (A2) determine $\widehat{c}$ and $\widehat{d}$ as functions of $\widehat{e}, \widehat{k}$ and θ. Moreover, equation (31) relates $\widehat{e}$ to w and r and thus to $\widehat{k}$. Substituting for $\widehat{e}$ as a function $\widehat{k}$ in (27) and (A2) then yields the solution for $\widehat{c}$ and $\widehat{d}$ as functions of $\widehat{k}$ and θ only: $\widehat{c}\left(\widehat{k}, \theta\right)$ and $\widehat{d}\left(\widehat{k}, \theta\right)$.

Combine equations (27) and (28), and substitute $\widehat{c}\left(\widehat{k}, \theta\right)$ and $\widehat{d}\left(\widehat{k}, \theta\right)$ in it to arrive at

$$(1+n)\left[\widehat{k}\varphi\left(\widehat{e}\left(\widehat{k}\right)\right) + \widehat{e}\left(\widehat{k}\right)\right] = w - (1+r)\frac{\widehat{e}\left(\widehat{k}\right)}{\varphi\left(\widehat{e}\left(\widehat{k}\right)\right)} - \widehat{c}\left(\widehat{k}, \theta\right) - \frac{\alpha}{1+n}\frac{\widehat{d}\left(\widehat{k}, \theta\right)}{\varphi\left(\widehat{e}\left(\widehat{k}\right)\right)}. \tag{A3}$$

Rewrite equation (A3) as

$$\begin{aligned}\Theta\left(\widehat{k}, \theta\right) \equiv &(1+n)\left[\widehat{k}\varphi\left(\widehat{e}\left(\widehat{k}\right)\right) + \widehat{e}\left(\widehat{k}\right)\right] - \\ &\left[w - (1+r)\frac{\widehat{e}\left(\widehat{k}\right)}{\varphi\left(\widehat{e}\left(\widehat{k}\right)\right)} - \widehat{c}\left(\widehat{k}, \theta\right) - \frac{\alpha}{1+n}\frac{\widehat{d}\left(\widehat{k}, \theta\right)}{\varphi\left(\widehat{e}\left(\widehat{k}\right)\right)}\right] = 0.\end{aligned}$$

The $\Theta\left(\widehat{k},\theta\right)=0$ relationship gives the equilibrium value of $\widehat{k}$ as a function of θ. Differentiating it totally with respect to θ yields

$$\begin{aligned}&\frac{\partial\Theta\left(\widehat{k},\theta\right)}{\partial\widehat{k}}\frac{d\widehat{k}}{d\theta}+\frac{\partial\Theta\left(\widehat{k},\theta\right)}{\partial\theta}=\\&\frac{\partial\Theta\left(\widehat{k},\theta\right)}{\partial\widehat{k}}\frac{d\widehat{k}}{d\theta}+\frac{\partial\widehat{c}\left(\widehat{k},\theta\right)}{\partial\theta}+\frac{\alpha}{1+n}\frac{1}{\varphi\left(\widehat{e}\right)}\frac{\partial\widehat{d}\left(\widehat{k},\theta\right)}{\partial\theta}=0.\end{aligned}$$

Rearranging and defining $\Gamma\equiv\partial\Theta\left(\widehat{k},\theta\right)/\partial\widehat{k}$ yields

$$\frac{d\widehat{k}}{d\theta}=-\frac{1}{\Gamma}\left[\frac{\partial\widehat{c}}{\partial\theta}+\frac{\alpha}{(1+n)\varphi\left(\widehat{e}\right)}\frac{\partial\widehat{d}}{\partial\theta}\right]. \tag{A4}$$

Now observe that one can think of the left-hand side of (A3) as the demand for $\widehat{k}$ and its right-hand side as the supply of $\widehat{k}$. The static stability condition then requires that the excess demand function $\Theta\left(\widehat{k},\theta\right)$ to be downward-sloping in r or upward-sloping in $\widehat{k}$. Consequently, $\Gamma>0$.

Next determine the bracketed expression on the right-hand side of (A4) by partially differentiating the system of equations (27) and (A2) with respect to θ. The resulting equations are, in matrix notation,

$$\begin{pmatrix}\widehat{u}_{cd}-\frac{\widehat{u}_d}{\widehat{u}_c}\widehat{u}_{cc} & \widehat{u}_{dd}-\frac{\widehat{u}_d}{\widehat{u}_c}\widehat{u}_{cd}\\ 1 & \frac{1-\alpha}{1+r}+\frac{\alpha}{(1+n)\varphi(\widehat{e})}\end{pmatrix}\begin{pmatrix}\frac{\partial\widehat{c}}{\partial\theta}\\ \frac{\partial\widehat{d}}{\partial\theta}\end{pmatrix}=\begin{pmatrix}\frac{\alpha\,\widehat{u}_c}{(1+n)\varphi(\widehat{e})}\\ 0\end{pmatrix}. \tag{A5}$$

Let

$$\Delta\equiv\left(\widehat{u}_{cd}-\frac{\widehat{u}_d}{\widehat{u}_c}\widehat{u}_{cc}\right)\left[\frac{1-\alpha}{1+r}+\frac{\alpha}{(1+n)\,\varphi\left(\widehat{e}\right)}\right]-\left(\widehat{u}_{dd}-\frac{\widehat{u}_d}{\widehat{u}_c}\widehat{u}_{cd}\right), \tag{A6}$$

denote the determinant of the 2×2 matrix that appears on the left-hand side of (A5). Premultiplying (A5) by the inverse of this matrix yields

$$\begin{pmatrix}\frac{\partial\widehat{c}}{\partial\theta}\\ \frac{\partial\widehat{d}}{\partial\theta}\end{pmatrix}=\frac{1}{\Delta}\begin{pmatrix}\frac{1-\alpha}{1+r}+\frac{\alpha}{(1+n)\varphi(\widehat{e})} & -\widehat{u}_{dd}+\frac{\widehat{u}_d}{\widehat{u}_c}\widehat{u}_{cd}\\ -1 & \widehat{u}_{cd}-\widehat{u}_{cc}\frac{\widehat{u}_d}{\widehat{u}_c}\end{pmatrix}\begin{pmatrix}\frac{\alpha\,\widehat{u}_c}{(1+n)\varphi(\widehat{e})}\\ 0\end{pmatrix}. \tag{A7}$$

Then rewrite (A7) as

$$
\begin{aligned}
\frac{\partial \widehat{c}}{\partial \widehat{\theta}} &= \frac{1}{\Delta}\left[\frac{1-\alpha}{1+r}+\frac{\alpha}{(1+n)\,\varphi\left(\widehat{e}\right)}\right]\frac{\alpha\,\widehat{u}_c}{(1+n)\,\varphi\left(\widehat{e}\right)}, && \text{(A8)}\\
\frac{\partial \widehat{d}}{\partial \widehat{\theta}} &= \frac{1}{\Delta}\frac{-\alpha\,\widehat{u}_c}{(1+n)\,\varphi\left(\widehat{e}\right)}. && \text{(A9)}
\end{aligned}
$$

To determine the sign of (A8)–(A9) one must determine the sign of Δ. Clearly, $\partial\widehat{c}/\partial\widehat{\theta}$ is of the same sign as Δ and $\partial\widehat{d}/\partial\widehat{\theta}$ is of the opposite sign to Δ. To do this, rewrite (A6) as

$$
\begin{aligned}
\Delta &= -\left[\frac{1-\alpha}{1+r}+\frac{\alpha}{(1+n)\,\varphi\left(\widehat{e}\right)}\right]\frac{\widehat{u}_d}{\widehat{u}_c}\widehat{u}_{cc}-\widehat{u}_{dd}+\left[\frac{1-\alpha}{1+r}+\frac{\alpha}{(1+n)\,\varphi\left(\widehat{e}\right)}+\frac{\widehat{u}_d}{\widehat{u}_c}\right]\widehat{u}_{cd}\\
&= \left[\frac{1-\alpha}{1+r}+\frac{\alpha}{(1+n)\,\varphi\left(\widehat{e}\right)}\right]\left(\widehat{u}_{cd}-\frac{\widehat{u}_d}{\widehat{u}_c}\widehat{u}_{cc}\right)+\left(\frac{\widehat{u}_d}{\widehat{u}_c}\widehat{u}_{cd}-\widehat{u}_{dd}\right).
\end{aligned}
$$

Assuming d and c are normal goods, one can easily show that $\widehat{u}_{cd}-\left(\widehat{u}_d/\widehat{u}_c\right)\widehat{u}_{cc}>0$ and $\left(\widehat{u}_d/\widehat{u}_c\right)\widehat{u}_{cd}-\widehat{u}_{dd}>0$ (where due to quasi-concavity of preferences, $-\left(\widehat{u}_d/\widehat{u}_c\right)^2\widehat{u}_{cc}-\widehat{u}_{dd}+2\left(\widehat{u}_d/\widehat{u}_c\right)\widehat{u}_{cd}>0$). Consequently, $\Delta>0$. Given the sign of Δ, it follows that $\partial\widehat{c}/\partial\widehat{\theta}>0$ and $\partial\widehat{d}/\partial\widehat{\theta}<0$. Intuitively, as θ increases the relative price of future goods to present consumption increases so that consumers respond by lowering their consumption of future goods and increasing their consumption of present goods.

Finally, substituting for $\partial\widehat{c}/\partial\theta$ and $\partial\widehat{d}/\partial\theta$ from (A8)–(A9) into the bracketed expression on the right-hand side of (A4) yields

$$
\frac{\partial \widehat{c}}{\partial \theta}+\frac{\alpha}{(1+n)\varphi\left(\widehat{e}\right)}\frac{\partial \widehat{d}}{\partial \theta}=\frac{1}{\Delta}\frac{\alpha\left(1-\alpha\right)\,\widehat{u}_c}{(1+r)\,(1+n)\,\varphi\left(\widehat{e}\right)}>0.
$$

It then follows from (A4) that

$$
\frac{d\widehat{k}}{d\theta}<0.
$$

Appendix B

Clower cash-in-advance constraint applied to expenditures in both periods

As previously, at the beginning of each period, the old hold all the stock of money M_t. Hence the young will have to borrow money from the old in order to finance their transactions (they have no assets of their own). The old will charge the young a rate of return on these lendings equal to what they can earn on real assets. Otherwise, rather than carrying cash for the young, they invest their savings in real assets and for their own transaction needs. Let m_t^y and m_t^o denote the amount of cash needed for transactions of each young and each old person at time t. A young person spends $(1+i_t)\, m_t^y$ when he buys m_t^y from the old at the *beginning* of period t (prior to the money injection into the economy that pushes prices from p_{t-1} to p_t). At the end of period t, he will also purchase the money used by the old for their own transactions. This is equal to $N_{t-1}m_t^o/N_t = m_t^o/\,(1+n)$ and is inclusive of the money distribution to the old. Consequently, the first-period budget constraint of the young, equation (11), will change to

$$c_t + s_t + (1+i_t)\,\frac{m_t^y}{p_t} + \frac{m_t^o/p_t}{1+n} = w_t h_t - e_t(1+r_t). \tag{B1}$$

Come the second period, the N_t individuals have grown old and, at the at the *beginning* of period $t+1$, will sell $N_{t+1}m_{t+1}^y$ money to the young of that period for their money transactions. Thus each old person sells $N_{t+1}m_{t+1}^y/N_t = (1+n)\, m_{t+1}^y$ for which he receives $(1+n)\, m_{t+1}^y\,(1+i_{t+1})$. Then, at the *end* of period $t+1$, each old person will sell what he used for transactions himself, m_{t+1}^o. Again, taking place at the end of the period, this is inclusive of money distributions. The second-period budget constraint for the young of period t, equation (12) of the text, now changes to

$$d_{t+1} = s_t\,(1+r_{t+1}) + (1+n)\,\frac{m_{t+1}^y}{p_{t+1}}\,(1+i_{t+1}) + \frac{m_{t+1}^o}{p_{t+1}}. \tag{B2}$$

Recall also that money grows in the economy according to the relationship

$$M_{t+1} = M_t + N_t b_{t+1},$$

which one can write as

$$N_{t+1}m_{t+1}^y + N_t m_{t+1}^o = N_t m_t^y + N_{t-1}m_t^o + N_t b_{t+1}.$$

Dividing by N_t, and rearranging,

$$(1+n)\, m_{t+1}^y + m_{t+1}^o = m_t^y + \frac{m_t^o}{1+n} + b_{t+1}. \tag{B3}$$

Substituting for $(1+n)\, m_{t+1}^y$ from (B3) into (B2), and writing $(1+i_{t+1})\, p_t/p_{t+1}$ for $(1+r_{t+1})$, one can rewrite the second-period budget constraint as

$$d_{t+1} = \left(p_t s_t + m_t^y + \frac{m_t^o}{1+n} + b_{t+1} - m_{t+1}^o \right) \frac{1+i_{t+1}}{p_{t+1}} + \frac{m_{t+1}^o}{p_{t+1}}. \tag{B4}$$

Eliminating s_t between (B1) and (B4), and simplifying yields

$$\begin{aligned} d_{t+1} &= \left[p_t w_t h_t - p_t e_t (1+r_t) - p_t c_t - i_t m_t^y + b_{t+1} - m_{t+1}^o \right] \frac{1+i_{t+1}}{p_{t+1}} + \frac{m_{t+1}^o}{p_{t+1}} \\ &= \left[w_t h_t - e_t (1+r_t) - c_t - i_t \frac{m_t^y}{p_t} + \frac{b_{t+1}}{p_t} - \frac{m_{t+1}^o}{p_t} \right] (1+r_{t+1}) + \frac{m_{t+1}^o}{p_{t+1}}. \end{aligned}$$

Dividing by $(1+r_{t+1})$, simplifying and rearranging the terms, one can write the above intertemporal budget constraint as

$$c_t + i_t \frac{m_t^y}{p_t} + \frac{1}{1+r_{t+1}} \left[d_{t+1} + i_{t+1} \frac{m_{t+1}^o}{p_{t+1}} \right] = w_t h_t - e_t (1+r_t) + \frac{b_{t+1}}{p_t}. \tag{B5}$$

The Clower cash-in-advance constraint in this case can be written as,

$$\frac{m_t^y}{p_t} = \gamma c_t, \tag{B6}$$

$$\frac{m_{t+1}^o}{p_{t+1}} = \alpha d_{t+1}, \tag{B7}$$

where I have allowed for the proportion of expenditures to be financed through cash to be different in the two periods. Substituting these expressions in (B5) yields,

$$(1+\gamma i_t)\, c_t + (1+\alpha i_{t+1}) \frac{d_{t+1}}{1+r_{t+1}} = w_t h_t - e_t (1+r_t) + \frac{b_{t+1}}{p_t}. \tag{B8}$$

This equation replaces (21) in the text. The individual's optimization problem then yields

$$\frac{\partial u\,(c_t, d_{t+1}) / \partial d_{t+1}}{\partial u\,(c_t, d_{t+1}) / \partial c_t} = \frac{1+\alpha i_{t+1}}{(1+\gamma i_t)\,(1+r_{t+1})}, \tag{B9}$$

which replaces (16).

Next, substituting

$$
\begin{aligned}
m_{t+1}^{y} &= \frac{1+\theta}{1+n} m_t^{y}, \\
m_{t+1}^{o} &= \frac{1+\theta}{1+n} m_t^{o},
\end{aligned}
$$

in (B3) and simplifying, then using (B6)–(B7), one finds the value of b_{t+1} to be

$$
\begin{aligned}
b_{t+1} &= \theta m_t^{y} + \frac{\theta}{1+\theta} m_{t+1}^{o}, \\
&= \theta \gamma p_t c_t + \frac{\theta}{1+\theta} \alpha p_{t+1} d_{t+1}.
\end{aligned}
$$

Substituting in (B8) yields, after simplification,

$$
\left[1+\gamma\left(i_t-\theta\right)\right] c_t + \left[1+\frac{\alpha\left(i_{t+1}-\theta\right)}{1+\theta}\right] \frac{d_{t+1}}{1+r_{t+1}} = w_t h_t - e_t(1+r_t).
$$

Divide this equation by h_t to get

$$
\left[1+\gamma\left(i_t-\theta\right)\right] \widehat{c}_t + \left[1+\frac{\alpha(i_{t+1}-\theta)}{1+\theta}\right] \frac{\widehat{d}_{t+1}}{1+r_{t+1}} = w_t - (1+r_t)\frac{\widehat{e}_t}{\varphi\left(\widehat{e}_t\right)}. \tag{B10}
$$

This equation replaces equation (22) in the text. The two equations (B9)–(B10), replacing (16) and (22), are the only equations relevant for determining the temporal equilibrium of the economy that change as result of having a cash-in-advance constraint for the young as well.

Turning to the laissez faire balanced-growth path of the economy, characterized by equations (26)–(33) in the text, equations (26)–(27) are replaced by the steady-state versions of (B9)–(B10), where in (B10) I have substituted for $i-\theta$ from (30),

$$
\frac{u_d\left(\widehat{c},\widehat{d}\right)}{u_c\left(\widehat{c},\widehat{d}\right)} = \frac{1+\alpha i}{\left(1+\gamma i\right)\left(1+r\right)}, \tag{B11}
$$

$$
\left\{1+\gamma\left(1+\theta\right)\left[\frac{1+r}{\left(1+n\right)\varphi\left(\widehat{e}\right)}-1\right]\right\}\widehat{c} + \left[\frac{1-\alpha}{1+r}+\frac{\alpha}{\left(1+n\right)\varphi\left(\widehat{e}\right)}\right]\widehat{d} = w - (1+r)\frac{\widehat{e}}{\varphi\left(\widehat{e}\right)}. \tag{B12}
$$

The implications of these changes for the results of the paper can now be checked.

If $\gamma \neq \alpha$, it follows from the above equations that unless $i = 0$ the first-best conditions $u_d\left(\widehat{c}, \widehat{d}\right)/u_c\left(\widehat{c}, \widehat{d}\right) = 1/(1+r) = (1+n)(1+\rho)\left[\varphi\left(\widehat{e}\right)\right]^{1-\beta}$ will not be satisfied in the laissez-faire equilibrium. On the other hand, if $\gamma = \alpha$, $u_d\left(\widehat{c}, \widehat{d}\right)/u_c\left(\widehat{c}, \widehat{d}\right) = 1/(1+r)$ and if capital is controlled through lump-sum taxes to ensure $1 + r = (1+n)(1+\rho)\left[\varphi\left(\widehat{e}\right)\right]^{1-\beta}$ it *appears* that the optimal monetary policy is indeterminate. However, I prove below that even in this case the Friedman rule must hold.

To see the necessity of the Friedman rule when capital is fully controlled, consider (B12). In this equation, only the price of $\widehat{c}$ depends on θ. It is plain that individuals attain their highest utility level if θ is set to minimize this price. Now with capital being set optimally, $1 + r = (1+n)(1+\rho)\left[\varphi\left(\widehat{e}\right)\right]^{1-\beta}$, and the price of $\widehat{c}$ becomes

$$1 + \gamma\left(1+\theta\right)\left[\frac{1+r}{\left(1+n\right)\varphi\left(\widehat{e}\right)} - 1\right] = 1 + \gamma\left(1+\theta\right)\left\{\left(1+\rho\right)\left[\varphi\left(\widehat{e}\right)\right]^{-\beta} - 1\right\}.$$

Moreover, from the transversality condition (50) for the optimal balanced growth path, $\left(1+\rho\right)\left[\varphi\left(\widehat{e}\right)\right]^{-\beta} - 1 > 0$. Hence the price of $\widehat{c}$ is minimal when $1+\theta$ assume its smallest value. But, because $i \geq 0$,

$$1 + i = \frac{\left(1+\theta\right)\left(1+r\right)}{\left(1+n\right)\varphi\left(\widehat{e}\right)} \geq 1,$$

or

$$1 + \theta \geq \frac{\left(1+n\right)\varphi\left(\widehat{e}\right)}{1+r}.$$

Consequently, the smallest value of $1+\theta$ is attained when $1+\theta = \left(1+n\right)\varphi\left(\widehat{e}\right)/\left(1+r\right)$. This, in turn, implies that $i = 0$.

These results sow that our findings in Propositions 1–4 remain valid. The same is true for the general message of Proposition 5 unless $\gamma = \alpha$. To see this, observe that the satisfaction of an undistorted intertemporal decision, condition (47), now requires are now replaced by

$$\begin{aligned}
i &= -\frac{\left(1+r\right) - \left(1+n\right)\left(1+\rho\right)\left[\varphi\left(\widehat{e}\right)\right]^{1-\beta}}{\gamma\left(1+r\right) - \alpha\left(1+n\right)\left(1+\rho\right)\left[\varphi\left(\widehat{e}\right)\right]^{1-\beta}}, \\
\theta &= -1 + \frac{\left(1-\alpha\right)\left(1+n\right)^2\left(1+\rho\right)\left[\varphi\left(\widehat{e}\right)\right]^{2-\beta} - \left(1-\gamma\right)\left(1+r\right)\left(1+n\right)\varphi\left(\widehat{e}\right)}{\gamma\left(1+r\right)^2 - \alpha\left(1+n\right)\left(1+\rho\right)\left(1+r\right)\left[\varphi\left(\widehat{e}\right)\right]^{1-\beta}}.
\end{aligned}$$

These conditions replace (60)–(61) in the text. Three possibilities arise. If $\gamma < \alpha$, we have a similar case to that in the text where $\gamma = 0$. Under this circumstance, if $1+r > (1+n)(1+\rho)[\varphi(\widehat{e})]^{1-\beta}$, condition (47) can be satisfied by opting for $i > 0$ and pushing $u_d\left(\widehat{c}, \widehat{d}\right)/u_c\left(\widehat{c}, \widehat{d}\right)$ upwards. And if $1+r < (1+n)(1+\rho)[\varphi(\widehat{e})]^{1-\beta}$, the Friedman rule is satisfied as a boundary condition. Second, one can have $\gamma > \alpha$. Then if $1+r < (1+n)(1+\rho)[\varphi(\widehat{e})]^{1-\beta}$, condition (47) can be satisfied by opting for $i > 0$ and pushing $u_d\left(\widehat{c}, \widehat{d}\right)/u_c\left(\widehat{c}, \widehat{d}\right) = (1+\alpha i)/(1+\gamma i)(1+r)$ downwards. On the other hand, if $1+r > (1+n)(1+\rho)[\varphi(\widehat{e})]^{1-\beta}$, the Friedman rule is satisfied as a boundary condition. The third possibility is that $\gamma = \alpha$. Under this circumstance, $u_d\left(\widehat{c}, \widehat{d}\right)/u_c\left(\widehat{c}, \widehat{d}\right)$ is independent of i. With $1+r \neq (1+n)(1+\rho)[\varphi(\widehat{e})]^{1-\beta}$, condition (47) can never be satisfied. Friedman rule is then again satisfied as a boundary condition.

Finally, consider Proposition 6. Substitute for i from equation (30) into (B11) to get

$$\frac{u_d\left(\widehat{c}, \widehat{d}\right)}{u_c\left(\widehat{c}, \widehat{d}\right)} = \frac{1}{1+r}\frac{(1-\alpha)(1+n)\varphi(\widehat{e}) + \alpha(1+r)(1+\theta)}{(1-\gamma)(1+n)\varphi(\widehat{e}) + \gamma(1+r)(1+\theta)}. \tag{B13}$$

Using (31), one can solve equations (B12)–(B13) to derive $\widehat{c}\left(\widehat{k}, \theta\right)$ and $\widehat{d}\left(\widehat{k}, \theta\right)$. Next combine (28) with equation (B12), the replacement for (27), and substitute $\widehat{c}\left(\widehat{k}, \theta\right)$ and $\widehat{d}\left(\widehat{k}, \theta\right)$ in the resulting equation to get

$$\begin{aligned}\Theta'\left(\widehat{k}, \theta\right) \equiv &(1+n)\left[\widehat{k}\varphi\left(\widehat{e}\left(\widehat{k}\right)\right) + \widehat{e}\left(\widehat{k}\right)\right] - \\ &\left\{w - (1+r)\frac{\widehat{e}\left(\widehat{k}\right)}{\varphi\left(\widehat{e}\left(\widehat{k}\right)\right)} - A\widehat{c}\left(\widehat{k}, \theta\right) - \frac{\alpha}{1+n}\frac{\widehat{d}\left(\widehat{k}, \theta\right)}{\varphi\left(\widehat{e}\left(\widehat{k}\right)\right)}\right\} = 0,\end{aligned} \tag{B14}$$

where

$$A \equiv \left[1 + \gamma(1+\theta)\left(\frac{1+r}{(1+n)\varphi(\widehat{e})} - 1\right)\right].$$

This equation replaces (A3). Differentiating (B14) totally with respect to θ yields

$$\begin{aligned}
&\frac{\partial\Theta'\left(\widehat{k},\theta\right)}{\partial\widehat{k}}\frac{d\widehat{k}}{d\theta}+\frac{\partial\Theta'\left(\widehat{k},\theta\right)}{\partial\theta}= \\
&\frac{\partial\Theta'\left(\widehat{k},\theta\right)}{\partial\widehat{k}}\frac{d\widehat{k}}{d\theta}+\gamma\left[\frac{1+r}{(1+n)\,\varphi\left(\widehat{e}\right)}-1\right]\widehat{c}\left(\widehat{k},\theta\right)+ \\
&A\frac{\partial\widehat{c}\left(\widehat{k},\theta\right)}{\partial\theta}+\frac{\alpha}{1+n}\frac{1}{\varphi\left(\widehat{e}\right)}\frac{\partial\widehat{d}\left(\widehat{k},\theta\right)}{\partial\theta}=0.
\end{aligned}\tag{B15}$$

Rearranging (B15)

$$\begin{aligned}
\frac{d\widehat{k}}{d\theta} &= -\frac{1}{\Gamma'}\gamma\left[\frac{1+r}{(1+n)\,\varphi\left(\widehat{e}\right)}-1\right]\widehat{c}\left(\widehat{k},\theta\right) \\
&\quad -\frac{1}{\Gamma'}\left\{A\frac{\partial\widehat{c}\left(\widehat{k},\theta\right)}{\partial\theta}+\frac{\alpha}{1+n}\frac{1}{\varphi\left(\widehat{e}\right)}\frac{\partial\widehat{d}\left(\widehat{k},\theta\right)}{\partial\theta}\right\}.
\end{aligned}\tag{B16}$$

Equation (B16) replaces equation (A4) where $\Gamma'\equiv\partial\Theta'\left(\widehat{k},\theta\right)/\partial\widehat{k}>0$ (due to the same stability condition).

Next, one can determine $\partial\widehat{c}\left(\widehat{k},\theta\right)/\partial\theta$ and $\partial\widehat{d}\left(\widehat{k},\theta\right)/\partial\theta$ by partially differentiating (B12)–(B13) with respect to θ. A bit of tedious algebraic calculations yields

$$\begin{aligned}
&\begin{pmatrix}\widehat{u}_{cd}-\frac{\widehat{u}_d}{\widehat{u}_c}\widehat{u}_{cc} & \widehat{u}_{dd}-\frac{\widehat{u}_d}{\widehat{u}_c}\widehat{u}_{cd}\\ A & \frac{1-\alpha}{1+r}+\frac{\alpha}{(1+n)\varphi(\widehat{e})}\end{pmatrix}\begin{pmatrix}\frac{\partial\widehat{c}}{\partial\theta}\\ \frac{\partial\widehat{d}}{\partial\theta}\end{pmatrix}= \\
&\begin{pmatrix}\frac{\alpha u_c\left(\widehat{c},\widehat{d}\right)-\gamma(1+r)u_d\left(\widehat{c},\widehat{d}\right)}{\{(1-\gamma)(1+n)\varphi(\widehat{e})+\gamma(1+r)(1+\theta)\}}\\ -\gamma\left[\frac{1+r}{(1+n)\varphi(\widehat{e})}-1\right]\widehat{c}\end{pmatrix}.
\end{aligned}\tag{B17}$$

Premultiplying (B17) by the inverse of the 2×2 matrix on its left-hand side yields

$$\begin{aligned}
\begin{pmatrix}\frac{\partial\widehat{c}}{\partial\theta}\\ \frac{\partial\widehat{d}}{\partial\theta}\end{pmatrix} &= \frac{1}{\Omega}\begin{pmatrix}\frac{1-\alpha}{1+r}+\frac{\alpha}{(1+n)\varphi(\widehat{e})} & -\widehat{u}_{dd}+\frac{\widehat{u}_d}{\widehat{u}_c}\widehat{u}_{cd}\\ -A & \widehat{u}_{cd}-\frac{\widehat{u}_d}{\widehat{u}_c}\widehat{u}_{cc}\end{pmatrix}\times \\
&\quad\begin{pmatrix}\frac{\alpha u_c\left(\widehat{c},\widehat{d}\right)-\gamma(1+r)u_d\left(\widehat{c},\widehat{d}\right)}{\{(1-\gamma)(1+n)\varphi(\widehat{e})+\gamma(1+r)(1+\theta)\}}\\ -\gamma\left[\frac{1+r}{(1+n)\varphi(\widehat{e})}-1\right]\widehat{c}\end{pmatrix}.
\end{aligned}\tag{B18}$$

where

$$\Omega \equiv \left[\frac{1-\alpha}{1+r}+\frac{\alpha}{(1+n)\,\varphi\left(\widehat{e}\right)}\right]\left(\widehat{u}_{cd}-\frac{\widehat{u}_d}{\widehat{u}_c}\widehat{u}_{cc}\right)+A\left(\frac{\widehat{u}_d}{\widehat{u}_c}\widehat{u}_{cd}-\widehat{u}_{dd}\right)>0.$$

A bit more algebraic manipulation reveals that at $(1+r)\,/\,(1+n)\,\varphi\left(\widehat{e}\right)=1$,

$$\begin{aligned}\frac{d\widehat{k}}{d\theta} &= -\frac{1}{\Gamma'}\left\{\frac{\partial\widehat{c}\left(\widehat{k},\theta\right)}{\partial\theta}+\frac{\alpha}{1+n}\frac{1}{\varphi\left(\widehat{e}\right)}\frac{\partial\widehat{d}\left(\widehat{k},\theta\right)}{\partial\theta}\right\} \\ &= -\frac{1}{\Gamma'}\frac{(\alpha-\gamma)\,(1-\alpha)}{\Omega\,(1+r)}\frac{(1+n)\,\varphi\left(\widehat{e}\right)}{\left\{(1-\gamma)\,(1+n)\,\varphi\left(\widehat{e}\right)+\gamma\,(1+r)\,(1+\theta)\right\}^2}u_c\left(\widehat{c},\widehat{d}\right).\end{aligned} \tag{B19}$$

so that $d\widehat{k}/d\theta$ is of opposite sign to $(\alpha-\gamma)$.

Notes

[1] Gahvari (1988) went further and showed that the optimality of the Friedman rule does not rest on the attainment of the modified golden rule. If a switch to the Friedman rule is accompanied by generation specific lump-sum taxes that neutralize the ensuing intergenerational wealth transfers, the Friedman rule becomes optimal given any initial steady state laissez faire equilibrium. This finding, and the importance of intergenerational wealth transfers appear to have gone unnoticed in the subsequent literature dealing with the Friedman rule until its rediscovery by Bhattacharya *et al.* (2005) who extended this result to models with different rationalizations for holding money. See, e.g., Freeman (1993), and Smith (1991, 2002). An exception is Ireland (2005); this issue has been discussed in Gahvari (2007).

[2] In van der Ploeg and Alogoskoufis' (1994) endogenous-growth model "there is learning by doing because then knowledge from one producer spills over and increases the output of rival firms" (p. 776). They consider a setup as in Weil (1989) wherein individuals are infinitely-lived but that new generations are born every period. They also rationalize money by putting real balances in the utility function.

[3] Gahvari (1988) shows that tax policy can be used in overlapping-generations models to offset the distributional effects of money creation across generations and neutralize the Tobin effect.

[4] That societies have more at stake in building up their human capital than what is forthcoming via market economies is a widely-accepted proposition. The literature has modeled this type of divergence between private and social valuation in a number of ways. Docquier *et al.*'s (2007) model is a particularly instructive approach in that it distinguishes between private and external effects of education in a simple and natural way. Education builds up one's own human

capital and individuals take this into account in their decision making process. But by educating oneself, one also helps the human capital of other agents in the economy. This remains an externality because it is hard to imagine that one can create markets for it.

[5]Docquier *et al.*'s (2007) result applies and the relationship between laissez faire and first-best solutions remains the same as in their paper.

[6]Another interesting recent study is Cunha (2008) who studies this question in the context of a two-sector open economy. He shows that the availability of a consumption tax on non-traded goods ensures the optimality of the Friedman rule, but if the tax is missing an inflation tax becomes a useful substitute for it.

[7]Other papers that discusses the effects of monetary growth on the real economy within an endogenous growth framework include Wang and Yip (1992), Gomme (1993), van der Ploeg and Alogoskoufis (1994), Jones and Manuelli (1995), Chang and Lai (2000), Dotsey and Sarte (2000), Chang *et al.* (2007), and Bhattacharya *et al.* (2009).

[8]Whether capital depreciates or not is of no consequence for the results. With depreciation, the relevant rate of return to savings is the net rate which is equal to the gross rate used in the paper net of the depreciation rate. Of course, in the steady state, investment in new capital should also cover the depreciation (in addition to the amount required due to population growth rate).

[9]This formulation is a "short cut" to stay within a two-period overlapping generations model. A more realistic formulation allows for three periods, the first of which is dedicated to the education of children.

[10]The Inada conditions ensure a positive level of investment in education. The assumption that $\varphi(0) = 1$ is tantamount to ruling out depreciation in inheriting one's parents' human capital. It ensures that a positive level of investment in education increases one's level of human capital over that of one's parents. Put differently, it ensures that the human capital of successive generations increases. Without this assumption, it is *possible* to have a steady-state equilibrium with declining human capital. See footnote 18 below.

[11]Given the paper's focus on the balanced growth/steady state equilibrium, the invariance of θ over time leads to no loss of generality.

[12]The exception is the effect of the rate of increase in money supply on capital accumulation and the growth rate of the economy; see Section 6.

[13]These budget constraints, plus the other equations of the model imply, by Walras Law, that the economy's resource constraint is satisfied; see Appendix A.

[14]In rewriting the young's first-period budget constraint, I have used

$$\begin{aligned}\frac{m_t}{p_t} &= \frac{1+i_{t+1}}{1+r_{t+1}}\frac{m_t}{p_{t+1}} = \frac{1+i_{t+1}}{1+r_{t+1}}\left(\alpha d_{t+1} - \frac{b_{t+1}}{p_{t+1}}\right) \\ &= \frac{1+i_{t+1}}{1+r_{t+1}}\left[\alpha\frac{s_t\left(1+r_{t+1}\right)}{1-\alpha} - \frac{b_{t+1}}{p_{t+1}}\right] = \frac{\alpha}{1-\alpha}\left(1+i_{t+1}\right)s_t - \frac{b_{t+1}}{p_t}.\end{aligned}$$

[15]As shown in the previous footnote,

$$\frac{m_t}{p_t} = \frac{\alpha}{1-\alpha}\left(1+i_{t+1}\right)s_t - \frac{b_{t+1}}{p_t}.$$

This determines the value of the real cash balances m_t/p_t as a function of r_{t+1}, i_{t+1} and b_{t+1}/p_t

as well. Observe also that, in equilibrium, $b_{t+1}/p_t = \theta m_t/p_t$ so that the equilibrium values of c_t, s_t, and m_t/p_t will all be determined as functions of r_{t+1} and i_{t+1}.

[16] The regularity conditions one imposes on preferences and the technology ensure only that positive values exist for the variables along a solution path. There is no guarantee, however, that there exists a steady-state solution to which the system is driven. See Appendix A for a discussion of local stability.

[17] This is obvious for all the stated relationships except for the steady-state version of (22) written as (27). The immediate steady-state version of (22) is

$$\widehat{c} + \left[1 + \frac{\alpha (i - \theta)}{1 + \theta}\right] \frac{\widehat{d}}{1 + r} = w - (1 + r)\frac{\widehat{e}}{\varphi(\widehat{e})}.$$

To arrive at (27), substitute for $i - \theta$ in the above equation from (30) according to

$$i - \theta = (1 + r)\frac{1 + \theta}{(1 + n)\varphi(\widehat{e})} - 1 - \theta = (1 + \theta)\left[\frac{1 + r}{(1 + n)\varphi(\widehat{e})} - 1\right].$$

[18] The value of b_{t+1}/p_t per h_t also remains constant over time. To see this, observe that with $b_{t+1} = \theta(M_t/N_t)$,

$$\frac{b_{t+1}}{b_t} = \frac{\theta M_t/N_t}{\theta M_{t-1}/N_{t-1}} = \frac{M_t}{M_{t-1}}\frac{N_{t-1}}{N_t} = \frac{1+\theta}{1+n}.$$

Consequently,

$$\frac{(b_{t+1}/p_t)/h_t}{(b_t/p_{t-1})/h_{t-1}} = \frac{b_{t+1}}{b_t}\frac{p_{t-1}}{p_t}\frac{h_{t-1}}{h_t} = \frac{1+\theta}{1+n}\frac{1}{1+\pi_{t+1}}\frac{1}{\varphi(\widehat{e}_t)} = \frac{\widehat{d}_{t+1}}{\widehat{d}_t},$$

where the last step follows from (20).

[19] As pointed out in footnote 9 above, with depreciation, one cannot rule out the *possibility* of a declining human capital over time. With $\varphi(0) < 1$, a positive level of $\widehat{e}$ and increasing $\varphi(\cdot)$ do not guarantee that $\varphi(\widehat{e}) > 1$. An steady-state equilibrium *might* then emerge wherein $\varphi(\widehat{e}) < 1$ so that $g < 0$. Under this circumstance, while $\widehat{c}, \widehat{d}, \widehat{e}, \widehat{k}, w, r, i$, and π remain constant, e_t, h_t, k_t, c_t, and d_t decline over time.

[20] Using (48), condition $[\varphi(\widehat{e})]^{\beta} < 1 + \rho$ also implies $\varphi(\widehat{e}) < (1+r)/(1+n)$ so that at the first best, $r > n$.

[21] This assumption makes the satisfaction of (50) easier; it is neither necessary nor sufficient for it.

[22] To satisfy its budget constraint, the government cannot set the values of both z^y and z^o freely.

[23] In the presence of the subsidy, the young choose $\widehat{e}_t$ at t to maximize $w_t\varphi(\widehat{e}_t) - (1 + r_t)(1 - \sigma)\widehat{e}_t$. This yields, in the steady state,

$$\varphi'(\widehat{e}) = \frac{(1 + r)(1 - \sigma)}{w}.$$

[24] In overlapping-generations model with cash-in-advance constraint, the Friedman rule is not unique. Allowing for differential commodity taxes across generations makes the optimal monetary rule indeterminate. One can use the commodity tax to balance any monetary growth rate

to ensure that (47) is satisfied in the face of (48). To rule out this indeterminacy, I do not allow for the possibility of differential commodity taxes. See Crettez *et al.* (2002) and Gahvari (2007).

[25] The intuition for (57) can best be gleaned by comparison with the corresponding familiar relationship for the standard overlapping-generations model. There, first-best capital accumulation is characterized by $1+r=(1+n)(1+\rho)$. Moreover, for $i=0$, money stock should grow at the rate of θ such that $1+\theta=(1+n)/(1+r)$. These equations imply

$$1+\theta=\frac{1}{1+\rho},$$

so that money should decline by the intertemporal discount rate.

With growth, the relationship that characterizes capital accumulation changes to $(1+g)^{\beta-1}(1+r)=(1+n)(1+\rho)$. The adjustment is due to the fact that marginal utility of future consumption increases per period by a rate equal to $(1+g)^{\beta-1}$ (utility increases by $(1+g)^{\beta}$). Similarly, the relationship that ensures $i=0$ changes to $1+\theta=(1+n)(1+g)/(1+r)$. These two latter relationships imply

$$1+\theta=\frac{(1+g)^{\beta}}{1+\rho}.$$

[26] The same result holds in Weiss (1980) and Gahvari (1988) who use a Diamond-type overlapping-generations model with money in the utility function (that does not allow for endogenous growth). An opposite result holds in Gahvari (2007) who also uses a Diamond-type overlapping-generations model but rationalizes money through a cash-in-advance constraint.

[27] The stark example is provided by the case when there are no fiscal instruments.

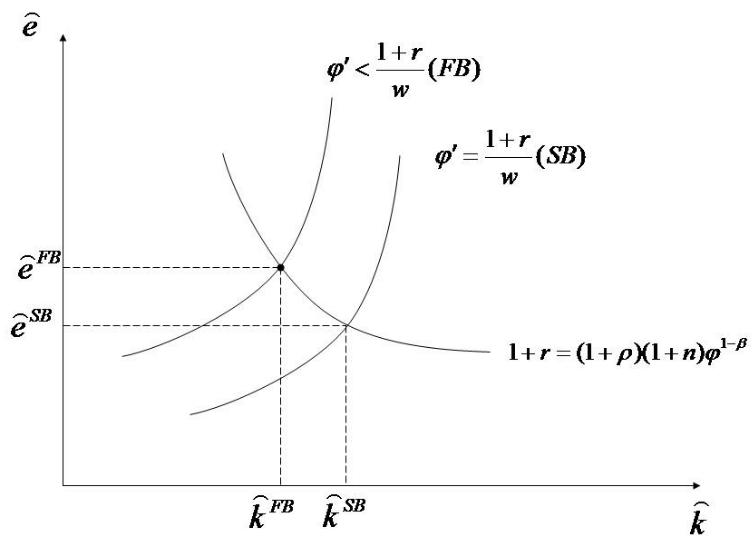
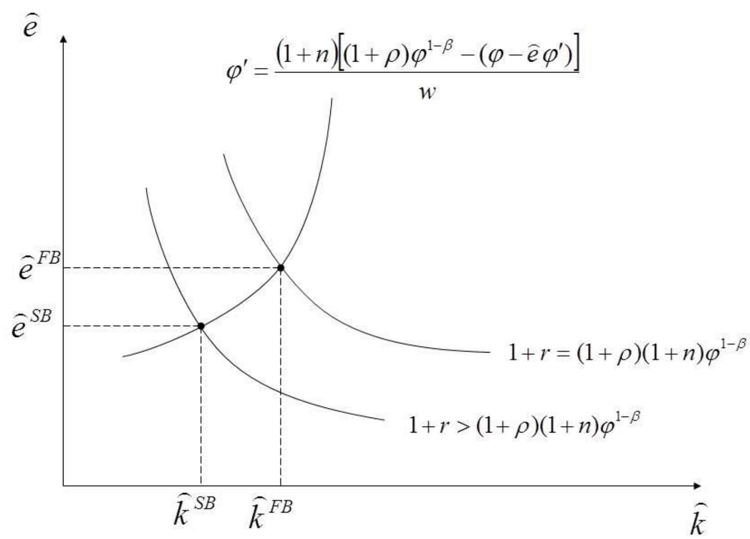

$\widehat{e}$
$\varphi' < \frac{1+r}{w}(FB)$
$\varphi' = \frac{1+r}{w}(SB)$
$\widehat{e}^{FB}$
$\widehat{e}^{SB}$
$1+r=(1+\rho)(1+n)\varphi^{1-\beta}$
$\widehat{k}^{FB}$
$\widehat{k}^{SB}$
$\widehat{k}$

Figure 1:



$\widehat{e}$
$\varphi' = \frac{(1+n)\left[(1+\rho)\varphi^{1-\beta} - (\varphi - \widehat{e}\,\varphi')\right]}{w}$
$\widehat{e}^{FB}$
$\widehat{e}^{SB}$
$1 + r = (1+\rho)(1+n)\varphi^{1-\beta}$
$1 + r > (1+\rho)(1+n)\varphi^{1-\beta}$
$\widehat{k}^{SB}$
$\widehat{k}^{FB}$
$\widehat{k}$

Figure 2:

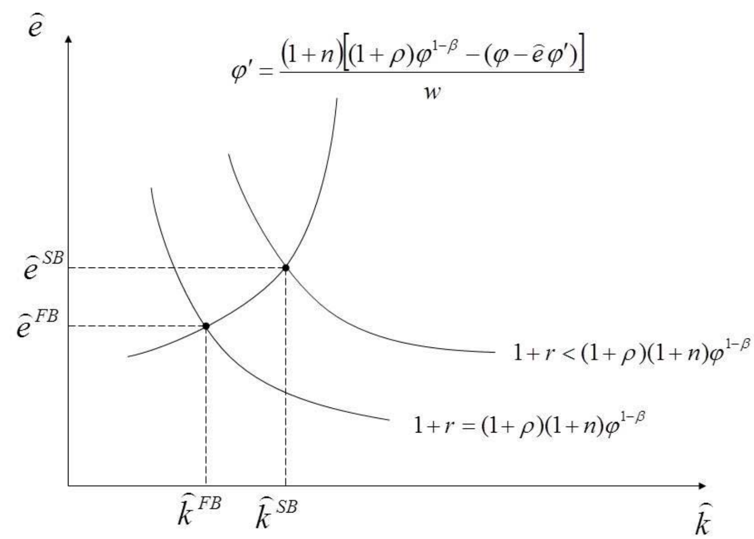

$\widehat{e}$
$\varphi' = \frac{(1+n)\left[(1+\rho)\varphi^{1-\beta} - (\varphi - \widehat{e}\,\varphi')\right]}{w}$
$\widehat{e}^{SB}$
$\widehat{e}^{FB}$
$1+r < (1+\rho)(1+n)\varphi^{1-\beta}$
$1+r = (1+\rho)(1+n)\varphi^{1-\beta}$
$\widehat{k}^{FB}$
$\widehat{k}^{SB}$
$\widehat{k}$

Figure 3:

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