The Friedman rule: old and new

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Abstract

In overlapping generations models, money growth creates intergenerational wealth effects and leads to the breakdown of the Friedman rule; the rule can be restored via lump-sum tax and transfers that neutralize these wealth transfers. Additionally, and in contrast to money-in-the-utility-function models, the Friedman rule is not unique in cash-in-advance-constraint models of money: A continuum of combinations of money growth rates and consumption taxes implement the first-best allocation. This paper traces through the intellectual origins of the first (old) result, which was recently restated in Bhattachrya, Haslag and Russell (2005), and formally demonstrates the second (new) result.

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1 Introduction

In the November 2005 issue of this *Journal*, Bhattachrya, Haslag and Russell (henceforth BHR) set out to explain the fundamental difference between infinitely-lived representative agent and overlapping generations models that lead to their (presumed) different prescriptions for optimal money supply. They summarize their finding as such: "the key difference between ILRA [infinitely-lived representative agent] and OG [overlapping generations] models is that in the latter, the standard method for constructing a monetary regime causes transactions involving money to become intergenerational transfers. Under alternative government fiscal/monetary regimes that offset these intergenerational transfers, the Friedman rule is always optimal" (p. 1401).

The current paper has two purposes. First, I point out that the BHR result is an old one, variants of which were demonstrated by Abel (1987) and Gahvari (1988). The basic difference between BHR and these earlier studies is in the way they rationalize money holdings. Whereas Abel and Gahvari utilized money-in-the-utility-function constructs, BHR employ two different models to justify the money demand: one on the basis of "reserve requirements" and the other "random relocation."[1]

Secondly, I will identify an important difference between cash-in-advance-constraint models of money, and the earlier money-in-the utility-function models, which result in different characterizations for the optimal money supply. I will show that whereas models with money-in-the-utility-function have a first-best that requires the satisfaction of Friedman rule, this is not necessarily the case for cash-in-advance-constraint models of money. Specifically, I will argue that while in these latter models Friedman rule is optimal, it is not unique. There exists a wide range of combinations of money growth

[1]I do not contend that BHR did nothing but duplicate this result. In a private correspondence, Steve Russell writes "The paper has eight propositions, two of which (propositions 3 and 7) are versions of your result in somewhat different environments. Two others (4 and 8) extend your result by showing that there is an equivalence between monetary regimes with the type of taxes and transfers you describe and monetary regimes with intermediated money. The other four propositions are less closely related to the contents of your paper."

rates and consumption taxes that achieve the same level of welfare.

The intuition behind this result must be sought in the observation that the opportunity cost of holding real balances manifests itself differently in monetary models with money-in-the-utility-function versus those with cash-in-advance constraints. In money-in-the-utility-function models, money growth affects the "price" (opportunity cost) of real balances that enter the utility function explicitly as an argument (and are treated like any other good). Changing this price does not change the relative prices of intertemporal consumption goods. This is not the case with cash-in-advance-constraint models of money. Here, the rate of money growth (positive or negative) directly affects the relative prices of intertemporal consumption goods—much in the same way as commodity taxes do. Consequently, in the former models, attaining first best requires *two* undistorted prices. This in turn requires that there will be no distortionary commodity taxes and that the nominal interest rate is pushed down to zero (so as to equate its opportunity cost to its marginal cost of production). On the other hand, attaining first-best in cash-in-advance-constraint models requires only *one* undistorted price; namely, the relative intertemporal price. With two available instruments, commodity tax and the rate of money growth, there will be a wide range of combinations of the two instruments that can achieve this.

2 Friedman rule and the "old" literature

The question of why infinitely-lived individual and overlapping generations models (with no operative bequest motive à la Barro (1974)) lead to different prescriptions for money growth (or, for that matter, for other policy instruments) may be answered in two interrelated ways. Transferring resources (with the same present value) across time are of no consequence in infinitely-lived individual models; they have no impact on the representative agent's budget constraint. This is not the case in overlapping generations models; such a transfer entails intergenerational wealth transfers thus affecting the economy's overall savings and capital intensity. As a consequence, any policy change that induces intertemporal transfer of resources will have different effects in the two settings unless the resulting intertemporal transfers in overlapping generations models are somehow neutralized. This way of looking at the problem, when applied to money growth rate, underlies BHR (2005) analysis. Gahvari's (1988) message was precisely the same. In explaining why overlapping generations models seem to reject the Friedman rule, he writes:

> The intuition behind this result is quite simple. An *unexpected* cut in the rate of increase in money supply will create an unexpected capital gain in the money holdings of those who hold money in their portfolios. In an infinitely-lived individual framework, the question of who receives the [any] newly created money does not arise. . . . In a life-cycle model, on the other hand, the question of who receives the newly created money assumes a central role. Indeed, this is the source of the sharply different results derived for life cycle models by Weiss and others as compared to the earlier result of Sidrauski [1967] and Friedman [1969] for infinitely-lived individual settings (p. 340).

Gahvari also demonstrated that introducing generation-specific lump-sum taxes in overlapping generations models can neutralize the adverse intergenerational wealth effects and restore the optimality of Friedman rule.[2] The abstract to his paper reads[3]

> It is demonstrated that the existing result in the literature regarding a positive relationship between money creation and the steady-state capital intensity and welfare [in overlapping generations models] is not due to monetary policy per se. On the contrary, this is shown to arise because of the intergenerational

[2]This point is a general one. Similar issues arise in contexts other than the optimality of Friedman rule. Taxation of pure rent is another example. Feldstein (1977) demonstrated that land rent taxes increase the economy's accumulation of real capital. His result was subsequently challenged by Calvo *et al.* (1979) on the basis of a Barro-type utility function. Later, Gahvari (1982) and Fane (1984) pointed out that the driving force behind Feldstein's result was not so much due to his stipulation of "selfish" preferences. Rather, the result was due to the intergenerational wealth transfers caused by taxation of rents. These taxes would be capitalized in the price of land (which is held by the old), leading to redistribution from current old to all future generations. They further demonstrated how levying appropriate generation-specific lump-sum taxes would make land rent taxes neutral in Feldstein's model.

[3]There is one additional feature of Gahvari's (1988) result which is worth emphasizing here. He shows that the existence of the lump-sum tax and transfer policies allows the economy to move *instantaneously* from one steady state to another as a result of a change in money creation.

> wealth effects embedded in the manner in which money is created. ... *any* other money supply rule is *Pareto inferior* to Chicago Rule [Friedman rule] provided that the government also embarks on a policy of wealth redistribution across generations through the imposition of appropriate lump-sum taxes and transfers" (p. 339).

Abel (1987) did not mention intergenerational wealth effects explicitly, but his setup allows for it to be neutralized. To understand his approach, observe that the competitive equilibrium is, given the customary assumptions, Pareto efficient in infinitely-lived individual models but not necessarily so in the overlapping generations models. In the same vein, welfare maximization requires no government intervention in the economy in infinitely-lived individual models, but it does do—in terms of generation-specific lump-sum taxes, or debt policy—in overlapping generations models. Thus the question of the optimal money growth in infinitely-lived individual constructs (when one rules out all taxes and transfers) is a study in the first-best. In an overlapping generations framework, on the other hand, ruling all tax and transfer instruments out implies a second-best environment. To have a first-best setting, one must allow for *lump-sum* taxes and transfers.

Allowing for generation specific lump-sum tax instruments, in addition to money creation, Abel (1987) proves that first-best optimality in overlapping generations models requires the satisfaction of Friedman rule. The welfare criterion he uses is the weighted average of utility of all future generations.[4] He traces the difference between his result and Weiss's (1980)—who had used an identical overlapping generations model to show that steady-state utility maximization requires a *positive* rate of money creation as opposed to the Friedman rule—to the inclusion of lump-sum tax instruments in his setup. He writes "In general, two independent policy instruments are required to allow the competitive economy to reach the first-best optimum" and that "I determine first-

[4] This is more general than steady-state utility maximization which corresponds to the maximization of an unweighted average of all utilities.

best rather than second-best policy" (p. 438).[5]

3 The OG Model with cash-in-advance constraint

Consider a standard two-period overlapping generations model. Individuals work in the first period supplying one unit of labor, and derive utility from consuming a composite consumption good, c, in both periods. There is no bequest motive, and population grows at a constant rate, n. Preferences are represented by

$$u_t^y = u(c_t^y, c_{t+1}^o), \tag{1}$$

where superscripts y and o refer to young and old, and subscript t denotes time. The utility function $u(\cdot)$ is strictly quasi-concave and twice differentiable.

The production technology exhibits constant returns to scale. Let Y_t, K_t, L_t denote aggregate output, aggregate capital, and aggregate labor (equivalent to the number of young persons) at time t. Define per capita output and capital according to $y_t = Y_t/L_t, k_t = K_t/L_t$, and represent the production function by $y_t = f(k_t)$, where $f(\cdot)$ is increasing and strictly concave. Assuming a competitive setting, the real wage w_t (measured in units of composite consumption good), and the real interest rate, r_t, are determined according to

$$w_t = y_t - k_t f'(k_t), \tag{2}$$

$$r_t = f'(k_t). \tag{3}$$

Output of each period can be used for consumption in the same period or retained, with no depreciation, to be used as an input (capital), K_{t+1}, in the next period production process. In addition to real savings, and in order to finance their consumption

[5] Levhari and Patinkin (1968) had earlier discussed the difference between first- and second-best environments in attaining the Friedman rule. They demonstrated, in the context of a Solow-Swan growth model, that Friedman rule will be satisfied if the government has, in addition to its inflationary policy, a "fiscal policy" tool to ensure that the economy's steady-state capital is at the Golden Rule, and not otherwise; see their discussion on pp 735–736.

in retirement, the young also purchase all the stock of money, M_t, from the old (before consumption takes place). The government policy instruments consist of generation-specific real lump-sum taxes, T_t^y, T_t^o, a tax on the young's consumption at the rate τ, and lump-sum money transfers to the old equal to s_t^o per person (before they sell their holdings of money to the young). Policy instruments can take positive or negative values.

The current young and old individuals face the budget constraints,

$$\begin{aligned} p_t\left[w_t - (1+\tau)c_t^y - T_t^y\right] &= \left(p_t K_{t+1} + M_t\right)/L_t, & (4)\\ p_t c_t^o &= \left[p_t K_t(1+r_t) + M_{t-1}\right]/L_{t-1} - p_t T_t^o + s_t^o, & (5)\end{aligned}$$

where w_t, r_t, and p_t are the real wage, the real interest rate (rate of return to capital), and the price level at time t. Denote the inflation rate with π_t, the monetary rate of interest with i_t, and the real cash balances held by an old person with x_t. Thus, define

$$\begin{aligned} \frac{p_{t+1}}{p_t} &\equiv 1 + \pi_{t+1}, & (6)\\ 1 + i_{t+1} &\equiv (1+r_{t+1})(1+\pi_{t+1}), & (7)\\ x_t &\equiv \frac{M_t/p_t}{L_t}. & (8)\end{aligned}$$

Using these variables, and combining equations (4) and (5), leads to the young's intertemporal budget constraint:

$$(1+\tau)c_t^y + \frac{c_{t+1}^o}{1+r_{t+1}} + \frac{i_{t+1}x_t}{1+i_{t+1}} = w_t - T_t^y - \frac{T_{t+1}^o}{1+r_{t+1}} + \frac{s_{t+1}^o/p_{t+1}}{1+r_{t+1}}, \tag{9}$$

where $i_{t+1}x_t/(1+i_{t+1})$ constitutes the opportunity cost of holding cash.

The government injects money into (or retires money from) the economy via lump-sum monetary transfers to (or monetary taxes on) the old (who hold all the stock of money in the economy). These transfers (or taxes) are set in proportion to the old's money holdings,

$$s_t^o = \theta(M_{t-1}/L_{t-1}), \tag{10}$$

such that

$$M_t = M_{t-1} + L_{t-1}s_t^o = (1+\theta)M_{t-1}, \tag{11}$$

where θ may be positive, zero, or negative. To rationalize the holding of money, I consider a cash-in-advance constraint as formulated by Hahn and Solow (1995) with

$$M_{t+1} \geqq \alpha L_t p_{t+1} c_{t+1}^o, \tag{12}$$

where $\alpha < 1$ denotes the proportion of future expenditures that must be financed from cash holdings. Assuming that the cash-in-advance constraint (12) is binding, the opportunity cost of holding cash can be rewritten, from (8), (6), (11), and (12), as

$$\frac{i_{t+1}}{1+i_{t+1}}x_t = \frac{i_{t+1}}{1+i_{t+1}}\frac{M_t/p_t}{L_t} = \frac{i_{t+1}}{1+i_{t+1}}\frac{1+\pi_{t+1}}{1+\theta}\frac{M_{t+1}}{L_t p_{t+1}} = \frac{i_{t+1}}{1+r_{t+1}}\frac{\alpha}{1+\theta}c_{t+1}^o. \tag{13}$$

Substitute for $i_{t+1}x_t/(1+i_{t+1})$ from (13) into (9) to rewrite the young's intertemporal budget constraint as,

$$(1+\tau)c_t^y + \left(1 + \frac{\alpha i_{t+1}}{1+\theta}\right)\frac{c_{t+1}^o}{1+r_{t+1}} = w_t - T_t^y - \frac{T_{t+1}^o}{1+r_{t+1}} + \frac{s_{t+1}^o/p_{t+1}}{1+r_{t+1}}. \tag{14}$$

Each young person maximizes the utility function (1) subject to the budget constraint (14), treating $T_t^y, T_{t+1}^o, s_{t+1}^o$ as lump sum. This results in the first-order condition,

$$\frac{\partial u/\partial c_{t+1}^o}{\partial u/\partial c_t^y} = \frac{1+\alpha i_{t+1}/\left(1+\theta\right)}{(1+\tau)(1+r_{t+1})}. \tag{15}$$

Equations (14)–(15) determine the young's demand functions for c_t^y and c_{t+1}^o as functions of $w_t, i_{t+1}, r_{t+1}, p_{t+1}$, and the policy variables $T_t^y, T_{t+1}^o, s_{t+1}^o, \tau, \theta$. The value of w_t is determined through the marginal productivity condition (2). Assuming perfect foresight, the values of r_{t+1}, π_{t+1}, and thus from (7) i_{t+1}, will be determined. Equations (6), (8), and (13) will then determine p_t and p_{t+1} (along with x_t).

Finally, observe that the tax instruments τ, T_t^y and T_t^o cannot all be set independently of one another. They are related through the government's budget constraint

$$(1+n)\left(\tau c_t^y + T_t^y\right) + T_t^o = 0, \tag{16}$$

so that the government has only two independent tax instruments. One can then use equation (16) to simplify the young's intertemporal budget constraint (14). The resulting equation can further be simplified by substitution from

$$\frac{s^o_{t+1}}{p_{t+1}} = \frac{\theta M_t/L_t}{p_{t+1}} = \frac{\theta M_{t+1}}{(1+\theta)L_t p_{t+1}} = \frac{\theta\alpha}{1+\theta} c^o_{t+1},$$

which is derived from equations (10)–(12). One arrives at

$$c^y_t + \left[1 + \frac{\alpha(i_{t+1}-\theta)}{1+\theta}\right] \frac{c^o_{t+1}}{1+r_{t+1}} = w_t + \frac{T^o_t}{1+n} - \frac{T^o_{t+1}}{1+r_{t+1}}, \tag{17}$$

in which neither T^y_t nor s^o_{t+1} appear. Similarly, and for future reference, use equations (8), (13), and the government's budget constraint (16), to rewrite equation (4) as

$$(1+n)k_{t+1} = w_t - c^y_t - \frac{\alpha}{1+n} c^o_t + \frac{T^o_t}{1+n}. \tag{18}$$

4 The Steady state

In the steady state, per-capita values are time invariant and equations (12)–(13), (15), (17)–(18) yield

$$\begin{aligned}
1+i &= \frac{(1+r)(1+\theta)}{1+n}, && (19)\\
\frac{\partial u/\partial c^o}{\partial u/\partial c^y} &= \frac{1+\alpha\, i/(1+\theta)}{(1+\tau)(1+r)}, && (20)\\
c^y + \left[1 + \frac{\alpha(r-n)}{1+n}\right]\frac{c^o}{1+r} &= w + \frac{(r-n)T^o}{(1+n)(1+r)}, && (21)\\
(1+n)k &= \frac{(1-\alpha)c^o + T^o}{1+r}. && (22)
\end{aligned}$$

With $r = f'(k)$ and $w = f(k) - kf'(k)$, equations (19)–(22) determine c^y, c^o, k and i as functions of T^o, θ, and τ. More specifically, substitute for i from (19) into (20), and simplify the resulting equation to arrive at

$$\frac{\partial u/\partial c^o}{\partial u/\partial c^y} = \frac{1}{(1+\tau)(1+r)}\left[1 + \alpha\,(\frac{1+r}{1+n} - \frac{1}{1+\theta})\right]. \tag{23}$$

Equations (21)–(23) determine the steady-state values of c^y, c^o, k, as well as r, w—the *real* variables of the economy.

The interesting point to note about equations (21)–(23) is that the parameters τ and θ appear only in (23). Consequently, for all combinations of τ and θ that leave the values of the other variables in (23) unchanged, the system of equations (21)–(23) yield the same solution for c^y, c^o, k. Put differently, there exists one extra degree of freedom in setting the policy instruments that determine the equilibrium values of the *real* sector of the economy.

Thus set $\tau = 0$ and $\theta = \theta_0$. These, in conjunction with the given values for n, α, and T^o, determine the equilibrium values of k, c^y, c^o through equations (21)–(23). Denote the resulting equilibrium values of the real sector variables with the subscript 0. Then, consider a different pair of policy parameters: $(\widehat{\tau}, \widehat{\theta})$, with the property that $\widehat{\tau} > -1$ is an arbitrary tax rate and $\widehat{\theta}$ is related to $\widehat{\tau}$ according to

$$\frac{1}{1+\widehat{\theta}} = \frac{1}{\alpha} + \frac{1+r_0}{1+n} + (1+\widehat{\tau})\left[\frac{1}{\alpha} + (\frac{1+r_0}{1+n} - \frac{1}{1+\theta_0})\right]. \qquad (24)$$

One can easily establish, by substitution from (24) into (21)–(23), that $(\widehat{\tau}, \widehat{\theta})$ supports the initial equilibrium $(k_0, w_0, r_0, c_0^y, c_0^o)$ under $(\tau = 0,\ \theta = \theta_0)$.

It is instructive to ponder why there is an extra degree of freedom in setting τ and θ in cash-in-advance-constraint models of money, but not in models with money-in-the-utility-function. Consider again equation (23) where τ and θ make their only appearance. This equation determines the relative price of intertemporal consumption goods. There is thus *two* policy instruments to determine *one* variable. The money-in-the-utility-function models, on the other hand, require the determination of *two* relative prices (as opposed to one here). There is the relative intertemporal price, *and* the (relative) price of real balances. To control them, one needs two instruments: τ and θ. Note also that, in both types of models, the third instrument T^o "controls" capital accumulation.[6]

[6] Crettez *et al.* (2002) and Bhattacharya *et al.* (2003) also discuss instrument indeterminacy. Crettez

4.1 Optimal steady state and the Friedman rule

In overlapping-generations models of the sort I have used here, maximization of steady-state utility requires T^o to be set in such a way as to ensure the economy is on its golden rule path satisfying the $r = n$ condition. It also requires that the Friedman rule ($i = 0$) is satisfied when $\tau = 0$. Setting $r = n$ and $i = 0$ in equation (19) implies $\theta = 0$. In turn, setting $r = n$ and $\theta = 0$ in (24) yields $\widehat{\theta} = \widehat{\tau}/(\alpha - \widehat{\tau})$. The upshot is that the steady-state utility attains its maximal value at $r = n$ and the continuum of combinations of τ and θ that satisfy,[7]

$$\theta = \frac{\tau}{\alpha - \tau}. \tag{25}$$

As long as there are no commodity taxes ($\tau = 0$), the Friedman rule satisfies (25). However, if $\tau \neq 0$, the optimal monetary policy requires a rule different from Friedman's. It requires, specifically, that $i = \theta = \tau/(\alpha - \tau)$. Of course, this does not mean that one can do better than the Friedman rule; only that the optimal monetary rule is indeterminate. There are infinitely many values of τ and θ (or τ and i) that satisfy (25). The Friedman rule is only one special case. Observe also that (25) is *not* a second-best rule; it does not characterize the optimal monetary growth rate in the presence of distortionary taxes. The rule is first-best.

To gain more intuition for this result, observe that in cash-in-advance-constraint models the opportunity cost of holding real balances—as indicated by equations (15) or (20)—works like a tax on consumption when old, much in the same way as τ is a tax on consumption when young. In the first-best, one wants to set these tax rates equally so that the relative prices are undistorted. This is precisely what setting $i = \theta = \tau/(\alpha - \tau)$ achieves, as it then results in

$$\frac{\partial u/\partial c^o}{\partial u/\partial c^y} = \frac{1}{1+r}. \tag{26}$$

et al.'s discussion centers around the choice of labor and capital income taxes versus debt and monetary policy, with its implication for Friedman rule; Bhattacharya *et al.*'s discussion is in terms of reserve requirements versus proportional taxes on savings.

[7] Gahvari (2006) contains a direct and formal demonstration of this result.

This is the relative intertemporal price in the absence of the commodity tax, $\tau = 0$, and when there is no cash-in-advance constraint, $\alpha = 0$.

Proposition 1 summarizes the main results of this section.

Proposition 1 *Introduce money in a standard two-period overlapping generations model and rationalize its demand on the basis of a cash-in-advance constraint of the form (12). Then:*

(i) A continuum of combinations of intertemporal commodity taxes, τ, and money growth rates, θ, in conjunction with generation-specific lump-sum taxes, T^o, support the same steady-state equilibrium (for the real variables).

(ii) If lump-sum taxes are set such that the economy is on its Golden Rule path, the steady-state welfare is maximal for a continuum values of τ and θ such that $\theta = \tau/(\alpha - \tau)$. The Friedman rule is only one special case.

5 Concluding remarks

The questions of the optimal money supply in general, and the validity of Friedman rule in particular, have been studied in monetary economics literature for more than four decades now, albeit in different guises. A particular question from the 1980s centered around the reasons for the presumed failure of Friedman's prescription in overlapping generations model as had been shown by Weiss (1980). Subsequently, Abel (1987) implied and Gahvari (1988) pointed out that this was due to the intergenerational wealth effects embedded in money creation, and the second-best nature of equilibrium in these models, in the absence of generation-specific lump-sum taxes. These authors also demonstrated that neutralizing the wealth effects through lump-sum taxation restores the optimality of Friedman rule. One aspect of this literature was its reliance on money-in-the-utility-function constructs to rationalize money holdings. The recent paper of Bhattachrya (2005) *et al.* re-affirms this result for monetary models with different justifications for money demand.

Leaving the intergenerational wealth effects aside, this paper has pointed out, using a cash-in-advance-constraint model, that a continuum of combinations of intertemporal commodity taxes and money growth rates support the same steady-state equilibrium (for the real variables). This pinpoints a crucial difference between (overlapping generations) models with money-in-the-utility-function and those with cash-in-advance constraint. Whereas the Friedman rule is the only first-best policy in the former, this is not the case in the latter. There exists, in cash-in-advance-constraint models, a continuum of combinations of money growth rates and consumption taxes that yield the same level of welfare as Friedman rule does. The reason is that, in these models, the rate of money growth (positive or negative) affects the relative prices of intertemporal consumption goods much in the same way as commodity taxes do. Consequently, any combination of the two policy instruments that result in an undistorted intertemporal price will be as good as any other.

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Appendix

Solve equations (21) and (23) for c^y and c^o, as functions of w, r, T^o, τ, θ. With $r = f'(k)$ and $w = f(k) - kf'(k)$, one may then write c^y and c^o simply as functions of k, T^o, τ, and θ. Of course, k itself is endogenously determined through equation (22). This allows us to write the steady state utility as

$$u = u\left(c^y\big(k(T^o,\tau,\theta),T^o,\tau,\theta\big), c^o\big(k(T^o,\tau,\theta),T^o,\tau,\theta\big)\right). \tag{A1}$$

The optimal values of the policy instruments T^o, τ, and θ are then determined through the maximization of (A1) with respect to these variables. The first-order conditions for this problem are,

$$\begin{aligned}
\frac{\partial u}{\partial T^o} &= \frac{\partial u}{\partial c^y}\left[\left(\frac{\partial c^y}{\partial k} + \frac{\partial u/\partial c^o}{\partial u/\partial c^y}\frac{\partial c^o}{\partial k}\right)\frac{dk}{dT^o} + \left(\frac{\partial c^y}{\partial T^o} + \frac{\partial u/\partial c^o}{\partial u/\partial c^y}\frac{\partial c^o}{\partial T^o}\right)\right] = 0, &\text{(A2)}\\
\frac{\partial u}{\partial \tau} &= \frac{\partial u}{\partial c^y}\left[\left(\frac{\partial c^y}{\partial k} + \frac{\partial u/\partial c^o}{\partial u/\partial c^y}\frac{\partial c^o}{\partial k}\right)\frac{dk}{d\tau} + \left(\frac{\partial c^y}{\partial \tau} + \frac{\partial u/\partial c^o}{\partial u/\partial c^y}\frac{\partial c^o}{\partial \tau}\right)\right] = 0, &\text{(A3)}\\
\frac{\partial u}{\partial \theta} &= \frac{\partial u}{\partial c^y}\left[\left(\frac{\partial c^y}{\partial k} + \frac{\partial u/\partial c^o}{\partial u/\partial c^y}\frac{\partial c^o}{\partial k}\right)\frac{dk}{d\theta} + \left(\frac{\partial c^y}{\partial \theta} + \frac{\partial u/\partial c^o}{\partial u/\partial c^y}\frac{\partial c^o}{\partial \theta}\right)\right] = 0. &\text{(A4)}
\end{aligned}$$

To evaluate (A2)–(A4), I next derive the expressions for $\partial c^y/\partial k, \partial c^o/\partial k, \partial c^y/\partial T^o, \partial c^o/\partial T^o, \partial c^y/\partial \tau, \partial c^o/\partial \tau, \partial c^y/\partial \theta, \partial c^o/\partial \theta$, and $dk/dT^o, dk/d\tau, dk/d\theta$.

Let, for ease in notation, subscripts 1 and 2 denote the partial derivatives of $u(c^y, c^o)$ with respect to c^y and c^o. It is a simple algebraic exercise to show, through partial differentiation of (20)–(21) with respect to k, T^o, τ, and θ, while making use of equation (19) and noting $dr/dk = f''(k), dw/dk = -kf''(k)$, that

$$\begin{aligned}
\boldsymbol{A}\begin{pmatrix} \frac{\partial c^y}{\partial k} \\ \frac{\partial c^o}{\partial k}\end{pmatrix} &= \frac{-f''(k)}{1+r}\begin{pmatrix} (r-n)k \\ \frac{(1+\theta-\alpha)u_1}{(1+\tau)(1+\theta)(1+r)}\end{pmatrix}, &\text{(A5)}\\
\boldsymbol{A}\begin{pmatrix} \frac{\partial c^y}{\partial T^o} \\ \frac{\partial c^o}{\partial T^o}\end{pmatrix} &= \begin{pmatrix} \frac{r-n}{(1+n)(1+r)} \\ 0\end{pmatrix}, &\text{(A6)}
\end{aligned}$$

$$\boldsymbol{A}\begin{pmatrix} \frac{\partial c^y}{\partial \tau} \\ \frac{\partial c^o}{\partial \tau} \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{-u_2}{(1+\tau)} \end{pmatrix}, \tag{A7}$$

$$\boldsymbol{A}\begin{pmatrix} \frac{\partial c^y}{\partial \theta} \\ \frac{\partial c^o}{\partial \theta} \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{\alpha u_1}{(1+\tau)(1+\theta)^2(1+r)} \end{pmatrix}, \tag{A8}$$

where,

$$\boldsymbol{A} = \begin{pmatrix} 1 & \left(1+\frac{\alpha(r-n)}{1+n}\right)\frac{1}{1+r} \\ u_{21} - \frac{1+\theta+\alpha i}{(1+\tau)(1+\theta)(1+r)}u_{11} & u_{22} - \frac{1+\theta+\alpha i}{(1+\tau)(1+\theta)(1+r)}u_{12} \end{pmatrix}. \tag{A9}$$

Then, premultiply (A5)–(A7) by $\boldsymbol{A^{-1}}$ to get

$$\begin{pmatrix} \frac{\partial c^y}{\partial k} \\ \frac{\partial c^o}{\partial k} \end{pmatrix} = \frac{-f''(k)}{1+r}\boldsymbol{A^{-1}}\begin{pmatrix} (r-n)k \\ \frac{(1+\theta-\alpha)u_1}{(1+\tau)(1+\theta)(1+r)} \end{pmatrix}, \tag{A10}$$

$$\begin{pmatrix} \frac{\partial c^y}{\partial T^o} \\ \frac{\partial c^o}{\partial T^o} \end{pmatrix} = \boldsymbol{A^{-1}}\begin{pmatrix} \frac{r-n}{(1+n)(1+r)} \\ 0 \end{pmatrix}, \tag{A11}$$

$$\begin{pmatrix} \frac{\partial c^y}{\partial \tau} \\ \frac{\partial c^o}{\partial \tau} \end{pmatrix} = \boldsymbol{A^{-1}}\begin{pmatrix} 0 \\ \frac{-u_2}{(1+\tau)} \end{pmatrix}, \tag{A12}$$

$$\begin{pmatrix} \frac{\partial c^y}{\partial \theta} \\ \frac{\partial c^o}{\partial \theta} \end{pmatrix} = \boldsymbol{A^{-1}}\begin{pmatrix} 0 \\ \frac{\alpha u_1}{(1+\tau)(1+\theta)^2(1+r)} \end{pmatrix}, \tag{A13}$$

where

$$\boldsymbol{A^{-1}} = \frac{1}{\Delta}\begin{pmatrix} \frac{1+\theta+\alpha i}{(1+\tau)(1+\theta)(1+r)}u_{12} - u_{22} & \left(1+\frac{\alpha(r-n)}{1+n}\right)\frac{1}{1+r} \\ u_{21} - \frac{1+\theta+\alpha i}{(1+\tau)(1+\theta)(1+r)}u_{11} & -1 \end{pmatrix}, \tag{A14}$$

and

$$\begin{aligned} \Delta \equiv & \left[1+\frac{\alpha(r-n)}{1+n}+\frac{1+\theta+\alpha i}{(1+\tau)(1+\theta)}\right]\frac{u_{12}}{1+r} \\ - & \left(1+\frac{\alpha(r-n)}{1+n}\right)\frac{1+\theta+\alpha i}{(1+\tau)(1+\theta)(1+r)}\frac{u_{11}}{1+r} - u_{22}. \end{aligned} \tag{A15}$$

Next, differentiate equation (22) partially with respect to T^o, τ, θ, while allowing for the changes in c^o and r with respect to k. This yields, upon simplification,

$$\frac{dk}{dT^o} = \frac{1+(1-\alpha)\,(\partial c^o/\partial T^o)}{(1+n)\,[1+r+kf''(k)]-(1-\alpha)\,(\partial c^o/\partial k)}, \tag{A16}$$

$$\frac{dk}{d\tau} = \frac{(1-\alpha)\,(\partial c^o/\partial \tau)}{(1+n)\,[1+r+kf''(k)]-(1-\alpha)\,(\partial c^o/\partial k)}, \tag{A17}$$

$$\frac{dk}{d\theta} = \frac{(1-\alpha)\,(\partial c^o/\partial \theta)}{(1+n)\,[1+r+kf''(k)]-(1-\alpha)\,(\partial c^o/\partial k)}. \tag{A18}$$

Finally, substitute from (A10)–(A18) into the first-order conditions (A2)–(A4) and simplify to get

$$\frac{\partial u}{\partial T^o} = (r-n)\,G + \left[\frac{\tau}{1+\tau} - \frac{\alpha i}{(1+\tau)\,(1+\theta)}\right] H = 0, \tag{A19}$$

$$\frac{\partial u}{\partial \tau} = (r-n)\,E + \left[\frac{\tau}{1+\tau} - \frac{\alpha i}{(1+\tau)\,(1+\theta)}\right] F = 0, \tag{A20}$$

$$\frac{\partial u}{\partial \theta} = (r-n)\,C + \left[\frac{\tau}{1+\tau} - \frac{\alpha i}{(1+\tau)\,(1+\theta)}\right] D = 0, \tag{A21}$$

where the expressions for G, H, E, F, C, D are,

$$\begin{aligned} G &\equiv \Gamma\frac{dk}{dT^o} + \frac{u_1}{\Delta\,(1+r)\,(1+n)}\left[-\left(\frac{1+\theta+\alpha i}{(1+\tau)(1+\theta)(1+r)}\right)^2 u_{11}\right. \\ &- \left. u_{22} + \frac{2\,(1+\theta+\alpha i)\,u_{12}}{(1+\tau)(1+\theta)(1+r)}\right], \end{aligned} \tag{A22}$$

$$H \equiv \frac{-(1+\theta-\alpha)u_1^2 f''(k)}{(1+\tau)(1+\theta)(1+r)^3\Delta}\frac{dk}{dT^o}, \tag{A23}$$

$$E \equiv \Gamma\frac{dk}{d\tau} - \frac{\alpha\,(1+\theta+\alpha i)\,u_1^2}{\Delta(1+\tau)^2(1+\theta)\,(1+r)^2\,(1+n)}, \tag{A24}$$

$$F \equiv \frac{(1+\theta-\alpha)u_1^2}{\Delta(1+\tau)(1+\theta)\,(1+r)^2}\left[\frac{-1}{1+\tau} - \frac{f''(k)}{1+r}\frac{dk}{d\tau}\right], \tag{A25}$$

$$C \equiv \Gamma\frac{dk}{d\theta} + \frac{\alpha^2 u_1^2}{\Delta(1+\tau)(1+\theta)^2\,(1+r)^2\,(1+n)}, \tag{A26}$$

$$D \equiv \frac{u_1^2}{\Delta(1+\tau)(1+\theta)\,(1+r)^2}\left[\frac{\alpha}{1+\theta} - \frac{(1+\theta-\alpha)f''(k)}{1+r}\frac{dk}{d\theta}\right], \tag{A27}$$

with

$$
\begin{aligned}
\Gamma &\equiv \frac{-u_1 k f''(k)}{\Delta\,(1+r)} \left[- \left(\frac{1+\theta+\alpha i}{(1+\tau)(1+\theta)(1+r)} \right)^2 u_{11} - u_{22} \right. \\
&+ \left. \frac{2\,(1+\theta+\alpha i)\,u_{12}}{(1+\tau)(1+\theta)(1+r)} + \frac{\alpha(1+\theta-\alpha)u_1}{(1+\tau)(1+\theta)(1+r)^2\,(1+n)\,k} \right]. \qquad \text{(A28)}
\end{aligned}
$$

It follows from (A19)–(A21) that the steady-state utility attains its maximum value at

$$
\begin{aligned}
r &= n, \qquad \text{(A29)} \\
i &= \frac{\tau\,(1+\theta)}{\alpha}. \qquad \text{(A30)}
\end{aligned}
$$

Setting $r = n$ in equation (19) yields $i = \theta$. Substituting θ for i in (A30) and simplifying results in

$$
\theta = \frac{\tau}{\alpha - \tau}. \qquad \text{(A31)}
$$