Second-best taxation of incomes and non-labor inputs in a model with endogenous wages

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Abstract

This paper considers a Mirrleesian optimal income tax model wherein labor inputs are not perfect substitutes and their wages are determined endogenously. It shows: (i) If skilled and unskilled workers are Edgeworth complements, skilled workers will necessarily face a marginal subsidy and unskilled workers a marginal tax on their incomes. These may not be the case if skilled and unskilled workers are Edgeworth substitutes. (ii) Redistributive concerns call for taxation of those inputs whose elasticity of complementarity with skilled labor is larger than with unskilled labor, and subsidization of those whose elasticity of complementarity is smaller.

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1 Introduction

Stiglitz's (1982) pioneering two-group model has become a workhorse for researchers in the optimal income tax literature. Although this literature has been centered almost exclusively on the circumstances where wages (or productivities) are determined exogenously, Stiglitz (1982) also studied the case where wages are determined endogenously. He showed that in this case skilled workers face a marginal subsidy and unskilled workers an additional marginal tax (on top of the tax they face with fixed wages). However, Stiglitz (1982) worked with a production function that had only skilled and unskilled workers as factors of production. The main aim of this paper is to demonstrate that the existence of other factors of production may reverse Stiglitz's (1982) results. A second aim of the paper is to examine the properties of the marginal tax rate on these other factors of production. This captures Naito's (1999) result on the breakdown of production efficiency in the presence of wage endogeneity within a very simple framework.

In achieving its first task, the paper provides a characterization for the optimal marginal income tax rates on skilled and unskilled workers.[1] It also links the signs of the tax rates to the direction of the change in the relative wage of unskilled to skilled workers when the labor supply of one or the other factor increases. It shows that Stiglitz's results continue to go through if this relative wage moves positively with the labor supply of skilled workers and negatively with the labor supply of unskilled workers. On the other hand, a negative relationship between this relative wage and the labor supply of skilled workers will reverse Stiglitz's (1982) result on the tax treatment of skilled workers. Similarly, a positive relationship between the relative wage of unskilled to skilled workers and the labor supply of unskilled workers will reverse Stiglitz's (1982) result on the tax treatment of unskilled workers.

The paper identifies Edgeworth substitutability between skilled and unskilled

[1] Micheletto (2004) gives a characterization of "effective" marginal income tax rates under wage endogeneity. These tax rates include commodity taxes that are constrained to be linear; see Edwards *et al.* (1994).

workers as the reason for why the relative wage of unskilled to skilled workers may decrease with a rise in the size of skilled workers and increase with a fall in the size of unskilled workers. Edgeworth complementarity rules out these possibilities. In turn, skilled and unskilled workers can be Edgeworth substitutes only if there are other factors of production in the economy. An example of a regular production function with skilled labor, unskilled labor, and capital establishes that not only Edgeworth substitutability is a possibility but also that it can indeed reverse Stiglitz's (1982) results. Specifically, if skilled and unskilled workers are Edgeworth complements, skilled workers will necessarily face a marginal subsidy on their income and unskilled workers will necessarily face an additional marginal tax on top of the traditional marginal tax. On the other hand, if skilled and unskilled workers are Edgeworth substitutes, it is possible for the skilled workers to face a marginal tax on their income and for the unskilled workers to face a countervailing marginal subsidy on their income (offsetting the traditional marginal income tax).

To be sure, this is not the first paper that studies the properties of the optimal marginal income tax rates where there is another factor of production in the economy besides skilled and unskilled labor. Pirttilä and Tuomala (2001) have studied this question, among others, in an overlapping-generations framework with endogenous capital formation. Yet, unlike this paper, they report identical results to Stiglitz's (1982) findings that skilled workers should face a marginal subsidy and unskilled workers an additional marginal tax [Proposition 1 on page 491 in Pirttilä and Tuomala (2001)]. The current paper demonstrates that while Pirttilä and Tuomala's (2001) results necessarily hold when skilled and unskilled workers are Edgeworth complements, Edgeworth substitutability *may* reverse the signs of the optimal marginal income tax rates they report.[2]

Another aspect of wage endogeneity that Stiglitz (1982) did not explore is its implication for the Diamond and Mirrlees (1971) celebrated production efficiency result. This holds that production efficiency is often desirable even in second-best

[2]It is of course also possible for the results of Stiglitz (1982) and Pirttilä and Tuomala (2001) to hold despite Edgeworth substitutability; see the example in Appendix B.

environments so that one should spare inputs and intermediate goods from (differential) taxation. Stiglitz (1985) proved the desirability of capital income taxes in a model with endogenous wages that had capital in addition to skilled and unskilled workers as the factors of production. Huber (1999) too studied the properties of capital income taxes in a similar one-sector model. The implications of these two studies for the breakdown of the production efficiency result is obvious. However, this result was not clearly spelled out until the publication of Naito (1999).[3]

The setup of this paper provides a simple framework for demonstrating the breakdown of the efficiency result. As such, it is pedagogically useful while also describing the properties of optimal factor taxes in light of their substitutability and complementarity relationships to labor inputs. Specifically, it shows that to address redistributive concerns, inputs whose elasticity of complementarity with skilled labor is larger than with unskilled labor must be taxed. On the other hand, inputs whose elasticity of complementarity with skilled labor is smaller than with unskilled labor must be subsidized.

2 The model

Consider an open economy which uses three factors of production to produce a composite consumption good, c. All markets are competitive. The factors of production consist of skilled workers, L^s, unskilled workers, L^u, and one non-labor input, K. The production technology, $O = \mathbf{O}(L^s, L^u, K)$, exhibits constant returns to scale with diminishing marginal product for all factors. Skilled and unskilled labor, who are *not* perfect substitutes, come from domestic sources with their wage rates, w^s and w^u, determined endogenously. The third input is imported from outside at the fixed world price of r. Denote the labor supply of a j-type worker, $j = s, u$, by h^j and the proportion of each type in total population

[3] This result was extended by to a small open economy and by Blackorby and Brett (2004) to the case of a strictly concave technology. Gaube (2005) corrected the literature's exclusive attention to "redistributive" and "regressive" cases by pointing out that the endogeneity of wages leads to additional configurations for the second-best solution. He also proved that these other cases too entail the breakdown of production efficiency.

by π^j. Normalize the population size at one so that $L^j = \pi^j h^j$. Preferences are defined over the composite consumption good, c, and the labor supply, h; represented by $U = \mathbf{U}(c,h)$ where $\mathbf{U}(c,h)$ is increasing in c, decreasing in h, and strictly quasi-concave.

2.1 The optimal tax problem

Assume that income, $I \equiv wh$, is publicly observable but that types and labor supplies are not. Given that types are unobservable, one cannot tax wage rates directly in production. Consequently, and given that markets are competitive, factor payment by firms to each type of labor input must be equal to its marginal product; that is, for $j = s, u$, $w^j = \mathbf{O}_{L^j}(L^s, L^u, K)$ where w^j is determined endogenously within the model. Introduce γ^j to denote a j-type worker's utility weight with the normalization that $\gamma^s + \gamma^u = 1$. Constrained Pareto-efficient allocations are found by maximizing $\gamma^s u^s + \gamma^u u^u$ subject to the economy's resource constraint, self-selection constraints, and two other constraints that ensure the equality of the firms' factor payments to labor inputs with their marginal products. Solving the optimal tax problem by imposing the equality of wage payments to marginal products as constraints distinguishes the method of the proof given in this paper from the traditional Stiglitz (1982, 1987) approach and makes the derivations much simpler.

Assume also that in equilibrium $w^s \geq w^u$ and redistribution is from skilled to unskilled workers.[4] One can then ignore the self-selection constraint corresponding to unskilled- mimicking skilled-workers and summarize the optimal tax problem

[4] As pointed out in footnote 2, Gaube (2005) has demonstrated that wage endogeneity may lead to second-best solutions other than the traditional "redistributive" and "regressive" cases. I nevertheless concentrate on the $w^s \geq w^u$ alone in light of the paper's goals.

by the Lagrangian

$$\begin{aligned}
\pounds \;=\; & \gamma^s \mathbf{U}\left(c^s, \frac{I^s}{w^s}\right) + \gamma^u \mathbf{U}\left(c^u, \frac{I^u}{w^u}\right) + \lambda\left[\mathbf{U}\left(c^s, \frac{I^s}{w^s}\right) - \mathbf{U}\left(c^u, \frac{I^u}{w^s}\right)\right] + \\
& \mu\left[\mathbf{O}\left(L^s, L^u, K\right) - \pi^s c^s - \pi^u c^u - rK - \overline{R}\right] + \\
& \delta^s\left[w^s - \mathbf{O}_{L^s}\left(L^s, L^u, K\right)\right] + \delta^u\left[w^u - \mathbf{O}_{L^u}\left(L^s, L^u, K\right)\right], \qquad (1)
\end{aligned}$$

where $\overline{R}$ is the government's external expenditures (non-transfers), and $\lambda, \mu, \delta^s, \delta^u$ are the Lagrangian multipliers associated with the downward self-selection constraint, the resource constraint, and the two labor demand constraints.

The first-order conditions to this problem can be written, after simplification, as (see Appendix A),

$$(\gamma^s + \lambda)\, \mathbf{U}_c(\cdot) = \mu\pi^s, \qquad (2)$$

$$(\gamma^s + \lambda)\, \frac{1}{w^s}\mathbf{U}_h(\cdot) = -\mu\pi^s + \delta^s\frac{\pi^s}{w^s}\mathbf{O}_{L^sL^s}(\cdot) + \delta^u\frac{\pi^s}{w^s}\mathbf{O}_{L^uL^s}(\cdot), \qquad (3)$$

$$\begin{aligned}
(\gamma^s + \lambda)\, \frac{I^s}{(w^s)^2}\mathbf{U}_h(\cdot) = {} & \lambda\frac{I^u}{(w^s)^2}\mathbf{U}_h(\cdot) - \mu\frac{\pi^s I^s}{w^s} + \\
& \delta^s\left[1 + \frac{\pi^s I^s}{(w^s)^2}\mathbf{O}_{L^sL^s}(\cdot)\right] + \delta^u\frac{\pi^s I^s}{(w^s)^2}\mathbf{O}_{L^uL^s}(\cdot), \qquad (4)
\end{aligned}$$

$$\gamma^u \mathbf{U}_c(\cdot) - \lambda \mathbf{U}_c(\cdot) = \mu\pi^u, \qquad (5)$$

$$\gamma^u\frac{1}{w^u}\mathbf{U}_h(\cdot) - \lambda\frac{1}{w^s}\mathbf{U}_h(\cdot) = -\mu\pi^u + \delta^s\frac{\pi^u}{w^u}\mathbf{O}_{L^sL^u}(\cdot) + \delta^u\frac{\pi^u}{w^u}\mathbf{O}_{L^uL^u}(\cdot), \qquad (6)$$

$$\gamma^u\frac{I^u}{(w^u)^2}\mathbf{U}_h(\cdot) = -\mu\frac{\pi^u I^u}{(w^u)^2}\mathbf{O}_{L^u}(\cdot) + \delta^s\frac{\pi^u I^u}{(w^u)^2}\mathbf{O}_{L^sL^u}(\cdot) + \delta^u\left[1 + \frac{\pi^u I^u}{(w^u)^2}\mathbf{O}_{L^uL^u}(\cdot)\right], \qquad (7)$$

$$\mu\left[\mathbf{O}_K(\cdot) - r\right] = \delta^s\mathbf{O}_{L^sK}(\cdot) + \delta^u\mathbf{O}_{L^uK}(\cdot). \qquad (8)$$

Equations (2)–(8), plus the constraints specified in the Lagrangian problem, characterize the optimal allocations for incomes, consumption levels, and the non-labor input.

3 Taxation of Incomes

Denote the implementing income tax function by $T(I) = I - c(I)$. At the taxpayer's equilibrium, his marginal income tax rate, $T'(I) = 1 - dc/dI$, is equal to[5]

$$1 - MRS_{cI}\left(c, I/w\right) \equiv 1 + \frac{\mathbf{U}_I\left(c, I/w\right)}{\mathbf{U}_c\left(c, I/w\right)}, \tag{9}$$

where MRS_{cI} denotes the marginal rate of substitution of consumption for income. Manipulating equations (2)–(7) yields (see the Appendix).

$$\begin{aligned} T'(I^s) &= \left[\frac{-\lambda I^u}{w^s}\mathbf{U}_h\left(c^u, \frac{I^u}{w^s}\right)\right]\frac{\frac{1}{w^s}\mathbf{O}_{L^sL^s}\left(\cdot\right) - \frac{1}{w^u}\mathbf{O}_{L^uL^s}\left(\cdot\right)}{\mu w^s}, & (10) \\ T'(I^u) &= \frac{\lambda \mathbf{U}_c\left(c^u, I^u/w^s\right)}{\mu \pi^u}\left[MRS_{cI}\left(c^u, \frac{I^u}{w^u}\right) - MRS_{cI}\left(c^u, \frac{I^u}{w^s}\right)\right] & (11) \\ &\quad + \left[\frac{-\lambda I^u}{w^s}\mathbf{U}_h\left(c^u, \frac{I^u}{w^s}\right)\right]\frac{\frac{1}{w^s}\mathbf{O}_{L^sL^u}\left(\cdot\right) - \frac{1}{w^u}\mathbf{O}_{L^uL^u}\left(\cdot\right)}{\mu w^u}. \end{aligned}$$

Given that $\left(-\lambda I^u/w^s\right)\mathbf{U}_h\left(c^u, I^u/w^s\right) > 0$, $T'(I^s)$ has the same sign as $\mathbf{O}_{L^sL^s}\left(\cdot\right)/w^s - \mathbf{O}_{L^uL^s}\left(\cdot\right)/w^u$. As to $T'(I^u)$ specified by equation (11), the first expression on its right-hand side is positive. This follows because with $I^u/w^u > I^u/w^s$, it must be the case that $MRS_{cI}(c^u,\ I^u/w^u) - MRS_{cI}(c^u, I^u/w^s) > 0$. There is also a second expression on the right-hand side of $T'(I^u)$ that has the same sign as $\mathbf{O}_{L^sL^u}\left(\cdot\right)/w^s - \mathbf{O}_{L^uL^u}\left(\cdot\right)/w^u$. Consequently, with $\mathbf{O}_{L^sL^s}\left(\cdot\right)$ and $\mathbf{O}_{L^uL^u}\left(\cdot\right)$ being negative, a sufficient condition for $T'(I^s) < 0$ and $T'(I^u) > 0$ is that $\mathbf{O}_{L^sL^u}\left(\cdot\right) > 0$. In other words, Edgeworth complementarity of skilled and unskilled labor guarantees $T'(I^s) < 0$ and $T'(I^u) > 0$.[6]

Stiglitz (1982) studied a model wherein skilled and unskilled labor are the only factors of production. In such a model, linear homogeneity of the production function coupled with diminishing marginal productivity of labor ensures that $\mathbf{O}_{L^sL^u}\left(\cdot\right) > 0$ so that the two factors are Edgeworth complements.[7] Consequently,

[5] If the implementing tax function is not differentiable at a point, its *left-hand* derivative will be equal to $1 - MRS_{cI}\left(c, I/w\right)$. The literature thus refers to $1 - MRS_{cI}\left(c, I/w\right)$ as the marginal tax rate; see Stiglitz (1987, p. 1003).

[6] Edgeworth complements and substitutes are defined in terms of the signs of cross partial derivatives of production (with respect to the factors of production) and utility functions (with respect to goods). A positive sign denotes a complement and a negative sign a substitute. See Edgeworth ([1897] 2001) and Weber (2005). When applied to production functions, Hicks (1970) refers to these concepts as q-complements and q-substitutes)

[7] Formally, linear homogeneity of $\mathbf{O}\left(\cdot\right)$ implies that $\mathbf{O}_{L^s}\left(\cdot\right)$ and $\mathbf{O}_{L^u}\left(\cdot\right)$ are homogeneous of degree zero. Consequently, $L^s\mathbf{O}_{L^sL^s}\left(\cdot\right) + L^u\mathbf{O}_{L^sL^u}\left(\cdot\right) = 0 \Rightarrow \mathbf{O}_{L^sL^u}\left(\cdot\right) = -\ L^s\mathbf{O}_{L^sL^s}\left(\cdot\right)/L^u > 0$.

he reached the conclusion that $T'(I^s) < 0$ and $T'(I^u) > 0$.

In models that include other inputs besides skilled and unskilled labor, however, there is no guarantee that skilled and unskilled labor are Edgeworth complements. Consider the generalized linear production function introduced by Diewert (1971) defined as,

$$\mathbf{O}(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n}\sum_{i=1}^{n} a_{ij}(x_i x_j)^{.5},$$

where x_i's are inputs and a_{ij}'s are constants. Here $\mathbf{O}_{x_i x_j}(\cdot)$ has the same sign as a_{ij} which Diewert (1971) allows to be negative. Of course, when some of the a_{ij}s are negative, the production function will not be nondecreasing or concave for all nonnegative input bundles. However, there is a range of x_i's for which the production function is nondecreasing and concave (and hence an adequate representation of production possibility sets).[8] Empirically too there is abundant evidence that factors can be Edgeworth substitutes and not necessarily Edgeworth complements. Thurston and Libby (2002), for example, applying the generalized linear function given above to the production of physician services, have concluded that "Technicians and aides are q-substitutes [Edgeworth substitutes] for administrative/clerical workers and nurses" (p 190).[9]

With Edgeworth substitutability, $\mathbf{O}_{L^s L^u}(\cdot) < 0$. This makes the sign of $\mathbf{O}_{L^s L^s}(\cdot)/w^s - \mathbf{O}_{L^u L^s}(\cdot)/w^u$ in (10) and the sign of $\mathbf{O}_{L^s L^u}(\cdot)/w^s - \mathbf{O}_{L^u L^u}(\cdot)/w^u$ in (11) indeterminate. In particular, if $\mathbf{O}_{L^s L^s}(\cdot)/w^s - \mathbf{O}_{L^u L^s}(\cdot)/w^u > 0$ then $T'(I^s) > 0$ and if $\mathbf{O}_{L^s L^u}(\cdot)/w^s - \mathbf{O}_{L^u L^u}(\cdot)/w^u < 0$ then the second component of $T'(I^u)$ will be negative contrary to the findings of Stiglitz (1982). To establish that these possibilities can occur, I present an example of Diewert's (1971) generalized linear production function in Appendix B wherein skilled and unskilled labor are Edgeworth substitutes with $\mathbf{O}_{L^s L^s}(\cdot)/w^s - \mathbf{O}_{L^u L^s}(\cdot)/w^u > 0$ or

[8] See Diewert (1971), pp 501–502, inequalities 3.15 and 3.16 with a_{ij} replacing b_{ij} and x_i replacing p_i.

[9] They have also found evidence for q-complements.

$\mathbf{O}_{L^sL^u}(\cdot)/w^s - \mathbf{O}_{L^uL^u}(\cdot)/w^u < 0$ (as well as with $\mathbf{O}_{L^sL^s}(\cdot)/w^s - \mathbf{O}_{L^uL^s}(\cdot)/w^u < 0$ and $\mathbf{O}_{L^sL^u}(\cdot)/w^s - \mathbf{O}_{L^uL^u}(\cdot)/w^u > 0$).

As a first step to understanding the economic reasons for these results, observe that the sign of $\mathbf{O}_{L^sL^s}(\cdot)/w^s - \mathbf{O}_{L^uL^s}(\cdot)/w^u$ determines the direction that the relative wage of unskilled to skilled workers, w^u/w^s, moves as the labor supply of skilled workers increases. Similarly, the sign of $\mathbf{O}_{L^sL^u}(\cdot)/w^s - \mathbf{O}_{L^uL^u}(\cdot)/w^u$ determines the direction that w^u/w^s moves as the labor supply of unskilled workers increases. This can be seen by differentiating $w^u/w^s = \mathbf{O}_{L^u}(\cdot)/\mathbf{O}_{L^s}(\cdot)$ partially with respect to L^s and L^u which yields

$$\begin{aligned}
\frac{\partial}{\partial L^s}\frac{\mathbf{O}_{L^u}(\cdot)}{\mathbf{O}_{L^s}(\cdot)} &= \frac{\mathbf{O}_{L^uL^s}(\cdot)\,w^s - \mathbf{O}_{L^sL^s}(\cdot)\,w^u}{(w^s)^2} \\
&= \frac{-w^u}{w^s}\left[\frac{1}{w^s}\mathbf{O}_{L^sL^s}(\cdot) - \frac{1}{w^u}\mathbf{O}_{L^uL^s}(\cdot)\right], \qquad (12) \\
\frac{\partial}{\partial L^u}\frac{\mathbf{O}_{L^u}(\cdot)}{\mathbf{O}_{L^s}(\cdot)} &= \frac{\mathbf{O}_{L^uL^u}(\cdot)\,w^s - \mathbf{O}_{L^sL^u}(\cdot)\,w^u}{(w^s)^2} \\
&= \frac{-w^u}{w^s}\left[\frac{1}{w^s}\mathbf{O}_{L^sL^u}(\cdot) - \frac{1}{w^u}\mathbf{O}_{L^uL^u}(\cdot)\right]. \qquad (13)
\end{aligned}$$

Using equations (12)–(13) one can then rewrite our earlier characterizations of $T'(I^s)$ and $T'(I^u)$ in (10)–(11) as

$$\begin{aligned}
T'(I^s) &= \frac{-1}{\mu w^u}\left[\frac{-\lambda I^u}{w^s}\mathbf{U}_h\left(c^u, \frac{I^u}{w^s}\right)\right]\frac{\partial}{\partial L^s}\left(\frac{w^u}{w^s}\right), \qquad (14) \\
T'(I^u) &= \frac{\lambda \mathbf{U}_c\left(c^u, I^u/w^s\right)}{\mu \pi^u}\left[MRS_{cI}\left(c^u, \frac{I^u}{w^u}\right) - MRS_{cI}\left(c^u, \frac{I^u}{w^s}\right)\right] \qquad (15) \\
&\quad - \frac{w^s}{\mu\left(w^u\right)^2}\left[\frac{-\lambda I^u}{w^s}\mathbf{U}_h\left(c^u, \frac{I^u}{w^s}\right)\right]\frac{\partial}{\partial L^u}\left(\frac{w^u}{w^s}\right).
\end{aligned}$$

Equation (14) shows $T'(I^s)$ assumes a sign opposite to $\partial\left(w^u/w^s\right)/\partial L^s$; similarly, equation (15) shows that $\partial\left(w^u/w^s\right)/\partial L^u$ has a negative effect on $T'(I^u)$. To see the intuition behind these results consider equation (14) first. If an increase in L^s increases w^u/w^s, it changes the relative wages in favor of the unskilled workers whom one wants to help. One thus tries to increase L^s by levying a marginal subsidy on skilled workers. On the other hand, if an increase in L^s lowers w^u/w^s, it changes the relative wages against unskilled workers. To discourage this, one resorts to a marginal tax on skilled workers. Turning to equation (15), if an increase in L^u decreases w^u/w^s, it changes the relative wages against unskilled workers. Increasing the marginal tax on unskilled workers discourages this. On the other hand, if an increase in L^u increases w^u/w^s, it changes the relative wages in favor of unskilled workers. To encourage this, one lowers the marginal tax on unskilled workers.

Next consider the relevance of Edgeworth complementarity/substitutability in determining what happens to w^u/w^s as L^s and L^u increase. Diminishing marginal productivity ensures that an increase in L^s lowers $w^s = \mathbf{O}_{L^s}(\cdot)$. Now, with Edgeworth complementarity between L^s and L^u, the increase in L^s increases $w^u = \mathbf{O}_{L^u}(\cdot)$ so that w^u/w^s unambiguously increases. On the other hand, with Edgeworth substitutability, $w^u = \mathbf{O}_{L^u}(\cdot)$ also decreases so that the effect on w^u/w^s is ambiguous. If the increase in L^s lowers w^u by a *lower* percentage than it lowers w^s then, as with Edgeworth complementarity, w^u/w^s continues to increase. On the other hand, if the increase in L^s lowers w^u by a *higher* percentage than it lowers w^s then w^u/w^s declines.

Similarly, diminishing marginal productivity ensures that an increase in L^u lowers $w^u = \mathbf{O}_{L^u}(\cdot)$. With Edgeworth complementarity between L^s and L^u, the increase in L^u increases $w^s = \mathbf{O}_{L^s}(\cdot)$ so that w^u/w^s unambiguously decreases. On the other hand, with Edgeworth substitutability, $w^s = \mathbf{O}_{L^s}(\cdot)$ also decreases and the effect on w^u/w^s is ambiguous. If the increase in L^u lowers w^s by a *lower* percentage than it lowers w^u then, as with Edgeworth complementarity, w^u/w^s continues to decline. On the other hand, if the increase in L^u lowers w^s by a *higher* percentage than it lowers w^u then w^u/w^s increases.

Finally, the role of Edgeworth complementarity/substitutability in determining how w^u/w^s changes as L^s and L^u increase provide the link between the results of this paper and the findings of Pirttilä and Tuomala (2001) who report identical results to Stiglitz's (1982). In proving $T'(I^s) < 0$, Pirttilä and Tuomala (2001) write "the increase in the labour supply of the high-ability type increases the relative wage of the low-ability workers" (p 491). Similarly, in proving $T'(I^u) > 0$, they write "since increasing labour supply of the low-ability type decreases their relative wage rate" (p 491). In light of the above results, it is apparent that in assuming these changes Pirttilä and Tuomala (2001) must have implicitly assumed that either skilled and unskilled workers are Edgeworth complements or that Edgeworth substitutability cannot cause w^u/w^s to decrease as L^s increases, nor cause w^u/w^s to increase as L^u increases.

These results are summarized in the following proposition.

Proposition 1 *Assume factors of production include inputs beyond skilled and unskilled workers whose wages are determined endogenously. Then:*

(i) The marginal income tax rates of skilled and unskilled workers are characterized by equations (10)–(11) and (14)–(15).

(ii) If an increase in the labor supply of skilled workers boosts the relative wage of the unskilled to skilled workers, skilled workers face a marginal subsidy on their income. On the other hand, if an increase in the labor supply of skilled workers lowers the relative wage of the unskilled to skilled workers, skilled workers face a marginal tax on their income.

(iii) If an increase in the labor supply of unskilled workers lowers the relative wage of the unskilled to skilled workers, unskilled workers face an additional marginal tax on their income on top of the traditional marginal tax in models with fixed wages. On the other hand, if an increase in the labor supply of unskilled workers boosts the relative wage of the unskilled to skilled workers, unskilled workers face a countervailing marginal subsidy on their income (as an offset to the traditional marginal income tax).

(iv) If skilled and unskilled workers are Edgeworth complements, increasing the labor supply of skilled workers will necessarily increase the relative wage of the unskilled to skilled workers and increasing the labor supply of unskilled workers will necessarily lower the relative wage of the unskilled to skilled workers. Consequently, skilled workers face a marginal subsidy on their income and unskilled

workers face an additional marginal tax.

(v) If skilled and unskilled workers are Edgeworth substitutes, increasing the labor supply of skilled workers may lower the relative wage of the unskilled to skilled workers. If this happens, skilled workers face a marginal tax on their income. Similarly, increasing the labor supply of unskilled workers may boost the relative wage of the unskilled to skilled workers. If this happens, unskilled workers face a countervailing marginal subsidy on their income.

4 Taxation of non-labor inputs

The optimal tax problem of subsection 2.1 can also be used as a simple pedagogical tool for demonstrating the breakdown of the production efficiency result when wages are endogenous. The model has, as in Huber (1999), only one sector which makes it simpler to work with. At the same time, its procedure of imposing the wage equilibrium conditions as constraints simplifies the derivations considerably.

Specifically, rewrite the optimality condition with respect to hiring of capital, first-order condition (8) as (see Appendix A),

$$\mathbf{O}_K(\cdot) - r = \left[\frac{-\lambda I^u}{w^s}\mathbf{U}_h\left(c^u, \frac{I^u}{w^s}\right)\right]\frac{\frac{1}{w^s}\mathbf{O}_{L^sK}(\cdot) - \frac{1}{w^u}\mathbf{O}_{L^uK}(\cdot)}{\mu}. \tag{16}$$

With exogenously fixed wage rates, $\mathbf{O}_{L^sK}(\cdot) = \mathbf{O}_{L^uK}(\cdot) = 0$ and $\mathbf{O}_K(\cdot) = r$. Then there should be no taxes on the non-labor input and production efficiency is satisfied. With endogenous wages, on the other hand, this is no longer the case because $\mathbf{O}_{L^sK}(\cdot) \neq 0$ and $\mathbf{O}_{L^uK}(\cdot) \neq 0$.[10]

When characterizing the conditions for taxing or subsidizing a factor of production in models with more than two factors, it is more useful to resort to the language of the elasticity of complementarity defined by Hicks (1970) than the elasticity of substitution used by Stiglitz (1982):[11]

[10] Being second best, $\lambda \neq 0$. Otherwise, in the first best, $\lambda = 0 \Rightarrow \mathbf{O}_K(\cdot) - r = 0$ and production efficiency holds.

[11] With constant returns to scale and two factors of production, the elasticity of complementarity is the inverse of both the direct and Allen Elasticity of substitution. See Sato and Koizumi (1973).

Definition 1 *Elasticity of complementarity between factors x_i and x_s in the production function $\mathbf{O}(x_1, x_2, \ldots, x_n,)$ is defined as*

$$\sigma_{x_i x_s} \equiv \frac{\mathbf{O}(\cdot)\,\mathbf{O}_{x_i x_s}(\cdot)}{\mathbf{O}_{x_i}(\cdot)\,\mathbf{O}_{x_s}(\cdot)}.$$

Thus multiply and divide the last expression on the right-hand side of (16) by $\mathbf{O}(\cdot)/\mathbf{O}_K(\cdot)$ and set $w^s = \mathbf{O}_{L^s}(\cdot)$, $w^u = \mathbf{O}_{L^u}(\cdot)$. Using Definition 1, one can rewrite (16) as

$$\mathbf{O}_K(\cdot) - r = \left[\frac{-\lambda I^u}{w^s}\mathbf{U}_h\left(c^u, \frac{I^u}{w^s}\right)\right]\frac{\mathbf{O}_K(\cdot)}{\mu\mathbf{O}(\cdot)}\left(\sigma_{L^s K} - \sigma_{L^u K}\right). \tag{17}$$

With $-\lambda I^u \mathbf{U}_h(c^u, I^u/w^s)/w^s > 0$, $\mathbf{O}_K(\cdot) - r$ has the same sign as $(\sigma_{L^s K} - \sigma_{L^u K})$. This sign is determined depending on whether the skilled or unskilled labor has a larger elasticity of complementarity with input K. It will be positive, i.e. K is taxed, if the skilled labor has the larger elasticity; and negative, i.e. K is subsidized, if the unskilled labor has the larger elasticity. The intuition is straightforward. If $\sigma_{L^s K} - \sigma_{L^u K} > 0$, K is "more of a complement" to L^s than to L^u; see Samuelson (1974). Under this circumstance, taxing K discourages the demand for skilled labor relatively more than the demand for unskilled labor. This lowers w^s relative to w^u. On the other hand, if $\sigma_{L^s K} - \sigma_{L^u K} < 0$, K is more of a complement to L^u than to L^s. In this case, subsidizing K encourages the demand for unskilled labor relatively more than the demand for skilled labor. This increases w^u relative to w^s. In both cases, redistribution takes place in the desired direction.

The above findings have direct links to Huber (1979) and Pirttilä and Tuomala (2001) who have shown that K should be taxed if increasing it increases w^s relative to w^u and subsidized if it decreases w^s relative to w^u. Their finding follows immediately from the complementarity results. Specifically, from the definition of the elasticity of complementarity, one has

$$\sigma_{L^s K} - \sigma_{L^u K} = \frac{\mathbf{O}(\cdot)}{\mathbf{O}_K(\cdot)}\left[\frac{\mathbf{O}_{L^s K}(\cdot)}{\mathbf{O}_{L^s}(\cdot)} - \frac{\mathbf{O}_{L^u K}(\cdot)}{\mathbf{O}_{L^u}(\cdot)}\right]. \tag{18}$$

Now consider how an increase in K affects the marginal rate of technical substitution between skilled and unskilled workers,

$$\frac{\partial}{\partial K}\left(MRTS_{L^s L^u}\right) = \frac{\partial}{\partial K}\left(\frac{\mathbf{O}_{L^s}(\cdot)}{\mathbf{O}_{L^u}(\cdot)}\right) = \frac{\mathbf{O}_{L^s}(\cdot)}{\mathbf{O}_{L^u}(\cdot)}\left[\frac{\mathbf{O}_{L^s K}(\cdot)}{\mathbf{O}_{L^s}(\cdot)} - \frac{\mathbf{O}_{L^u K}(\cdot)}{\mathbf{O}_{L^u}(\cdot)}\right]. \quad (19)$$

With $w^s = \mathbf{O}_{L^s}(\cdot)$ and $w^u = \mathbf{O}_{L^u}(\cdot)$, it follows from expressions (18)–(19) that $\sigma_{L^s K} \gtreqless \sigma_{L^u K} \iff \partial\left(w^s/w^u\right)/\partial K \gtreqless 0$.

This also shows that if the production function is weakly separable between labor (skilled and unskilled) and the non-labor input K,

$$\sigma_{L^s K} - \sigma_{L^u K} = \frac{\mathbf{O}(\cdot)}{\mathbf{O}_K(\cdot)}\frac{\mathbf{O}_{L^u}(\cdot)}{\mathbf{O}_{L^s}(\cdot)}\frac{\partial}{\partial K}\left(MRTS_{L^s L^u}\right) = 0,$$

so that production efficiency is restored.[12]

Finally, observe that $\sigma_{L^s K} - \sigma_{L^u K}$ is necessarily positive if K is an Edgeworth complement to skilled workers and an Edgeworth substitute to unskilled workers so that K must be taxed. On the other hand, $\sigma_{L^s K} - \sigma_{L^u K}$ is necessarily negative if K is an Edgeworth complement to unskilled workers and an Edgeworth substitute to skilled workers so that K must be subsidized. Thus Edgeworth complementarity with skilled labor and Edgeworth substitutability with unskilled labor calls for a tax; while Edgeworth complementarity with unskilled labor and Edgeworth substitutability with skilled labor calls for a subsidy. These results are summarized as:

Proposition 2 *Assume that skilled and unskilled workers are not perfect substitutes and that their wages are determined endogenously. Then:*

(i) Input taxes/subsidies are part of second-best tax structures; they are levied for redistributive purposes and characterized by equation (17).

(ii) If the non-labor input has a higher (lower) elasticity of complementarity with skilled labor as compared to unskilled labor, it must be taxed (subsidized).

(iii) If the production function is weakly separable between labor (skilled and unskilled) and the non-labor input, the non-labor input has the same elasticity

[12] This last result is Proposition 2 in Huber (1999).

of complementarity with skilled and unskilled labor and it should not be taxed or subsidized. The production efficiency result is restored.

(iv) The non-labor input must be taxed if it is an Edgeworth complement to skilled labor and an Edgeworth substitute to unskilled labor; it must be subsidized if it is an Edgeworth complement to unskilled labor and an Edgeworth substitute to skilled labor.

5 Concluding remarks

This paper has studied the properties of optimal income and non-labor input taxes in a Mirrleesian model with endogenously determined wages, and when skilled and unskilled workers are not perfect substitutes. It has extended Stiglitz's (1982) result on the properties of the marginal income tax rates to models wherein production requires inputs besides skilled and unskilled workers. It has shown that if skilled and unskilled workers are Edgeworth complements, skilled workers necessarily face a marginal subsidy on their income and unskilled workers necessarily face an additional marginal tax on top of the traditional marginal tax in models with fixed wages. If skilled and unskilled workers are Edgeworth substitutes, it is possible for the skilled workers to face a marginal tax on their income and for the unskilled workers to face a countervailing marginal subsidy on their income (offsetting the traditional marginal income tax). The former result occurs if an increase in the labor supply of skilled workers lowers the relative wage of unskilled to skilled workers. The latter result occurs if an increase in the labor supply of unskilled workers increases the relative wage of unskilled to skilled workers. An example has established that with Edgeworth substitutability either of these possibilities can occur.

Second, the model is particularly simple and thus pedagogically useful for demonstrating the breakdown of Diamond and Mirrlees (1971) production efficiency result when wage rates are determined endogenously. In this context, it has shown that redistributive concerns call for taxation of inputs that are complements to skilled workers and substitutes to unskilled workers, and subsidization of factor inputs that are substitutes to skilled workers and complements to unskilled workers. Specifically, if any factor input has a greater elasticity of complementarity with skilled workers than with unskilled workers that factor must be taxed at the margin. On the other hand, if the factor input has a greater elasticity of complementarity with unskilled workers than with skilled workers the factor must be subsidized at the margin.

Appendix A

Derivation of equations (2)–(8): Differentiate the Lagrangian expression (1) with respect to c^j, I^j, w^j, $j = s, u$, and K to get:

$$\begin{aligned}
\frac{\partial \pounds}{\partial c^s} &= (\gamma^s + \lambda)\,\mathbf{U}_c\left(c^s, \frac{I^s}{w^s}\right) - \mu\pi^s = 0, && \text{(A1)}\\
\frac{\partial \pounds}{\partial I^s} &= (\gamma^s + \lambda)\,\frac{1}{w^s}\mathbf{U}_h\left(c^s, \frac{I^s}{w^s}\right) + \mu\mathbf{O}_{L^s}(L^s, L^u, K)\,\frac{\partial L^s}{\partial I^s} - \\
&\quad \delta^s\mathbf{O}_{L^sL^s}(L^s, L^u, K)\,\frac{\partial L^s}{\partial I^s} - \delta^u\mathbf{O}_{L^uL^s}(L^s, L^u, K)\,\frac{\partial L^s}{\partial I^s} = 0, && \text{(A2)}\\
\frac{\partial \pounds}{\partial w^s} &= (\gamma^s + \lambda)\,\frac{-I^s}{(w^s)^2}\mathbf{U}_h\left(c^s, \frac{I^s}{w^s}\right) - \lambda\mathbf{U}_h\left(c^u, \frac{I^u}{w^s}\right)\frac{-I^u}{(w^s)^2} + \mu\mathbf{O}_{L^s}(L^s, L^u, K)\,\frac{\partial L^s}{\partial w^s} + \\
&\quad \delta^s\left[1 - \mathbf{O}_{L^sL^s}(L^s, L^u, K)\,\frac{\partial L^s}{\partial w^s}\right] - \delta^u\mathbf{O}_{L^uL^s}(L^s, L^u, K)\,\frac{\partial L^s}{\partial w^s} = 0, && \text{(A3)}\\
\frac{\partial \pounds}{\partial c^u} &= \gamma^u\mathbf{U}_c\left(c^u, \frac{I^u}{w^u}\right) - \lambda\mathbf{U}_c\left(c^u, \frac{I^u}{w^s}\right) - \mu\pi^u = 0, && \text{(A4)}\\
\frac{\partial \pounds}{\partial I^u} &= \gamma^u\frac{1}{w^u}\mathbf{U}_h\left(c^u, \frac{I^u}{w^u}\right) - \lambda\frac{1}{w^s}\mathbf{U}_h\left(c^u, \frac{I^u}{w^s}\right) + \mu\mathbf{O}_{L^u}(L^s, L^u, K)\,\frac{\partial L^u}{\partial I^u} - \\
&\quad \delta^s\mathbf{O}_{L^sL^u}(L^s, L^u, K)\,\frac{\partial L^u}{\partial I^u} - \delta^u\mathbf{O}_{L^uL^u}(L^s, L^u, K)\,\frac{\partial L^u}{\partial I^u} = 0, && \text{(A5)}\\
\frac{\partial \pounds}{\partial w^u} &= \gamma^u\frac{-I^u}{(w^u)^2}\mathbf{U}_h\left(c^u, \frac{I^u}{w^u}\right) + \mu\mathbf{O}_{L^u}(L^s, L^u, K)\,\frac{\partial L^u}{\partial w^u} - \\
&\quad \delta^s\mathbf{O}_{L^sL^u}(L^s, L^u, K)\,\frac{\partial L^u}{\partial w^u} + \delta^u\left[1 - \mathbf{O}_{L^uL^u}(L^s, L^u, K)\,\frac{\partial L^u}{\partial w^u}\right] = 0, && \text{(A6)}\\
\frac{\partial \pounds}{\partial K} &= \mu\left[\mathbf{O}_K(L^s, L^u, K) - r\right] - \\
&\quad \delta^s\mathbf{O}_{L^sK}(L^s, L^u, K) - \delta^u\mathbf{O}_{L^uK}(L^s, L^u, K) = 0. && \text{(A7)}
\end{aligned}$$

Next, partially differentiate L^j once with respect to I^j and once with respect to w^j, $j = s, u$,

$$\begin{aligned}
\frac{\partial L^j}{\partial I^j}|_{w^j} &= \frac{\partial}{\partial I^j}\left(\pi^j I^j / w^j\right) = \frac{\pi^j}{w^j}, && \text{(A8)}\\
\frac{\partial L^j}{\partial w^j}|_{I^j} &= \frac{\partial}{\partial w^j}\left(\pi^j I^j / w^j\right) = \frac{-\pi^j I^j}{(w^j)^2}. && \text{(A9)}
\end{aligned}$$

Then substitute from equations (A8)–(A9) into the first-order conditions (A1)–(A7), using $\mathbf{O}_{L^j}(L^s, L^u, K) = w^j$, $j = s, u$, and simplify.

Derivation of equations (10)–(11): The derivations use the values of the Lagrange multipliers δ^s and δ^u given in the following lemma.

Lemma 1 *The values of the Lagrange multipliers associated with the constraints* $w^s = \mathbf{O}_{L^s}(L^s, L^u, K)$ *and* $w^u = \mathbf{O}_{L^u}(L^s, L^u, K)$ *are*

$$\delta^s = \frac{-1}{w^s}\left[\frac{\lambda I^u}{w^s}\mathbf{U}_h\left(c^u, \frac{I^u}{w^s}\right)\right], \tag{A10}$$

$$\delta^u = \frac{1}{w^u}\left[\frac{\lambda I^u}{w^s}\mathbf{U}_h\left(c^u, \frac{I^u}{w^s}\right)\right]. \tag{A11}$$

Proof. To derive (A10), multiply equation (3) by I^s/w^s, subtract the resulting equation from (4), and simplify. Similarly, to derive (A11), multiply equation (6) by I^u/w^u, subtract the resulting equation from (7), and simplify. ■

To derive (10), divide equation (3) by equation (2) and substitute $\mathbf{U}_I(c^s, I^s/w^s)$ for $\mathbf{U}_h(c^s, I^s/w^s)/w^s$. This yields

$$T'(I^s) = \frac{\delta^s\mathbf{O}_{L^sL^s}(L^s, L^u, K) + \delta^u\mathbf{O}_{L^uL^s}(L^s, L^u, K)}{\mu w^s}.$$

Substituting for δ^s and δ^u from (A10)–(A11) into above results in (10).

To derive (11), divide equation (6) by equation (5) to get

$$-\frac{1}{w^u}\frac{\mathbf{U}_h(c^u, I^u/w^u)}{\mathbf{U}_c(c^u, I^u/w^u)} = \frac{1-\lambda\mathbf{U}_h(c^u, I^u/w^s)/\mu\pi^u w^s}{1+\lambda\mathbf{U}_c(c^u, I^u/w^s)/\mu\pi^u} - \frac{[\delta^s\mathbf{O}_{L^sL^u}(L^s, L^u, K) + \delta^u\mathbf{O}_{L^uL^u}(L^s, L^u, K)]/\mu w^u}{1+\lambda\mathbf{U}_c(c^u, I^u/w^s)/\mu\pi^u}.$$

Multiply this expression through by $1+\lambda\mathbf{U}_c(c^u, I^u/w^s)/\mu\pi^u$ and collect terms to arrive at

$$-\frac{1}{w^u}\frac{\mathbf{U}_h(c^u, I^u/w^u)}{\mathbf{U}_c(c^u, I^u/w^u)} = 1 + \frac{\lambda\mathbf{U}_c(c^u, I^u/w^s)}{\mu\pi^u}\left[\frac{1}{w^u}\frac{\mathbf{U}_h(c^u, I^u/w^u)}{\mathbf{U}_c(c^u, I^u/w^u)} - \frac{1}{w^s}\frac{\mathbf{U}_h(c^u, I^u/w^s)}{\mathbf{U}_c(c^u, I^u/w^s)}\right] - \frac{\delta^s\mathbf{O}_{L^sL^u}(L^s, L^u, K) + \delta^u\mathbf{O}_{L^uL^u}(L^s, L^u, K)}{\mu w^u}.$$

Then substitute $\mathbf{U}_I(c, I/w^j)$ for $\mathbf{U}_h(c, I/w^j)/w^j$ and $MRS_{cI}(c, I/w^j)$ for $-\mathbf{U}_I(c, I/w^j)/\mathbf{U}_c(c, I/w^j)$ to get

$$T'(I^u) = \frac{\lambda\mathbf{U}_c(c^u, I^u/w^s)}{\mu\pi^u}\left[MRS_{cI}\left(c^u, \frac{I^u}{w^u}\right) - MRS_{cI}\left(c^u, \frac{I^u}{w^s}\right)\right] + \frac{\delta^s\mathbf{O}_{L^sL^u}(L^s, L^u, K) + \delta^u\mathbf{O}_{L^uL^u}(L^s, L^u, K)}{\mu w^u}.$$

Substituting for δ^s and δ^u from (A10)–(A11) into above yields (11).

Derivation of equation (16): Divide both sides of equations (8) by μ to get

$$\mathbf{O}_K(\cdot) - r = \frac{\delta^s \mathbf{O}_{L^s K}(\cdot) + \delta^u \mathbf{O}_{L^u K}(\cdot)}{\mu}.$$

Then substitute the values of δ^s and δ^u from (A10)–(A11) into above.

Appendix B

Consider the following example from the class of generalized linear production functions introduced by Diewert (1971):

$$O = \mathbf{O}\left(L^s, L^u, K\right) = 2A\left(KL^s\right)^{.5} + 2B\left(KL^u\right)^{.5} - 2\left(L^sL^u\right)^{.5}, \tag{B1}$$

where $A, B,$ and C are positive constants. The various first- and second-order partial derivatives of $\mathbf{O}\left(L^s, L^u, K\right)$ are:[13]

$$\begin{aligned}
\mathbf{O}_{L^s}(\cdot) &= \left(\frac{L^u}{L^s}\right)^{.5}\left[A\left(\frac{K}{L^u}\right)^{.5} - 1\right], && \text{(B2)}\\
\mathbf{O}_{L^u}(\cdot) &= \left(\frac{L^s}{L^u}\right)^{.5}\left[B\left(\frac{K}{L^s}\right)^{.5} - 1\right], && \text{(B3)}\\
\mathbf{O}_{K}(\cdot) &= (K)^{-.5}\left[A\left(L^s\right)^{.5} + B\left(L^u\right)^{.5}\right], && \text{(B4)}\\
\mathbf{O}_{L^sL^s}(\cdot) &= -.5\left(L^s\right)^{-1.5}\left(L^u\right)^{.5}\left[A\left(\frac{K}{L^u}\right)^{.5} - 1\right], && \text{(B5)}\\
\mathbf{O}_{L^sL^u}(\cdot) &= -.5\left(L^s\right)^{-.5}\left(L^u\right)^{-.5}, && \text{(B6)}\\
\mathbf{O}_{L^sK}(\cdot) &= .5A\left(K\right)^{-.5}\left(L^s\right)^{-.5}, && \text{(B7)}\\
\mathbf{O}_{L^uL^u}(\cdot) &= -.5\left(L^u\right)^{-1.5}\left(L^s\right)^{.5}\left[B\left(\frac{K}{L^s}\right)^{.5} - 1\right], && \text{(B8)}\\
\mathbf{O}_{L^uK}(\cdot) &= .5B\left(K\right)^{-.5}\left(L^u\right)^{-.5}, && \text{(B9)}\\
\mathbf{O}_{KK}(\cdot) &= -.5\left(K\right)^{-1.5}\left[A\left(L^s\right)^{.5} + B\left(L^u\right)^{.5}\right]. && \text{(B10)}
\end{aligned}$$

Observe that in this example $\mathbf{O}_{L^sL^u}(\cdot) < 0$ so that skilled and unskilled workers are Edgeworth substitutes.

Condition 1: Production function (B1) must be increasing in $L^s, L^u,$ and K. It is clear from (B4) that $\mathbf{O}\left(L^s, L^u, K\right)$ is always increasing in K. To have it also increasing in L^s and L^u in the relevant range, it follows from (B2)–(B3) that the following conditions must be satisfied

$$\left(\frac{K}{L^u}\right)^{.5} > \frac{1}{A} \text{ and } \left(\frac{K}{L^s}\right)^{.5} > \frac{1}{B}. \tag{B11}$$

[13] Clearly, if $C < 0$, we have a “normal” production function with each pair of factors of productions being Edgeworth complements.

Condition 2: Production function (B1) must be concave in the relevant range. First, from (B10), $\mathbf{O}_{KK}(\cdot) < 0$. Additionally, the two conditions in (B11) ensure that $\mathbf{O}_{L^sL^s}(\cdot) < 0$ and $\mathbf{O}_{L^uL^u}(\cdot) < 0$. Second, given that $\mathbf{O}(L^s, L^u, K)$ is linear homogeneous, its principal minor of order 3 is zero. Third, thus remains the condition that $\mathbf{O}(L^s, L^u, K)$'s principal minor of order 2 is non-negative. That is,

$$\mathbf{O}_{L^sL^s}(\cdot)\,\mathbf{O}_{L^uL^u}(\cdot) - \mathbf{O}^2_{L^uL^s}(\cdot) =$$
$$.25\,(L^s)^{-1}(L^u)^{-1}\left[A\left(\frac{K}{L^u}\right)^{.5} - 1\right]\left[B\left(\frac{K}{L^s}\right)^{.5} - 1\right] - .25\,(L^s)^{-1}(L^u)^{-1} \geq 0.$$

Simplifying one can rewrite this condition as,

$$\frac{A}{\left(\frac{K}{L^s}\right)^{.5}} + \frac{B}{\left(\frac{K}{L^u}\right)^{.5}} \leq AB. \tag{B12}$$

Property 1: Edgeworth substitutability and the effect of increasing L^s on w^u/w^s. From equations (B2)–(B3), (B5)–(B6),

$$\frac{\mathbf{O}_{L^sL^s}(\cdot)}{\mathbf{O}_{L^s}(\cdot)} - \frac{\mathbf{O}_{L^uL^s}(\cdot)}{\mathbf{O}_{L^u}(\cdot)} = .5\,(L^s)^{-1}\left[\frac{2/B - (K/L^s)^{.5}}{(K/L^s)^{.5} - 1/B}\right],$$

where the denominator is positive (see (B11)). Consequently, from (12), $\partial(w^u/w^s)/\partial L^s$ is of opposite sign to $2/B - (K/L^s)^{.5}$. If this expression is negative, $\partial(w^u/w^s)/\partial L^s > 0$ and from (14) $T'(I^s) < 0$. This is the result derived by Stiglitz (1982) and Pirttilä and Tuomala (2001). On the other hand, if $2/B - (K/L^s)^{.5} > 0$ then $\partial(w^u/w^s)/\partial L^s > 0$ and $T'(I^s) > 0$. This is the opposite of Stiglitz's (1982) and Pirttilä and Tuomala's (2001) result. Observe that the required

$$\left(\frac{K}{L^s}\right)^{.5} < \frac{2}{B} \tag{B13}$$

condition for $T'(I^s) > 0$ is consistent with conditions (B11)–(B12) so that it is indeed *possible* for high-skilled workers to face a positive marginal income tax rate.[14] Of course, $(K/L^s)^{.5} > 2/B$ is also consistent with (B11)–(B12) so that one may have $\partial(w^u/w^s)/\partial L^s > 0$ and $T'(I^s) < 0$.

[14] One may choose the units of measurement for L^s, L^u, and K such that $L^s = L^u = K = 1$

Property 2: Edgeworth substitutability and the effect of increasing L^u on w^u/w^s. Substituting from equations (B2)–(B3), (B6), and (B8),

$$\frac{\mathbf{O}_{L^sL^u}(\cdot)}{\mathbf{O}_{L^s}(\cdot)} - \frac{\mathbf{O}_{L^uL^u}(\cdot)}{\mathbf{O}_{L^u}(\cdot)} = .5\,(L^u)^{-1}\left[\frac{(K/L^u)^{.5} - 2/A}{(K/L^u)^{.5} - 1/A}\right],$$

where the denominator is positive (see (B11)). Consequently, from (13), $\partial\,(w^u/w^s)\,/\partial L^u$ is of opposite sign to $(K/L^u)^{.5} - 2/A$. If this expression is positive, $\partial\,(w^u/w^s)\,/\partial L^u < 0$ and from (15) the second component of $T'(I^u)$ is positive—the result derived by Stiglitz (1982) and Pirttilä and Tuomala (2001). On the other hand, if $(K/L^u)^{.5} - 2/A < 0$ then $\partial\,(w^u/w^s)\,/\partial L^u > 0$ and the corresponding component in $T'(I^u)$ is negative. Under this circumstance, one gets a result opposite to that derived by Pirttilä and Tuomala (2001). The important point is that, as with condition (B13), condition

$$\left(\frac{K}{L^u}\right)^{.5} < \frac{2}{A} \tag{B14}$$

is consistent with conditions (B11)–(B12).[15] Of course, $(K/L^u)^{.5} > 2/A$ is also consistent with (B11)–(B12) and one may have $\partial\,(w^u/w^s)\,/\partial L^u < 0$ and $T'(I^u) > 0$.

One final point about this example is worth pointing out. Conditions (B13) and (B14) cannot hold simultaneously in the face of conditions (B11)–(B12). To see this, observe that one can rewrite condition (B13) as $A/\,(K/L^s)^{.5} > AB/2$ and condition (B14) as $B/\,(K/L^u)^{.5} > AB/2$. Now combining $A/\,(K/L^s)^{.5} > AB/2$ with (B12) implies $B/\,(K/L^u)^{.5} < AB/2$. Similarly, combining $B/\,(K/L^u)^{.5} > AB/2$ with (B12) implies $A/\,(K/L^s)^{.5} < AB/2$. Consequently, if $T'(I^s) > 0$ then

at their equilibrium values. Then conditions (B11)–(B13) for $T'(I^s) > 0$ become conditions on the size of the parameters A and B. These are given by

$$A > 1,\ 1 < B < 2,\ \text{and}\ AB \geq A + B,$$

which can readily be satisfied.

[15] With the units of measurement such that $L^s = L^u = K = 1$, the three conditions are

$$1 < A < 2,\ B > 1,\ \text{and}\ AB \geq A + B,$$

and can readily be satisfied.

the second component of $T'(I^u)$ is also positive. Similarly, if the second component of $T'(I^u)$ is negative then $T'(I^s) < 0$. Put differently, in this example, either of the two results of Stiglitz (1982) and Pirttilä and Tuomala (2001) may be violated but only one at a time. Both cannot be violated simultaneously. Being an example though, one does not know if this is a general property or not.

References

[1] Atkinson, Anthony B., Stiglitz, Joseph E., 1976. The design of tax structure: direct versus indirect taxation. Journal of Public Economics 6, 55–75.

[2] Blackorby, Charles, Brett, Craig, 2004. Production efficiency and the direct-indirect tax mix. Journal of Public Economic Theory 6, 165–180.

[3] Diamond, Peter A., Mirrlees, James A., 1971. Optimal taxation and public production I: production efficiency and II: tax rules. American Economic Review 61, 8–27 and 261–278.

[4] Edgeworth, Francis Y. [1987] 2001. The pure theory of monopoly. Bridel, Pascal, ed., 6 vols. London: Pickering and Chatto.

[5] Edwards, Jeremy, Keen, Michael, Tuomala, Matti, 1994. Income tax, commodity taxes, and public good provision: a brief guide. FinanzArchiv 51, 472–487.

[6] Gaube, Thomas, 2005. Income taxation, endogenous factor prices and production efficiency. Scandinavian Journal of Economics 107, 335–352.

[7] Hicks, John R., 1970. Elasticities of substitution again: substitutes and complements. Oxford Economic Papers 22, 289–296.

[8] Huber, Bernd, 1999. Tax competition and tax coordination in an optimum income tax model. Journal of Public Economics 71, 441–458.

[9] Naito, Hisahiro, 1999. Re-examination of uniform commodity taxes under a non-linear income tax system and its implication for production efficiency. Journal of Public Economics 71, 165–188.

[10] Micheletto, Luca, 2004. Optimal redistributive policy with endogenous wages. FinanzArchiv 60, 141–159.

[11] Pirttilä, Jukka, Tuomala, Matti, 2001. On optimal non-linear taxation and public good provision in an overlapping generations economy. Journal of Public Economics 79, 485–501.

[12] Samuelson, Paul A., 1974. Complementarity: An essay on the 40th anniversary of the Hicks-Allen revolution in demand theory. Journal of Economic Literature 12, 1255–1289.

[13] Sato, Ryuzo, Koizumi, Tetsunori, 1973. On the elasticities of substitution and complementarity. Oxford Economic Papers 25, 44–56.

[14] Stiglitz, Joseph E., 1982. Self-selection and Pareto efficient taxation. Journal of Public Economics 17, 213–240.

[15] Stiglitz, Joseph E., 1985. Inequality and capital taxation. IMSSS Technical Report No. 457, Stanford University.

[16] Stiglitz, Joseph E., 1987. Pareto efficient and optimal taxation and the new new welfare economics. Auerbach, Alan J. and Feldstein, Martin S., eds, Handbook of Public Economics, Vol 2, North-Holland, Amsterdam, 991–1042.

[17] Thurston, Norman K., Libby, Anne M., 2002. A production function for physician services revisited. The Review of Economics and Statistics 84, 184–191.

[18] Weber, Christian E., 2005. Edgeworth on complementarity, or Edgeworth, Auspitz-Lieben, and Pareto de-homogenized.