Principle of targeting in environmental taxation

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Abstract

This paper re-examines Sandmo's (1975) celebrated "additive property" and the principle of targeting in environmental taxation. It argues that, in the absence of direct emission taxes, one cannot in general divide commodity taxes into two mutually exclusive separate components of Pigouvian externality-correcting and Ramsey revenue-raising. Externality-correcting terms appear also in the expressions for the taxes on non-polluting goods—as well as in the expressions for taxes on the polluting goods—unless preferences are additively quasilinear either in one of the non-polluting goods or in the labor supply. On the other hand, in the presence of direct emission taxes, one can use emission taxes for externality correction and leave commodity taxes for revenue raising. Nevertheless the optimal emission tax is, in general, different from the marginal social damage of emissions.

JEL classification: H21; H23

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1 Introduction

The Pigouvian prescription for correcting an externality is to levy a tax on it equal to its marginal social damage. This is a first-best remedy which may have to be modified in second-best environments. Sandmo (1975) made this point in a pioneering work some twenty five years ago in the context of an economy where there are distortionary taxes in the system and when emissions are not taxed directly. Sandmo's main finding, dubbed the "additivity property," was that the presence of externality alters only the tax formula for the externality-generating good, leaving other tax formulas unaffected. Dixit (1985) later referred to Sandmo's result as an instance of the more general "principle of targeting". The idea is that one should best counter a distortion by the tax instrument that acts on it directly. Bovenberg and van der Ploeg (1994) also emphasize this principle in their finding that, in addition to tax formulas for other goods, the formula for the labor income tax must also remain unaffected. Over the years, a number of authors have refined and extended Sandmo's results in a number of ways.[1]

Two aspects of Sandmo's result remain unresolved. One concerns the observation that tax formulas change only in the case of polluting goods. This has invariably been interpreted to mean that one should not tamper with the tax on non-polluting goods for the purpose of correcting externalities. Dixit's interpretation appears to be based on this view. In turn, this view has lead to arguments about the externality-correcting versus revenue raising roles of different tax instruments. Untangling this "Ramsey-plus-Pigou" tax structure into two separate components, however, appears to be questionable. The second aspect is that Sandmo's analysis was limited to a setting where emissions are not taxed directly. Rather, the policy makers combat emissions by taxing goods that emit pollutants. Yet this question lies at the heart of tax treatment of externalities.

[1]See, among others, Bovenberg and van der Ploeg (1994), Bovenberg and de Mooij (1994, 1997), Bovenberg and Goulder (1996), Fullerton (1997), Schöb (1997), Cremer, Gahvari and Ladoux (1998,2001), Cremer and Gahvari (2001), Kopczuk (2003), Boadway and Tremblay (2008), and Micheletto (2008). See also the survey by Bovenberg and Goulder (2002) and the references therein.

The aim of this paper is to address both of these problems. In addressing the first question, I characterize the structure of optimal commodity taxes on polluting and non-polluting goods in a model containing many polluting and many non-polluting goods. I show how these taxes differ from Ramsey tax formulas in the absence of externalities. More importantly I prove that, contrary to the generally-accepted view in the literature, it is not just the taxes on polluting goods that have Pigouvian elements; the taxes on non-polluting goods too contain Pigouvian features. Nor can one separate these taxes into two mutually exclusive components, one for revenue raising and the other for Pigouvian considerations. In this sense, the principle of targeting fails.

As a follow-up to this general finding, I will examine if there are preference structures that allow one to separate the Pigouvian role from the Ramsey role. I show this will be true for preferences that are additive and quasilinear either in labor supply or in one of the non-polluting goods. With these preferences, the marginal social damage of emissions appear only in the optimal commodity taxes on polluting goods, but not in the optimal commodity taxes for non-polluting goods or in the optimal wage tax.

Turning to the question of emission taxes, I will show when and how the Pigouvian prescription needs to be modified when applied to direct taxation of emissions. Specifically, I will show that if preferences are separable in emissions and goods including labor supply, no adjustment is required. The optimal emission tax must be set equal to the marginal social damage of emissions (when the disutility of emission damage is translated into dollars via the shadow cost of public funds rather than the private marginal utility of income). Without the separability, this equality will no longer hold. The emission tax must be adjusted by a term that reflects the indirect effects of emissions on commodity tax revenues when demands for goods are functions of emission levels.

Finally, I will also show that whether emissions can be taxed directly or not has important implications for the structure and role of commodity taxes. Specifically, in the presence of an optimal emission tax, the formulas for optimal commodity taxes on polluting and non-polluting goods are identical to the Ramsey tax formulas in the absence of externalities. The principle of targeting applies in that emission taxes are levied to correct for the externality and commodity taxes are levied for raising revenues.

2 The model

Consider an economy with N identical individuals each endowed with one unit of time. Each person has preferences over consumer goods, labor supply and total emissions of pollutants into the atmosphere. There are $n+m$ consumer goods. The first n goods are non-polluting or "clean" goods whose production entails no emissions. These goods are produced by a linear technology subject to constant returns to scale by firms operating in a competitive environment. Denote the vector of private goods by $\underline{x} = (x_1, x_2, \ldots, x_n)$, their consumer prices by $\underline{p} = (p_1, p_2, \ldots, p_n)$, and the commodity taxes levied on them by $\underline{t} = (t_1, t_2, \ldots, t_n)$. Normalize the producer prices of these goods at one so that $p_i = 1 + t_i\,;\ i = 1, 2, \ldots, n$.

The second m goods are polluting or "dirty" goods whose production entails emissions of certain pollutants (e.g. CO_2, SO_2, etc.) into the atmosphere. This results in a negative consumption externality. Denote the vector of polluting goods by $\underline{y} = (y_1, y_2, \ldots, y_m)$, and emission per unit of output in the polluting industry s by e_s, with $\underline{e} = (e_1, e_2, \ldots, e_m)$. Assume the resource cost of producing one unit of y_s is a function of e_s. Denote this by $C_s(e_s)$ and assume that $C_s(\cdot)$ is continuously differentiable with $C_s''(\cdot) > 0$, $C_s'(\cdot) < 0$ for all e_s up to some limit $\bar{e}_s$, and $C_s'(\bar{e}_s) = 0\,;\ s = 1, 2, \ldots, m.$[2]

[2] The assumption that the production cost of y_s is negatively correlated to its emissions captures the fact that technologies which cut emissions are generally more expensive to employ. This must necessarily hold if emissions are reduced through abatement. When one is concerned with the choice between technologies, or the choice between inputs, each with its own emission characteristics, $C_s(e_s)$ may not necessarily be downward-sloping everywhere. This function is the lower frontier of a production set where different technologies are represented by points in the e, C space. Now it may well be the case that some "cleaner" technologies are also less expensive to employ. For example, it is less polluting to use natural gas for electricity generation than coal. In recent years, it has also been less expensive to do so. However, one can simply ignore any such possible upward-sloping parts of $C_s(e_s)$. They pose no conflict of interest between firms and the society.

Finally, assume that the production cost of y_s, for a given $C_s(e_s)$, exhibits constant returns to scale. Thus, $C_s(e_s)$ denotes the average and the marginal cost of producing y_s. Denote the consumer price of $\underline{y}$ by $\underline{q} = (q_1, q_2, \ldots, q_m)$ and the commodity taxes on $\underline{y}$ by $\underline{\tau} = (\tau_1, \tau_2, \ldots, \tau_m)$.

Preferences are represented by

$$u = \mathbf{u}\left(\underline{x}, \underline{y}, L, E\right), \tag{1}$$

where $\mathbf{u}(\cdot)$ is strictly quasi-concave, twice continuously differentiable and strictly increasing in $\underline{x}, \underline{y}$, and decreasing in L and E. With industry s producing Ny_s units of polluting good s, and each unit of y_s entailing e_s units of emissions, industry s generates $N\ e_s y_s$ units of emissions. Aggregate emissions by all industries are then equal to

$$E = N \sum_{s=1}^{m} e_s y_s. \tag{2}$$

I study two different scenarios. In one, emissions are observable and can be taxed directly in addition to the tax that may be levied on the polluting good. In the other, emissions are not observable and the only way they may be taxed is indirectly via the tax on the polluting good. I will explore the ways the structure, and the role, of taxes on polluting goods may differ under these two different scenarios and for the application of the principle of targeting. To give a unified presentation of the two scenarios, I introduce the notion of the "effective" tax on a commodity and define it as the difference between its consumer price and its marginal cost. In the case of non-polluting goods, the effective tax is the same as the "statutory" tax levied by the government on these goods. The same is true for polluting goods if there are no direct taxes on emissions. When emissions are taxed directly, however, the effective tax on a polluting good differs from its commodity tax. Denote the emission tax rate by θ and the effective tax on y_s

by T_s. Then,

$$T_s = q_s - C_s(e_s), \tag{3}$$

$$= \tau_s + \theta e_s. \tag{4}$$

The effective tax on polluting goods thus consists of two components. The first is the commodity tax τ_s and the second θe_s arises because of the emission tax. Observe that one can use the same notation for the case when emissions are not subject to a direct tax. Under this latter circumstance, $\theta = 0$ and T_s reduces to τ_s. That is,

$$T_s = \tau_s. \tag{5}$$

2.1 Emission taxes and emission per unit of output

As with clean goods, firms producing polluting goods operate in a competitive environment. A firm producing y_s chooses its emission level to maximize its profit

$$[q_s - C_s(e_s) - \theta e_s - \tau_s] y_s.$$

For any $y_s > 0$, the firm thus chooses e_s to minimize

$$C_s(e_s) + \theta e_s.$$

The firm's choice of e_s is thus found from[3]

$$-C_s'(e_s) = \theta. \tag{6}$$

Denote the solution to equation (6) by $\tilde{e}_s$. With zero profit condition in equilibrium, it must then be the case that

$$q_s = C_s(\tilde{e}_s) + \theta \tilde{e}_s + \tau_s. \tag{7}$$

[3]The second-order condition for the firm's optimization problem is satisfied from the convexity assumption $C_s''(\cdot) > 0$.

Observe that as long as all industries face the same emission tax, one will have

$$-C_s'(e_s) = -C_k'(e_k), \text{ for all } s \text{ and } k = 1, 2, \ldots, m. \tag{8}$$

That is, marginal (private) benefit of emissions is equalized across industries whether or not this marginal benefit is equal to the marginal social damage of emissions. If industries have different unit cost functions, $C_s(\cdot)$, they will choose different per unit emission levels e_s. On the other hand, with identical unit cost functions, $C(\cdot)$, $-C'(e_s) = -C'(e_k)$ implies that $e_s = e_k$.

3 Optimal effective commodity taxes

The discussion in this section applies whether or not emissions are taxed. Denote the wage rate by w, the tax rate on wages by t_w, and the lump-sum rebate, if any, by M. (Thus, if $M < 0$ we have a lump-sum tax). The representative consumer maximizes utility subject to the budget constraint:

$$\sum_{i=1}^{n} p_i x_i + \sum_{s=1}^{m} q_s y_s = w_n L + M, \tag{9}$$

where $w_n = w(1 - t_w)$ is the net of tax wage. The maximization problem yields the demand functions for $\underline{x}$ and $\underline{y}$ and the supply function for L. The individual's indirect utility function can then be defined as

$$\begin{aligned} v &= \mathbf{v}\left(\underline{p}, \underline{q}, w_n, M, E\right) \\ &\equiv \mathbf{u}\left(\underline{x}(\underline{p}, \underline{q}, w_n, M, E), \underline{y}(\underline{p}, \underline{q}, w_n, M, E), L(\underline{p}, \underline{q}, w_n, M, E), E\right). \end{aligned} \tag{10}$$

To determine the optimal tax rates, maximize the indirect utility function (10) with respect to the available instruments and subject to the government's budget constraint

$$\sum_{i=1}^{n} t_i x_i + \sum_{s=1}^{m} T_s y_s + t_w w L - M = \overline{R}, \tag{11}$$

where $\overline{R}$ is the government's per-capita external revenue requirement. The following Lemma simplifies the exposition of our results below.

Lemma 1 *Consider the Ramsey tax problem summarized by the Lagrangian*

$$\mathcal{L} = \mathbf{v}\left(\underline{p}, \underline{q}, w_n, M, E\right) + \mu \left[\sum_{i=1}^{n} t_i x_i + \sum_{s=1}^{m} T_s y_s + t_w w L - M - \overline{R}\right]. \tag{12}$$

Denote the marginal utility of income by $\alpha \equiv \partial v / \partial M$ *and define*

$$\Gamma \equiv N \left\{ \frac{\sum_{i=1}^{n} t_i \frac{\partial x_i}{\partial E} + \sum_{s=1}^{m} \left[T_s + N \frac{\mathbf{v}_E}{\mu} e_s\right] \frac{\partial y_s}{\partial E} + t_w w \frac{\partial L}{\partial E}}{1 - N \sum_{s=1}^{m} e_s \frac{\partial y_s}{\partial E}} \right\}, \tag{13}$$

where the denominator in Γ *is positive; see equation (A13) in the Appendic. Regardless of the availability of an emission tax:*

(i) The first-order conditions with respect to commodity tax instruments, $\underline{t}, \underline{\tau}$*, the wage tax* t_w*, and the lump-sum rebate* M *are given by, for all* $j = 1, 2, \ldots, n$*, and* $k = 1, 2, \ldots, m$*:*

$$\frac{\partial \mathcal{L}}{\partial t_j} = (\mu - \alpha) x_j + \mu \left\{ \sum_{i=1}^{n} t_i \frac{\partial x_i}{\partial p_j} + \sum_{s=1}^{m} \left[T_s + \left(N \frac{\mathbf{v}_E}{\mu} + \Gamma\right) e_s\right] \frac{\partial y_s}{\partial p_j} + t_w w \frac{\partial L}{\partial p_j} \right\} = 0, \tag{14}$$

$$\frac{\partial \mathcal{L}}{\partial \tau_k} = (\mu - \alpha) y_k + \mu \left\{ \sum_{i=1}^{n} t_i \frac{\partial x_i}{\partial q_k} + \sum_{s=1}^{m} \left[T_s + \left(N \frac{\mathbf{v}_E}{\mu} + \Gamma\right) e_s\right] \frac{\partial y_s}{\partial q_k} + t_w w \frac{\partial L}{\partial q_k} \right\} = 0, \tag{15}$$

$$\frac{-1}{w} \frac{\partial \mathcal{L}}{\partial t_w} = (\mu - \alpha)(-L) + \mu \left\{ \sum_{i=1}^{n} t_i \frac{\partial x_i}{\partial w_n} + \sum_{s=1}^{m} \left[T_s + \left(N \frac{\mathbf{v}_E}{\mu} + \Gamma\right) e_s\right] \frac{\partial y_s}{\partial w_n} + t_w w \frac{\partial L}{\partial w_n} \right\} = 0, \tag{16}$$

$$\frac{\partial \mathcal{L}}{\partial M} = -(\mu - \alpha) + \mu \left\{ \sum_{i=1}^{n} t_i \frac{\partial x_i}{\partial M} + \sum_{s=1}^{m} \left[T_s + \left(N \frac{\mathbf{v}_E}{\mu} + \Gamma\right) e_s\right] \frac{\partial y_s}{\partial M} + t_w w \frac{\partial L}{\partial M} \right\} = 0. \tag{17}$$

(ii) Let the sign $\sim$ *on a demand or a supply variable denote its "compensated" version. The first-order conditions with respect to commodity tax instruments,* $\underline{t}, \underline{\tau}$*, and the wage tax* t_w *can also be written as*

$$\sum_{i=1}^{n} t_i \left(\frac{\partial \widetilde{x}_i}{\partial p_j}\right) + \sum_{s=1}^{m} \left[T_s + \left(N \frac{\mathbf{v}_E}{\mu} + \Gamma\right) e_s\right] \left(\frac{\partial \widetilde{y}_s}{\partial p_j}\right) + t_w w \left(\frac{\partial \widetilde{L}}{\partial p_j}\right) = x_j \left(\frac{1}{\mu} \frac{\partial \mathcal{L}}{\partial M}\right), \tag{18}$$

$$\sum_{i=1}^{n} t_i \left(\frac{\partial \widetilde{x}_i}{\partial q_k}\right) + \sum_{s=1}^{m} \left[T_s + \left(N\frac{\mathbf{v}_E}{\mu} + \Gamma\right) e_s\right] \left(\frac{\partial \widetilde{y}_s}{\partial q_k}\right) + t_w w \left(\frac{\partial \widetilde{L}}{\partial q_k}\right) = y_k \left(\frac{1}{\mu}\frac{\partial \mathcal{L}}{\partial M}\right), \tag{19}$$

$$\sum_{i=1}^{n} t_i \left(\frac{\partial \widetilde{x}_i}{\partial w_n}\right) + \sum_{s=1}^{m} \left[T_s + \left(N\frac{\mathbf{v}_E}{\mu} + \Gamma\right) e_s\right] \left(\frac{\partial \widetilde{y}_s}{\partial w_n}\right) + t_w w \left(\frac{\partial \widetilde{L}}{\partial w_n}\right) = (-L) \left(\frac{1}{\mu}\frac{\partial \mathcal{L}}{\partial M}\right). \tag{20}$$

Proof. See the Appendix. ■

The interesting point to note about equations (14)–(17) is the way the taxes on non-polluting goods and polluting goods appear in them. Corresponding to t_i or t_w one has the following expression for the tax on polluting good,

$$T_s + \left(N\frac{\mathbf{v}_E}{\mu} + \Gamma\right) e_s.$$

This has two terms in addition to T_s. The first can be written as $Ne_s\,(\mathbf{v}_E/\mu)$ with N being the number of people affected by the pollutant and e_s the amount of emissions per unit of output. The term $\mathbf{v}_E/\mu$ is the marginal utility of emissions to individuals discounted by the shadow cost of public funds to the society μ. Put differently, $-\mathbf{v}_E/\mu$ is how the society assesses the marginal damage of emissions in terms of public dollars. This conception of the "social" cost accounts for both the damage of the emissions, as perceived by the individuals themselves, as well as the fact that in the absence of lump-sum taxes the cost of a dollar to the society μ is not the same as its private cost to the individual α.[4] The second term Γe_s captures the effect of a change in emission on consumer demands and through them on tax revenues. This term disappears if

[4] Alternatively, one can go from private cost to social cost in two steps reflecting two different phenomena. Write $-\mathbf{v}_E/\mu = (-\mathbf{v}_E/\alpha)\,/\,(\mu/\alpha)$ where $-\mathbf{v}_E/\alpha$ is the marginal damage of emissions discounted by the marginal utility of income. This is how an individual assesses the marginal damage of emissions in terms of dollars. In this way, one considers $N\,(-\mathbf{v}_E/\alpha)$ to represent the marginal social damage of emissions (as seen by the individuals themselves). This latter way of arriving at social cost by decomposing the terms into two different concepts, and labeling $-\mathbf{v}_E/\alpha$ as the marginal social damage of emissions, is behind the definition of the so-called "Pigouvian tax" by Bovenberg and van der Ploeg (1994), Fullerton (1997), and others in the literature. The direct way of going to the social cost by labeling $-\mathbf{v}_E/\mu$ as the marginal social damage of emissions, on the other hand, is behind the definition adopted by Cremer *et al.* (1998).

The two definitions apply equally to models with heterogeneous agents. One can define the marginal preferences are separable in emissions and the rest of the goods (including labor supply) so that there is no Edgeworth complementarity or substitutability relationships between emissions and goods.

Finally, I should note that the first-order condition with respect to M, equation (17), holds only in the first best when lump-sum taxes are available. In the second-best lump-sum taxes are unavailable and this equation must be deleted from the set of first-order conditions.

3.1 Normalization and tax characterization

Observe that because demand and the labor supply functions are homogeneous of degree zero in $\underline{p}, \underline{q}, w_n$, and M, one of the consumer prices (including the wage) can be normalized to one. Equivalently, one of the commodity or labor tax rates can be set to zero. This also implies that the equation corresponding to the tax which is normalized at zero must then be deleted from the set of first-order conditions.

Introduce $\widetilde{\Delta}$ to denote the Slutsky matrix. This is the matrix associated with the derivatives of the compensated demands functions $\underline{\widetilde{x}}$, $\underline{\widetilde{y}}$, and the compensated labor supply $\widetilde{L}$ with respect to consumer prices and the net wage,

$$\widetilde{\Delta} \equiv \begin{pmatrix} \frac{\partial \widetilde{x}_1}{\partial p_1} & \cdots & \frac{\partial \widetilde{x}_n}{\partial p_1} & \frac{\partial \widetilde{y}_1}{\partial p_1} & \cdots & \frac{\partial \widetilde{y}_m}{\partial p_1} & \frac{\partial \widetilde{L}}{\partial p_1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{\partial \widetilde{x}_1}{\partial p_n} & \cdots & \frac{\partial \widetilde{x}_n}{\partial p_n} & \frac{\partial \widetilde{y}_1}{\partial p_n} & \cdots & \frac{\partial \widetilde{y}_m}{\partial p_n} & \frac{\partial \widetilde{L}}{\partial p_n} \\ \frac{\partial \widetilde{x}_1}{\partial q_1} & \cdots & \frac{\partial \widetilde{x}_n}{\partial q_1} & \frac{\partial \widetilde{y}_1}{\partial q_1} & \cdots & \frac{\partial \widetilde{y}_m}{\partial q_1} & \frac{\partial \widetilde{L}}{\partial q_1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{\partial \widetilde{x}_1}{\partial q_m} & \cdots & \frac{\partial \widetilde{x}_n}{\partial q_m} & \frac{\partial \widetilde{y}_1}{\partial q_m} & \cdots & \frac{\partial \widetilde{y}_m}{\partial q_m} & \frac{\partial \widetilde{L}}{\partial q_m} \\ \frac{\partial \widetilde{x}_1}{\partial w_n} & \cdots & \frac{\partial \widetilde{x}_n}{\partial w_n} & \frac{\partial \widetilde{y}_1}{\partial w_n} & \cdots & \frac{\partial \widetilde{y}_m}{\partial w_n} & \frac{\partial \widetilde{L}}{\partial w_n} \end{pmatrix}. \tag{21}$$

social damage of emissions as $\sum_{h=1}^{H} \pi^h \left(\mathbf{v}_E^h / \alpha^h\right)$ or as $\sum_{h=1}^{H} \left(\pi^h \mathbf{v}_E^h\right) / \mu$, when h denotes a household of a particular type and π^h the number of such household types in the economy. Observe also that, as with all definitions, it is not always obvious which one is "better" or "more appropriate". Each has its own merit. In any event, the choice of the definition changes one's choice of terminology only and not the actual results. As long as one realizes what definition one is working with, and sticks to it throughout one's analysis, the terminology should not matter. Finally, note that in the first-best $\mu = \alpha$ and the two definitions amount to the same thing.

Let $\widetilde{\Delta}_w$ denote the matrix derived from $\widetilde{\Delta}$ by deleting its last rows and columns; this corresponds to a tax system where the wage tax is normalized to zero. Similarly, let $\widetilde{\Delta}_1$ denote the matrix derived from $\widetilde{\Delta}$ by deleting its first rows and columns (corresponding to a tax system wherein the tax on the first consumption good is normalized to zero). Finally, introduce

$$\begin{aligned}\Psi &\equiv \sum_{i=1}^{n} t_i \left(\frac{\partial x_i}{\partial M}\right) + t_w w \left(\frac{\partial L}{\partial M}\right) + \sum_{s=1}^{m} \left[T_s + \left(N\frac{\mathbf{v}_E}{\mu} + \Gamma\right) e_s\right] \left(\frac{\partial y_s}{\partial M}\right), \qquad (22)\\ &= \frac{1}{\mu}\frac{\partial \mathcal{L}}{\partial M} + \frac{\mu-\alpha}{\mu},\end{aligned}$$

where the second equality follows from equation (17). Observe that Ψ reflects the impact of income effects on tax revenues. In the absence of income effects on any particular good, x_i, L, or y_s, this channel will be closed. Let Ψ_w and Ψ_1 denote the expressions obtained from Ψ by setting $t_w = 0$ and $t_1 = 0$. Define Γ_w and Γ_1 in a similar fashion. The following proposition gives a characterization for the optimal effective commodity taxes, $\underline{t}, t_w, \underline{T}$, under the two normalization rules.

Proposition 1 *Consider the Ramsey tax problem of Lemma 1. Regardless of the availability of the emission tax, the optimal effective commodity taxes are characterized by:*

(i) If t_w is normalized to zero,

$$\begin{pmatrix} \underline{t} \\ \underline{T} + \left(N\frac{\mathbf{v}_E}{\mu} + \Gamma_w\right)\underline{e} \end{pmatrix} = \left(\Psi_w - \frac{\mu-\alpha}{\mu}\right)\widetilde{\Delta}_w^{-1}\begin{pmatrix} \underline{x} \\ \underline{y} \end{pmatrix}. \qquad (23)$$

(ii) If t_1 is normalized to zero,

$$\begin{pmatrix} t_2 \\ \vdots \\ t_n \\ \underline{T} + \left(N\frac{\mathbf{v}_E}{\mu} + \Gamma_1\right)\underline{e} \\ t_w w \end{pmatrix} = \left(\Psi_1 - \frac{\mu-\alpha}{\mu}\right)\widetilde{\Delta}_1^{-1}\begin{pmatrix} x_2 \\ \vdots \\ x_n \\ \underline{y} \\ -L \end{pmatrix}. \qquad (24)$$

Proof. To derive (23), observe that with the normalization $t_w = 0$ one maximizes welfare subject to $\underline{t}$ and $\underline{T}$ only. Hence equation (20) does not hold. Rewrite the applicable equations (18)–(19) in matrix form as

$$\widetilde{\Delta}_w \begin{pmatrix} \underline{t} \\ \underline{T} + \left(N \frac{\mathbf{v}_E}{\mu} + \Gamma_w \right) \underline{e} \end{pmatrix} = \left(\Psi_w - \frac{\mu - \alpha}{\mu} \right) \begin{pmatrix} \underline{x} \\ \underline{y} \end{pmatrix},$$

where Ψ_w is defined in the text. Pre-multiplying this equation by $\widetilde{\Delta}_w^{-1}$ yields (23).[5]

To derive (24), observe that with the normalization $t_1 = 0$ equation (18) does not hold for $j = 1$; the maximization is over $t_2, \ldots, t_n, \underline{T}$, and t_w. These equations can then be written in matrix form as $\widetilde{\Delta}_1 \begin{pmatrix} t_2 \\ \vdots \\ t_n \\ \underline{T} + \left(N \frac{\mathbf{v}_E}{\mu} + \Gamma_1 \right) \underline{e} \\ t_w w \end{pmatrix} = \left(\Psi_1 - \frac{\mu - \alpha}{\mu} \right) \begin{pmatrix} x_2 \\ \vdots \\ x_n \\ \underline{y} \\ -L \end{pmatrix}$,
where Ψ_1 is also defined in the text. Pre-multiplying this equation by $\widetilde{\Delta}_1^{-1}$ yields (24).[6]
■

It is important to point out here that the system of equations (23) and (24) are characterizations and not closed-form solutions for the optimal taxes. In particular, the expressions that appear in the right-hand sides of (23) and (24) are themselves functions of the vector of taxes, $\underline{t}, \underline{T}$, and t_w. I will come back to this point later on when discussing the principle of targeting. Observe also that in these characterizations, as with Lemma 1, corresponding to t_i or t_w one has the expression $T_s + (N\mathbf{v}_E/\mu + \Gamma)\, e_s$ and not T_s when it comes to the polluting goods.

4 Optimal emission tax

The preceding material apply regardless of the availability of the emission tax. This section studies the nature of the optimal emission tax if the government is able to levy

[5] Observe that $\widetilde{\Delta}_w$ is of full rank so that its inverse exists.
[6] Matrix $\widetilde{\Delta}_1$ is also of full rank.

a direct emission tax. The following Lemma simplifies the derivation of the optimal emission tax.

Lemma 2 *Consider the Ramsey tax problem of Lemma 1 and assume that emissions can be taxed directly at the rate* θ*.*

(i) The first-order conditions with respect to commodity tax instruments, $\underline{t}, \underline{\tau}$*, the wage tax* t_w*, and the lump-sum rebate* M*, for all* $j = 1, 2, \ldots, n$*, and* $k = 1, 2, \ldots, m$*, continue to be characterized by equations (14)–(17).*

(ii) When $\partial\mathcal{L}/\partial\tau_k$ *is set equal to zero for all* $k = 1, 2, \ldots, m$*, the first-order condition with respect to the emission tax* θ *is characterized by*

$$\frac{\partial\mathcal{L}}{\partial\theta} = \mu\left(\theta + N\frac{\mathbf{v}_E}{\mu} + \Gamma\right)\sum_{s=1}^{m}\frac{\partial e_s}{\partial\theta}y_s = 0, \tag{25}$$

where Γ *is defined by equation (13).*

Proof. See the Appendix. ■

Armed with Lemma 2, I now present a proposition for the characterization of the optimal emission tax. This proposition and Proposition 1 form the basis for my discussion of the principle of targeting.

Proposition 2 *Consider the Ramsey tax problem of Lemma 1 and assume one can directly tax emissions. Then:*

(i) In the presence of lump-sum taxes, the Pigouvian prescription holds so that the optimal emission tax is equal to the marginal social damage of emissions:

$$\theta = N\left(\frac{-\mathbf{v}_E}{\mu}\right) = N\left(\frac{-\mathbf{v}_E}{\alpha}\right). \tag{26}$$

Commodity taxes are set equal to zero and all tax revenues are raised from the lump-sum tax.

(ii) In the absence of lump-sum taxes, the Pigouvian prescription is modified. The optimal emission tax is characterized by

$$\theta = N\left(\frac{-\mathbf{v}_E}{\mu}\right) - \Gamma, \tag{27}$$

$$= N\left(\frac{-\mathbf{v}_E}{\mu}\right) - N\left(\sum_{i=1}^{n} t_i \frac{\partial x_i}{\partial E} + \sum_{s=1}^{m} \tau_s \frac{\partial y_s}{\partial E} + t_w w \frac{\partial L}{\partial E}\right), \tag{28}$$

where Γ is defined by equation (13). Consequently, the optimal emission tax differs from the marginal social damage of emissions, $N(-\mathbf{v}_E/\mu)$.

(iii) In the absence of lump-sum taxes, the Pigouvian prescription holds if preferences are separable in emissions and goods including labor supply. The optimal emission tax is equal to the social marginal damage of emissions, $N(-\mathbf{v}_E/\mu)$. However, unlike the first best, $N(-\mathbf{v}_E/\mu) \neq N(-\mathbf{v}_E/\alpha)$.

Proof. To prove result (i), one can easily check that a value of zero for all commodity taxes, $\underline{t} = \underline{\tau} = t_w = \underline{0}$, coupled with $\theta = N(-\mathbf{v}_E/\mu)$ for the emission tax, and $-M = \overline{R}$ for the lump-sum tax, constitute a solution to the first-order conditions (14)–(17) and (25) of Lemma 2. Observe also that in this case $\Gamma = 0$ and $\mu = \alpha$.

To prove (ii), set the expression for $\partial\mathcal{L}/\partial\theta$ in (25) equal to zero:

$$\mu\left(\theta + N\frac{\mathbf{v}_E}{\mu} + \Gamma\right)\sum_{s=1}^{m}\frac{\partial e_s}{\partial\theta} y_s = 0.$$

Now recall from equation (6) that $\theta = -C_s'(e_s)$. Differentiating this relationship with respect to θ and rearranging the terms,

$$\frac{\partial e_s}{\partial\theta} = -\frac{1}{C_s''(e_s)} < 0,$$

where the sign follows from the assumption that $C_s''(e_s) > 0$. Result (27) follows immediately. To derive (28), substitute for Γ from (13) into (27), multiply through by

$1-N\sum_{s=1}^{m}e_s\left(\partial y_s/\partial E\right)$, and rearrange the terms to get

$$\begin{aligned}
\theta-N\left(\frac{-\mathbf{v}_E}{\mu}\right) &= -N\left[\sum_{i=1}^{n}t_i\frac{\partial x_i}{\partial E}+\sum_{s=1}^{m}\left(T_s+N\frac{\mathbf{v}_E}{\mu}e_s\right)\frac{\partial y_s}{\partial E}+t_w w\frac{\partial L}{\partial E}\right]+N\left(\theta-N\frac{-\mathbf{v}_E}{\mu}\right)\sum_{s=1}^{m}e_s\frac{\partial y_s}{\partial E}, \\
&= -N\left\{\sum_{i=1}^{n}t_i\frac{\partial x_i}{\partial E}+\sum_{s=1}^{m}\left[T_s+N\frac{\mathbf{v}_E}{\mu}e_s-e_s\left(\theta-N\frac{-\mathbf{v}_E}{\mu}\right)\right]\frac{\partial y_s}{\partial E}+t_w w\frac{\partial L}{\partial E}\right\}, \\
&= -N\left[\sum_{i=1}^{n}t_i\frac{\partial x_i}{\partial E}+\sum_{s=1}^{m}\left(T_s-\theta e_s\right)\frac{\partial y_s}{\partial E}+t_w w\frac{\partial L}{\partial E}\right],
\end{aligned}$$

where, from (4), $T_s-\theta e_s=\tau_s$.

Finally, to prove (iii), observe that with separability, demand and labor supply functions are independent of emissions. This results in $\Gamma=0$ so that $\theta=N\left(-\mathbf{v}_E/\mu\right)$.

■

The result for the first-best is obvious; it forms the basis for the concept of principle of targeting which I will discuss in the next section. Result (ii) shows how the Pigouvian prescription is modified when there are distortionary taxes in the economy. To see its implication, observe that when faced with an emission tax θ, a firm in industry s sets its emissions such that $-C_s'(e_s)=\theta$. With the socially optimal emission tax being given by $\theta=N\left(-\mathbf{v}_E/\mu\right)-\Gamma$, this implies that the marginal (private) benefit of emissions to the firm must be set equal to the marginal social damage, $N\left(-\mathbf{v}_E/\mu\right)$, *plus* an adjustment term, $-\Gamma$. Thus marginal (private) benefit and marginal social damage of emissions differ (unless preferences are separable in emissions and other goods so that $\Gamma=0$.) This seems, at first blush, rather counter-intuitive. Further reflection, however, makes sense of it. The crucial point to note is that, with distortionary taxes, a change in emissions affects welfare not only through the private benefit to firms and the social damage caused by emissions, but also through its impact on tax revenues. The adjustment term consists precisely of these additional effects. They will appear as long as there exist complementarity or substitutability relationships between emissions and private goods (clean as well as dirty). If increased emissions lead to increased demand for private goods, the tax revenues that government collects increase. In turn, this would imply a reduction in resources to be collected from the emission tax. Thus revenue increases from commodity taxes constitute a "benefit" to increased emissions and works to countervail its negative effect (of increased social damage). This would translate, in terms of equation (27), into an increase in Γ and a reduction in the right-hand side of (27).

Observe also that in the absence of any complementarity and substitutability relationships between emissions and goods, a change in emissions leaves the demand and labor supply functions intact. Consequently, there will be no effect on tax revenues either. Under this circumstance, the benefit and cost of a change in marginal emissions will be confined to its private benefit to firms and damages imposed on the society. Hence the optimum will be characterized by the equality between the two. This explains (iii).

5 The principle of targeting

The nature of first-best taxes serves as the starting point for distinguishing the role of emission and commodity taxes and understanding the concept of the "principle of targeting." It is clear that in the first best, when lump-sum taxation is feasible and emissions are publicly observable, the emission tax is levied to correct for the externality; that is without this tax, the emission level will not be optimal. The lump-sum tax, on the other hand, is used to cover the rest of the government's external revenue requirement. It is also the case that the lump-sum tax is the only instrument used in the absence of emissions. When these properties—namely, (i) using the emission tax for correcting the externalities and (ii) identical structure for the other tax instruments in the presence and absence of externalities—hold, one can think of the tax instruments to have different and distinct roles.

Observe also that although the lump sum tax is the instrument used with and without externality to raise revenues, this does not mean that the *value* of the lump-sum remains the same in the two cases. If the government is to raise $\overline{R}$ with and without the emission tax, the fact some revenues are raised from emission taxes implies that less revenues will have to be raised from the lump-sum tax. Consequently, in stating that the tax instruments have different roles, one does not mean that the values of the tax instruments remain the same (with and without the externality).

A similar attempt for carving out different roles for different tax instruments in second-best environments was first attempted by Sandmo (1975). He found that, in a Ramsey tax model without the emission tax, the presence of externality alters only the tax formula for the externality generating good, leaving other tax formulas unaffected. He dubbed this the "additivity property." Dixit (1985) later referred to Sandmo's result as an instance of the more general "principle of targeting". Other studies discussing this property and expounding over it include, among others, Bovenberg and van der Ploeg (1994), Cremer et al. (1998, 2001), and Cremer and Gahvari (2001).

In this section, I investigate what Sandmo's result actually tells us, whether the tax formulas actually remain the same, and whether or not one can meaningfully talk about separation of tax roles.

5.1 With emission taxes

Consider first the case where emissions can be taxed directly and their tax is set optimally. With the optimal emission tax being given by (27), the effective tax on the polluting good y_s is equal to

$$\begin{aligned} T_s &= \tau_s + \theta e_s \\ &= \tau_s + \left[N \left(\frac{-\mathbf{v}_E}{\mu} \right) - \Gamma \right] e_s. \end{aligned}$$

Substituting this in the optimal commodity tax characterizations (23) and (24), yields

$$\left(\begin{array}{c} \underline{t} \\ \underline{\tau} \end{array} \right) = \left(\Psi_w - \frac{\mu - \alpha}{\mu} \right) \widetilde{\Delta}_w^{-1} \left(\begin{array}{c} \underline{x} \\ \underline{y} \end{array} \right), \tag{29}$$

if t_w is normalized to zero, and

$$\begin{pmatrix} t_2 \\ \vdots \\ t_n \\ \underline{\tau} \\ t_w w \end{pmatrix} = \left(\Psi_1 - \frac{\mu - \alpha}{\mu}\right) \widetilde{\Delta}_1^{-1} \begin{pmatrix} x_2 \\ \vdots \\ x_n \\ \underline{y} \\ -L \end{pmatrix}, \tag{30}$$

if t_1 is normalized to zero. Observe also that from (22) the expression for Ψ will be simplified to

$$\Psi \equiv \sum_{i=1}^{n} t_i \left(\frac{\partial x_i}{\partial M}\right) + t_w w \left(\frac{\partial L}{\partial M}\right) + \sum_{s=1}^{m} \tau_s \left(\frac{\partial y_s}{\partial M}\right). \tag{31}$$

Equations (29) and (30) are precisely the expressions one gets for the characterizations of optimal commodity taxes in the absence of emissions. In this sense, one can say that there is a separation in roles for emission and commodity taxes. While commodity taxes do affect the level of emissions, they cannot ensure that emission levels are optimal. It is the emission tax that is levied for the attainment of optimal emissions. Even though they also raise revenues, this is not their primary role. That role is assigned to commodity taxes. The principle of targeting applies.

It is also interesting to note that, unlike in the first-best, the emission tax now differs from the marginal social damage of emissions. The difference is captured by the expression Γ. As argued earlier, in the presence of distortionary taxes, a change in emissions, caused by the emission tax, affects welfare not only through the private benefit to firms and the social damage caused by emissions, but also through its impact on tax revenues. This feedback changes the amount of resources to be collected from the emission tax and must be taken into account when setting the optimal emission tax. There will be no such feedbacks in the absence of a complementarity or substitutability relationship between emissions and goods. Under this circumstance, $\Gamma = 0$ and the optimal emission tax is equal to the marginal social damage of emissions.

5.2 Without emission taxes

I now turn to the case where emissions are not observable and cannot be taxed directly. Commodity taxes are the only available tax instruments. This is the case Sandmo (1975) studied. Clearly, in the absence of emission taxes, emissions can be controlled only indirectly through the available commodity taxes. The interesting question is: Which commodity taxes? In particular, should the government adjust only the tax on polluting goods or on the tax on non-polluting goods as well? To change emissions one needs to change the consumption of polluting goods. It is plain that changing the consumption of any particular good can most directly be achieved by changing the price of that commodity itself. Put differently, by levying a tax on that commodity. However, this alone does not mean that one should not tamper with the tax on the non-polluting goods as a way to affect emissions. If a non-polluting good happens to be a close complement of a polluting good, taxing the complementary good too will reduce the consumption of the polluting good and with it the level of aggregate emissions.

Ever since the appearance of Sandmo (1975), however, many writers appear to have interpreted his exposition of the additivity property as an argument for adjusting the tax on the polluting goods only. To show that this interpretation is incorrect, I turn to the optimal effective commodity tax characterizations (23) and (24) of Section 3. Observe that with $\theta = 0$, the effective tax on good y_s is the same as the statutory commodity tax on it, τ_s. The optimal commodity tax characterizations (23) and (24) can then be rewritten in terms of $\underline{\tau}$ rather than $\underline{T}$ as

$$\begin{pmatrix} \underline{t} \\ \underline{\tau} + \left(N \frac{\mathbf{v}_E}{\mu} + \Gamma_w \right) \underline{e} \end{pmatrix} = \left(\Psi_w - \frac{\mu - \alpha}{\mu} \right) \widetilde{\Delta}_w^{-1} \begin{pmatrix} \underline{x} \\ \underline{y} \end{pmatrix}. \tag{32}$$

if t_w is normalized to zero, and

$$\begin{pmatrix} t_2 \\ \vdots \\ t_n \\ \underline{\tau} + \left(N\frac{\mathbf{v}_E}{\mu} + \Gamma_1 \right) \underline{e} \\ t_w w \end{pmatrix} = \left(\Psi_1 - \frac{\mu - \alpha}{\mu} \right) \widetilde{\Delta}_1^{-1} \begin{pmatrix} x_2 \\ \vdots \\ x_n \\ \underline{y} \\ -L \end{pmatrix}. \tag{33}$$

if t_1 is normalized to zero. Observe also that one can also rewrite the expression for Ψ, on which Ψ_w and Ψ_1 in (32)–(33) are based, as

$$\Psi \equiv \sum_{i=1}^{n} t_i \left(\frac{\partial x_i}{\partial M} \right) + t_w w \left(\frac{\partial L}{\partial M} \right) + \sum_{s=1}^{m} \left[\tau_s + \left(N\frac{\mathbf{v}_E}{\mu} + \Gamma \right) e_s \right] \left(\frac{\partial y_s}{\partial M} \right), \tag{34}$$

with Ψ_w and Ψ_1 denoting the expressions obtained from Ψ by setting $t_w = 0$ and $t_1 = 0$.

Comparing tax characterizations (32)–(33) with the ones one gets in the absence of externalities, one finds that the tax on non-polluting goods have stayed put while the tax on polluting goods have acquired an additional term $(N\mathbf{v}_E/\mu + \Gamma)\, e_s$.[7] This is the finding of Sandmo (1975) and what he called "additive property". As stated, this comparison of "tax formulas" is correct; however, it is also misleading. In particular, this comparison and the statement that the tax formulas for non-polluting goods *as written above* remain the same, does *not* tell us that the presence of emissions requires an adjustment in the polluting goods taxes only. The point is that the tax characterizations (32)–(33) are *not* closed-form solutions for $\underline{t}, \underline{\tau}$, and t_w. Given that the right-hand side of these equations are themselves functions of $\underline{t}, \underline{\tau}$, and t_w, no such deductions are warranted. Put differently, if one could derive a closed form solution for $\underline{t}, \underline{\tau}$, and t_w, it will *not* be the case that the terms reflecting the marginal social damage of emissions appear only in $\underline{\tau}$. To make the point absolutely clear, I resort to a specific example.

[7] A similar type of argument is made with respect to the "difference" between how $\underline{t}$ and $\underline{\tau}$ appear in equations (32)–(33) (as opposed to how these equations compare with the corresponding ones in the absence of emissions). Again, the two sets of tax instruments $\underline{t}$ and $\underline{\tau}$ differ by $(N\mathbf{v}_E/\mu + \Gamma)\, \underline{e}$ indicating an adjustment in the latter to reflect marginal social damage of emissions.

5.3 CES preferences

Assume that preferences are of the CES variety given by

$$u = \frac{-1}{\gamma}\left(x^{-\gamma} + y^{-\gamma} + l^{-\gamma}\right) - \varphi(E),$$

where $l = 1 - L$ is leisure and $E = Ny$ (where I have normalized emissions per unit of output e to one so that producing one unit of y results in one of emissions.) One can then easily determine the demand functions for l, x, and y as follows,

$$l = \frac{1 + \frac{M}{w_n}}{\left(\frac{p}{w_n}\right)^{1-\rho} + \left(\frac{q}{w_n}\right)^{1-\rho} + 1}, \tag{35}$$

$$x = \frac{\left(1 + \frac{M}{w_n}\right)}{\left(\frac{p}{w_n}\right)^{1-\rho} + \left(\frac{q}{w_n}\right)^{1-\rho} + 1}\left(\frac{p}{w_n}\right)^{-\rho}, \tag{36}$$

$$y = \frac{\left(1 + \frac{M}{w_n}\right)}{\left(\frac{p}{w_n}\right)^{1-\rho} + \left(\frac{q}{w_n}\right)^{1-\rho} + 1}\left(\frac{q}{w_n}\right)^{-\rho}. \tag{37}$$

where $\rho \equiv 1/(1+\gamma)$ denotes the elasticity of substitution.

To calculate second-best taxes, I set $M = 0$ and normalize the wage tax t_w to zero and w to one. Consequently,

$$\begin{aligned}\begin{pmatrix} t \\ \tau + N\frac{\mathbf{v}_E}{\mu} \end{pmatrix} &= \left(\Psi_w - \frac{\mu - \alpha}{\mu}\right)\widetilde{\Delta}_w^{-1}\begin{pmatrix} x \\ y \end{pmatrix} \\ &= \frac{\frac{\mu-\alpha}{\mu}\left(1 + p^{1-\rho} + q^{1-\rho}\right) - \left[tp^{-\rho} + \left(\tau + N\frac{\mathbf{v}_E}{\mu}\right)q^{-\rho}\right]}{\rho + 2\left(p^{1-\rho} + q^{1-\rho}\right)}\begin{pmatrix} p \\ q \end{pmatrix}. \end{aligned} \tag{38}$$

Rearranging and manipulation yield,

$$\frac{\tau + N\frac{\mathbf{v}_E}{\mu}}{q} = \frac{t}{p} = \frac{\mu - \alpha}{\mu}\frac{1}{3 + \frac{\rho - 3}{1 + p^{1-\rho} + q^{1-\rho}}},$$

which one can rewrite as

$$q - \frac{\mu-\alpha}{\mu}\frac{q}{3+\frac{\rho-3}{1+p^{1-\rho}+q^{1-\rho}}} = 1 + N\frac{-\mathbf{v}_E}{\mu}, \tag{39}$$

$$p - \frac{\mu-\alpha}{\mu}\frac{p}{3+\frac{\rho-3}{1+p^{1-\rho}+q^{1-\rho}}} = 1. \tag{40}$$

It follows from these two equations that $N(-\mathbf{v}_E/\mu)$ appears not only in the expression for q but also in the expression for p and thus t (unless $\rho = 1$ or $\rho = 3$). That is, the structure of the tax on non-polluting good is also affected by the emissions.

It is only in the special cases of $\rho = 1$, which corresponds to Cobb-Douglas preferences, and $\rho = 3$ that $N(-\mathbf{v}_E/\mu)$ appears only in q and τ but not in p and t. With $\rho = 1$, it follows from equations (39)–(40) that[8]

$$\begin{aligned}\tau &= \frac{3(\mu-\alpha)}{4\mu+3\alpha} + \frac{7\mu N(-\mathbf{v}_E/\mu)}{4\mu+3\alpha}, \\ t &= \frac{3(\mu-\alpha)}{4\mu+3\alpha}.\end{aligned}$$

With $\rho = 3$,[9]

$$\begin{aligned}\tau &= \frac{\mu-\alpha}{2\mu+\alpha} + \frac{3\mu N(-\mathbf{v}_E/\mu)}{2\mu+\alpha}, \\ t &= \frac{\mu-\alpha}{2\mu+\alpha}.\end{aligned}$$

The main results of this section are summarized as

[8] Observe also that in this simple case with Cobb-Douglas preferences, the "tax differential" between τ and t is

$$\frac{7}{4+3\alpha/\mu}N(-\mathbf{v}_E/\mu) = \frac{7}{3+4\mu/\alpha}N(-\mathbf{v}_E/\alpha).$$

This is different from either $N(-\mathbf{v}_E/\mu)$ or $N(-\mathbf{v}_E/\alpha)$, the two different measures of the marginal social damage of emissions. With $\mu > \alpha$ in the presence of distortionary taxation, this tax differential exceeds $N(-\mathbf{v}_E/\mu)$ and falls short of $N(-\mathbf{v}_E/\alpha)$. There is a small literature that attempts to compare the optimal tax differential with the "Pigouvian tax" (defined by one or the other measure of the marginal social damage of emissions). See, among others, Bovenberg and de Mooij (1994, 1997), Fullerton (1997), Schöb (1997), Cremer et al. (2001).

[9] The tax differential is now

$$\frac{3}{2+\alpha/\mu}N(-\mathbf{v}_E/\mu) = \frac{3}{1+2\mu/\alpha}N(-\mathbf{v}_E/\alpha).$$

Again, with $\mu > \alpha$, this exceeds $N(-\mathbf{v}_E/\mu)$ and falls short of $N(-\mathbf{v}_E/\alpha)$.

Proposition 3 *Consider the Ramsey tax problem of Lemma 1.*

(i) In the presence of an optimal emission tax, the formulas for optimal commodity taxes on polluting and non-polluting goods are identical to the Ramsey tax formulas in the absence of externalities. The principle of targeting applies in that emission taxes are levied to correct for the externality and commodity taxes are levied for raising revenues.

(ii) In the absence of direct emission taxes, the structure of optimal commodity taxes on polluting and non-polluting goods in general differ from Ramsey tax formulas. The tax on non-polluting goods have Pigouvian elements. One cannot identify separate components for revenue raising and Pigouvian considerations in the tax formulas. In this sense, the principle of targeting fails.

6 Separating Ramsey and Pigouvian considerations

In the previous section, I showed that if preferences are Cobb-Douglas (unitary elasticity of substitution), or CES with an elasticity of substitution equal to three, the tax on the non-polluting good will be levied independently of the marginal social damage of emissions so that there is a separation of roles for the tax instruments and the principle of targeting holds. In this section, I investigate what preference structures lead to such separability of roles when there are no emission taxes.

6.1 Quasilinear preferences in labor

Assume preferences that are additive and quasi-linear in labor supply, so that (1) is written as

$$u = \sum_{i=1}^{n} f_i(x_i) + \sum_{s=1}^{m} g_s(y_s) - L - \varphi(E). \tag{41}$$

Maximizing (41) with respect to $\underline{x}$, $\underline{y}$, and L, subject to the individual's budget constraint (9) yields the following first-order conditions

$$f_i'(x_i) = \alpha p_i, \quad i = 1, 2, \ldots, n,$$

$$
\begin{aligned}
g_s'(y_s) &= \alpha q_s, \quad s = 1, 2, \ldots, m, \\
1 &= \alpha w_n.
\end{aligned}
$$

In this case, demand for all non-leisure goods have two desirable properties. First, they are independent of income so that demands and compensated demands are the same. Second, each demand curve is a function of its own price and w_n. Thus when labor is the numeraire, i.e. $t_w \equiv 0$ and $w_n \equiv 1$, demands are functions of their own price only. I will thus follow this normalization.

Given the above two properties, it follows from equation (21) for $\widetilde{\Delta}$ and the definition of $\widetilde{\Delta}_w$ that in this case $\widetilde{\Delta}_w$ is a diagonal matrix given by

$$
\widetilde{\Delta}_w = \begin{pmatrix}
\frac{\partial \widetilde{x}_1}{\partial p_1} & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & \frac{\partial \widetilde{x}_n}{\partial p_n} & 0 & \cdots & 0 \\
0 & \cdots & 0 & \frac{\partial \widetilde{y}_1}{\partial q_1} & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & \frac{\partial \widetilde{y}_m}{\partial q_m}
\end{pmatrix}. \tag{42}
$$

Moreover, the additivity of preferences in emissions imply that $\underline{x}, \underline{y}$, and L are independent of E. It then follows from the expression for Γ in (13) that $\Gamma = 0$. Similarly, with preferences being quasilinear in labor supply, $\underline{x}$ and $\underline{y}$ are independent of lump-sum income M; all the income effects show up in labor supply L. Consequently, the expression for Ψ simplifies, from (22), to $\Psi = t_w w \left(\partial L / \partial M\right)$. However, with the normalization of $t_w = 0$, Ψ will also be equal to zero: $\Psi = 0$. Thus, from equation (32), the optimal tax rates $\underline{t}$ and $\underline{\tau}$ are given by

$$
\begin{pmatrix} \underline{t} \\ \underline{\tau} + \left(N \frac{\mathbf{v}_E}{\mu}\right) \underline{e} \end{pmatrix} = \left(-\frac{\mu - \alpha}{\mu}\right) \widetilde{\Delta}_w^{-1} \begin{pmatrix} \underline{x} \\ \underline{y} \end{pmatrix}, \tag{43}
$$

where $\widetilde{\Delta}_w$ is the diagonal matrix in (42).

It now follows from (43) that one can write the optimal tax rates as

$$
\begin{aligned}
t_i &= \left(1-\frac{\alpha}{\mu}\right)\frac{x_i}{\left(-\partial x_i/\partial p_i\right)}, \quad i=1,2,\ldots,n,\\
\tau_s &= \left(1-\frac{\alpha}{\mu}\right)\frac{y_s}{\left(-\partial y_s/\partial q_s\right)}+N\frac{-\mathbf{v}_E}{\mu}e_s, \quad s=1,2,\ldots,m.
\end{aligned}
$$

Alternatively, one can rewrite the above equations in elasticity terms. Thus define $\varepsilon_j^x \equiv \left(-\partial x_j/\partial p_j\right)\left(p_j/x_j\right), \varepsilon_k^y \equiv \left(-\partial y_k/\partial q_k\right)\left(q_k/y_k\right)$ and rewrite the equation for t_j and τ_k, after a bit of algebraic manipulations, as

$$
t_i = \frac{\left(1-\frac{\alpha}{\mu}\right)}{\varepsilon_i^x-\left(1-\frac{\alpha}{\mu}\right)}, \quad i=1,2,\ldots,n, \tag{44}
$$

$$
\tau_s = \frac{\left(1-\frac{\alpha}{\mu}\right)}{\varepsilon_s^y-\left(1-\frac{\alpha}{\mu}\right)}+\frac{\varepsilon_s^y}{\varepsilon_s^y-\left(1-\frac{\alpha}{\mu}\right)}\left(N\frac{-\mathbf{v}_E}{\mu}e_s\right), \quad s=1,2,\ldots,m. \tag{45}
$$

With demand for goods being functions of their own prices only, the elasticities are functions of their own prices only. Consequently, the emission terms appear only in the tax formulas for taxes on polluting goods, equation (45) but not in the tax formulas for taxes on non-polluting goods, equation (44).

6.2 Quasilinear preferences in goods

Assume now that preferences are additive and quasi-linear in one of the non-polluting goods, say good one. Hence write (1) as

$$
u = x_1+\sum_{i=2}^{n} f_i\left(x_i\right)+\sum_{s=1}^{m} g_s\left(y_s\right)-\phi(L)-\varphi\left(E\right). \tag{46}
$$

Maximizing (46) with respect to $\underline{x}$, $\underline{y}$, and L, subject to the individual's budget constraint (9) yields the following first-order conditions

$$
1=\alpha p_1,
$$

$$
\begin{aligned}
f_i'(x_i) &= \alpha p_i, \quad i = 2, 3, \ldots, n, \\
g_s'(y_s) &= \alpha q_s, \quad s = 1, 2, \ldots, m, \\
\phi'(L) &= \alpha w_n.
\end{aligned}
$$

Now, it is the labor supply and all the demand functions, except for good one, that are independent of income and are equal to their corresponding compensated labor supply and compensated demand functions. Secondly, each is a function of its own price and p_1. In this case, I use good one as the numeraire and follow the normalization $t_1 \equiv 0$ and $p_1 \equiv 1$. Consequently, with the exception of good one, all demands are functions of their own price only and the labor supply is a function of w_n only.

Given these properties, it is now $\widetilde{\Delta}_1$ which is a diagonal matrix. From its definition and equation (21) for $\widetilde{\Delta}$, one has

$$
\widetilde{\Delta}_1 = \begin{pmatrix}
\frac{\partial \widetilde{x}_2}{\partial p_2} & \cdots & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \cdots & \frac{\partial \widetilde{x}_n}{\partial p_n} & 0 & \cdots & 0 & 0 \\
0 & \cdots & 0 & \frac{\partial \widetilde{y}_1}{\partial q_1} & \cdots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & \frac{\partial \widetilde{y}_m}{\partial q_m} & 0 \\
0 & \cdots & 0 & 0 & \cdots & 0 & \frac{\partial \widetilde{L}}{\partial w_n}
\end{pmatrix}. \tag{47}
$$

Again, the additivity of preferences in emissions imply that $\underline{x}, \underline{y}$, and L are independent of E so that $\Gamma = 0$. Additionally, in this case, all the income effects show up in good one. Consequently, the expression for Ψ simplifies, from (22), to $\Psi = t_1 (\partial x_1 / \partial M)$. Thus, with the normalization of $t_1 = 0$, $\Psi = 0$. It now follows from equation (33) that

$$
\begin{pmatrix}
t_2 \\ \vdots \\ t_n \\ \underline{\tau} + \left(N \frac{\mathbf{v}_E}{\mu} \right) \underline{e} \\ t_w w
\end{pmatrix} = \left(-\frac{\mu - \alpha}{\mu} \right) \widetilde{\Delta}_1^{-1} \begin{pmatrix} x_2 \\ \vdots \\ x_n \\ \underline{y} \\ -L \end{pmatrix}, \tag{48}
$$

where $\widetilde{\Delta}_1$ is the diagonal matrix in (47).

Equation (48) leads to the following expressions for the optimal tax rates

$$\begin{aligned}
t_i &= \left(1-\frac{\alpha}{\mu}\right)\frac{x_i}{(-\partial x_i/\partial p_i)}, \quad i=2,3,\ldots,n, \\
\tau_s &= \left(1-\frac{\alpha}{\mu}\right)\frac{y_s}{(-\partial y_s/\partial q_s)} - N\frac{\mathbf{v}_E}{\mu}e_s, \quad s=1,2,\ldots,m, \\
t_w w &= \left(1-\frac{\alpha}{\mu}\right)\frac{L}{(\partial L/\partial w_n)}.
\end{aligned}$$

Defining elasticity for goods, as previously, and the elasticity of labor supply by $\varepsilon^L \equiv (\partial L/\partial w_n)(w_n/L)$, one can and rewrite the equation for t_i, τ_s, and t_w, after a bit of algebraic manipulations, as

$$t_i = \frac{1-\frac{\alpha}{\mu}}{\varepsilon_i^x - \left(1-\frac{\alpha}{\mu}\right)}, \quad i=2,3,\ldots,n, \tag{49}$$

$$\tau_s = \frac{\left(1-\frac{\alpha}{\mu}\right)}{\varepsilon_s^y - \left(1-\frac{\alpha}{\mu}\right)} + \frac{\varepsilon_s^y}{\varepsilon_s^y - \left(1-\frac{\alpha}{\mu}\right)}\left(N\frac{-\mathbf{v}_E}{\mu}e_s\right), \quad s=1,2,\ldots,m, \tag{50}$$

$$t_w = \frac{1-\frac{\alpha}{\mu}}{\varepsilon^L + \left(1-\frac{\alpha}{\mu}\right)}. \tag{51}$$

With demands for goods (other than the untaxed good one) and labor supply being functions of their own prices only, the corresponding elasticities are functions of their own prices only. Consequently, the emission terms appear only in the tax formulas for taxes on polluting goods, equation (50) but not in the formulas for non-polluting goods or labor supply, equations (49) and (51).

The results of this section are summarized as

Proposition 4 *Consider the Ramsey tax problem of Lemma 1 and assume there are no direct emission taxes. A sufficient condition for the terms containing the marginal social damage of emissions to appear in the optimal commodity tax on polluting goods, but not in the optimal commodity tax for non-polluting goods and labor supply, is for preferences to be additive and quasilinear in labor supply, as in (41), or in one of the* *non-polluting goods, as in (46). The optimal tax rates are given by (44)–(45) in the former case and (49)–(51) in the latter case.*

7 Concluding remarks

This paper has revisited the question of the principle of targeting in environmental taxation. One dimension of this question that the paper has studied concerns the widely-accepted interpretation of Sandmo's (1975) additivity result (derived in a Ramsey tax model where only goods can be taxed). According to this view, Pigouvian considerations affect only the tax treatment of polluting goods but not non-polluting goods, with the former reflecting both externality-correcting and revenue-raising functions and the latter only a revenue-raising function. The paper has challenged this view and argued that such separation is in general not possible. Pigouvian elements also enter in the taxes on non-polluting goods. To be able to separate these functions, one needs to impose severe restrictions on the structure of preferences. Additively quasilinear preferences, either in one of the non-pollting goods or in labor supply, allow this.

The second dimension of this question that has been studied concerns direct taxation of emissions. The paper has shown that when emissions are taxed directly, and optimally, they are levied for Pigouvian reasons leaving revenue raising to commodity taxes (on polluting as well as non-polluting goods). In this case, the formulas for optimal commodity taxes on polluting and non-polluting goods are identical to the Ramsey tax formulas in the absence of externalities. The principle of targeting thus applies. Nevertheless, the emission tax differs from the Pigouvian prescription and is not equal to the marginal social damage of emissions. A modification is necessary to account for the indirect effects of emissions on commodity tax revenues when some kind of complementarity or substitutability relationship exists between demands for goods and emissions. Only in the absence of such relationships the emission tax is equal to the marginal social damage of emissions.

As a suggestion for future research, note that this paper is based on a representative consumer model. This approach was chosen deliberately to match Sandmo's original formulation and the bulk of literature on this subject. However, a richer and more satisfactory approach to these issues should be based on the more modern optimal tax theory à la Mirrlees (1971). This theory allows for individuals to be heterogeneous and justifies the absence of first-best taxes by the existence of informational asymmetries between tax authorities and taxpayers (rather than through an ad-hoc restriction as is done in the Ramsey tax model). This approach also allows income to be taxed nonlinearly, as well as linearly, rather than proportionally as in the Ramsey tax model.[10] These aspects lead to a better understanding of the role and the properties of the various feasible tax instruments the government can employ to achieve its goals of externality correction, efficiency and redistribution.

[10] See Cremer and Gahvari (2001) for some steps in this direction.

Appendix

Proof of Lemma 1: Part (i). Differentiate the Lagrangian expression (12) with respect to $\underline{t}, \underline{\tau}, t_w$, and M to get,

$$\frac{\partial \mathcal{L}}{\partial t_j} = \frac{\partial \mathbf{v}}{\partial p_j} + \mu \left[x_j + \sum_{i=1}^{n} t_i \frac{\partial x_i}{\partial p_j} + \sum_{s=1}^{m} T_s \frac{\partial y_s}{\partial p_j} + t_w w \frac{\partial L}{\partial p_j} \right] + \tag{A1}$$

$$\left\{ \mathbf{v}_E + \mu \left[\sum_{i=1}^{n} t_i \frac{\partial x_i}{\partial E} + \sum_{s=1}^{m} T_s \frac{\partial y_s}{\partial E} + t_w w \frac{\partial L}{\partial E} \right] \right\} \frac{dE}{dt_j} = 0, \quad j = 1, 2, \ldots, n,$$

$$\frac{\partial \mathcal{L}}{\partial \tau_k} = \frac{\partial \mathbf{v}}{\partial q_k} + \mu \left[y_k + \sum_{i=1}^{n} t_i \frac{\partial x_i}{\partial q_k} + \sum_{s=1}^{m} T_s \frac{\partial y_s}{\partial q_k} + t_w w \frac{\partial L}{\partial q_k} \right] + \tag{A2}$$

$$\left\{ \mathbf{v}_E + \mu \left[\sum_{i=1}^{n} t_i \frac{\partial x_i}{\partial E} + \sum_{s=1}^{m} T_s \frac{\partial y_s}{\partial E} + t_w w \frac{\partial L}{\partial E} \right] \right\} \frac{dE}{d\tau_k} = 0, \quad k = 1, 2, \ldots, m,$$

$$\frac{\partial \mathcal{L}}{\partial t_w} = \left\{ \frac{\partial \mathbf{v}}{\partial w_n} + \mu \left[\sum_{i=1}^{n} t_i \frac{\partial x_i}{\partial w_n} + \sum_{s=1}^{m} T_s \frac{\partial y_s}{\partial w_n} + t_w w \frac{\partial L}{\partial w_n} \right] \right\} \frac{\partial w_n}{\partial t_w} + \mu w L + \tag{A3}$$

$$\left\{ \mathbf{v}_E + \mu \left[\sum_{i=1}^{n} t_i \frac{\partial x_i}{\partial E} + \sum_{s=1}^{m} T_s \frac{\partial y_s}{\partial E} + t_w w \frac{\partial L}{\partial E} \right] \right\} \frac{dE}{dt_w} = 0,$$

$$\frac{\partial \mathcal{L}}{\partial M} = \left\{ \frac{\partial \mathbf{v}}{\partial M} + \mu \left[-1 + \sum_{i=1}^{n} t_i \frac{\partial x_i}{\partial M} + \sum_{s=1}^{m} T_s \frac{\partial y_s}{\partial M} + t_w w \frac{\partial L}{\partial M} \right] \right\} + \tag{A4}$$

$$\left\{ \mathbf{v}_E + \mu \left[\sum_{i=1}^{n} t_i \frac{\partial x_i}{\partial E} + \sum_{s=1}^{m} T_s \frac{\partial y_s}{\partial E} + t_w w \frac{\partial L}{\partial E} \right] \right\} \frac{dE}{dM} = 0.$$

Next, recall that

$$E = N \sum_{s=1}^{m} e_s y_s.$$

Differentiating this equation with respect to $\underline{t}, \underline{\tau}, t_w$, and M yields

$$\frac{dE}{dt_j} = N \sum_{s=1}^{m} e_s \left(\frac{\partial y_s}{\partial p_j} + \frac{\partial y_s}{\partial E} \frac{dE}{dt_j} \right), \quad j = 1, 2, \ldots, n, \tag{A5}$$

$$\frac{dE}{d\tau_k} = N \sum_{s=1}^{m} e_s \left(\frac{\partial y_s}{\partial q_k} + \frac{\partial y_s}{\partial E} \frac{dE}{d\tau_k} \right), \quad k = 1, 2, \ldots, m, \tag{A6}$$

$$\frac{dE}{dt_w} = N\sum_{s=1}^{m} e_s\left(\frac{\partial y_s}{\partial w_n}\frac{\partial w_n}{\partial t_w}+\frac{\partial y_s}{\partial E}\frac{dE}{dt_w}\right), \tag{A7}$$

$$\frac{dE}{dM} = N\sum_{s=1}^{m} e_s\left(\frac{\partial y_s}{\partial M}+\frac{\partial y_s}{\partial E}\frac{dE}{dM}\right). \tag{A8}$$

Manipulate equation (A5):

$$\begin{aligned}\frac{dE}{dt_j} &= N\left(\sum_{s=1}^{m} e_s\frac{\partial y_s}{\partial p_j}+\sum_{s=1}^{m} e_s\frac{\partial y_s}{\partial E}\frac{dE}{dt_j}\right)\\ &= N\sum_{s=1}^{m} e_s\frac{\partial y_s}{\partial p_j}+\frac{dE}{dt_j}N\sum_{s=1}^{m} e_s\frac{\partial y_s}{\partial E}.\end{aligned}$$

"Solving" this equation for dE/dt_j yields

$$\frac{dE}{dt_j} = \frac{N\sum_{s=1}^{m} e_s\frac{\partial y_s}{\partial p_j}}{1-N\sum_{s=1}^{m} e_s\frac{\partial y_s}{\partial E}}. \tag{A9}$$

Manipulating equations (A6), (A7), and (A8) in a similar fashion results in

$$\frac{dE}{d\tau_k} = \frac{N\sum_{s=1}^{m} e_s\frac{\partial y_s}{\partial q_k}}{1-N\sum_{s=1}^{m} e_s\frac{\partial y_s}{\partial E}}, \tag{A10}$$

$$\frac{dE}{dt_w} = \frac{N\sum_{s=1}^{m} e_s\frac{\partial y_s}{\partial w_n}\frac{\partial w_n}{\partial t_w}}{1-N\sum_{s=1}^{m} e_s\frac{\partial y_s}{\partial E}}, \tag{A11}$$

$$\frac{dE}{dM} = \frac{N\sum_{s=1}^{m} e_s\frac{\partial y_s}{\partial M}}{1-N\sum_{s=1}^{m} e_s\frac{\partial y_s}{\partial E}}. \tag{A12}$$

Observe that E would change with M in the same direction as the aggregate polluting goods change with M so that

$$1-N\sum_{s=1}^{m} e_s\frac{\partial y_s}{\partial E} > 0. \tag{A13}$$

The next step consists of substituting the expressions for $dE/dt_j, dE/d\tau_k, dE/dt_w$, and dE/dM from (A9)–(A12) into the first-order conditions (A1)–(A4) and simplifying. Start with substituting dE/dt_j in (A9), using Roy's identity. This yields

$$\frac{\partial\mathcal{L}}{\partial t_j} = (\mu-\alpha)\,x_j+\mu\left(\sum_{i=1}^{n} t_i\frac{\partial x_i}{\partial p_j}+\sum_{s=1}^{m} T_s\frac{\partial y_s}{\partial p_j}+t_w w\frac{\partial L}{\partial p_j}\right)+$$

$$\left[\mathbf{v}_E+\mu\left(\sum_{i=1}^n t_i\frac{\partial x_i}{\partial E}+\sum_{s=1}^m T_s\frac{\partial y_s}{\partial E}+t_w w\frac{\partial L}{\partial E}\right)\right]\frac{N}{1-N\sum_{s=1}^m e_s\frac{\partial y_s}{\partial E}}\sum_{s=1}^m e_s\frac{\partial y_s}{\partial p_j}$$

$$=(\mu-\alpha)\,x_j+\mu\left(\sum_{i=1}^n t_i\frac{\partial x_i}{\partial p_j}+t_w w\frac{\partial L}{\partial p_j}\right)+\mu\sum_{s=1}^m T_s\frac{\partial y_s}{\partial p_j}+$$

$$\sum_{s=1}^m\frac{N\left[\mathbf{v}_E+\mu\left(\sum_{i=1}^n t_i\frac{\partial x_i}{\partial E}+\sum_{s=1}^m T_s\frac{\partial y_s}{\partial E}+t_w w\frac{\partial L}{\partial E}\right)\right]}{1-N\sum_{s=1}^m e_s\frac{\partial y_s}{\partial E}}e_s\frac{\partial y_s}{\partial p_j}$$

$$=(\mu-\alpha)\,x_j+\mu\left(\sum_{i=1}^n t_i\frac{\partial x_i}{\partial p_j}+t_w w\frac{\partial L}{\partial p_j}\right)+$$

$$\mu\sum_{s=1}^m\left[\frac{N\left(\frac{\mathbf{v}_E}{\mu}+\sum_{i=1}^n t_i\frac{\partial x_i}{\partial E}+\sum_{s=1}^m T_s\frac{\partial y_s}{\partial E}+t_w w\frac{\partial L}{\partial E}\right)}{1-N\sum_{s=1}^m e_s\frac{\partial y_s}{\partial E}}e_s+T_s\right]\frac{\partial y_s}{\partial p_j},$$

$$=(\mu-\alpha)\,x_j+\mu\left(\sum_{i=1}^n t_i\frac{\partial x_i}{\partial p_j}+t_w w\frac{\partial L}{\partial p_j}\right)+$$

$$\mu\sum_{s=1}^m\left[\frac{N\left(\frac{\mathbf{v}_E}{\mu}+\sum_{i=1}^n t_i\frac{\partial x_i}{\partial E}+\sum_{s=1}^m T_s\frac{\partial y_s}{\partial E}+t_w w\frac{\partial L}{\partial E}\right)}{1-N\sum_{s=1}^m e_s\frac{\partial y_s}{\partial E}}e_s+T_s+N\frac{\mathbf{v}_E}{\mu}e_s-N\frac{\mathbf{v}_E}{\mu}e_s\right]\frac{\partial y_s}{\partial p_j},$$

$$=(\mu-\alpha)\,x_j+\mu\left(\sum_{i=1}^n t_i\frac{\partial x_i}{\partial p_j}+t_w w\frac{\partial L}{\partial p_j}\right)+\mu\sum_{s=1}^m\left(T_s+N\frac{\mathbf{v}_E}{\mu}e_s\right)\frac{\partial y_s}{\partial p_j}+$$

$$\mu N\sum_{s=1}^m\left[\frac{\left(\frac{\mathbf{v}_E}{\mu}+\sum_{i=1}^n t_i\frac{\partial x_i}{\partial E}+\sum_{s=1}^m T_s\frac{\partial y_s}{\partial E}+t_w w\frac{\partial L}{\partial E}\right)-\frac{\mathbf{v}_E}{\mu}\left(1-N\sum_{s=1}^m e_s\frac{\partial y_s}{\partial E}\right)}{1-N\sum_{s=1}^m e_s\frac{\partial y_s}{\partial E}}\right]e_s\frac{\partial y_s}{\partial p_j},$$

$$=(\mu-\alpha)\,x_j+\mu\left[\sum_{i=1}^n t_i\frac{\partial x_i}{\partial p_j}+t_w w\frac{\partial L}{\partial p_j}+\sum_{s=1}^m\left(T_s+N\frac{\mathbf{v}_E}{\mu}e_s\right)\frac{\partial y_s}{\partial p_j}\right]+$$

$$\mu\sum_{s=1}^m\left[N\frac{\sum_{i=1}^n t_i\frac{\partial x_i}{\partial E}+\sum_{s=1}^m\left(T_s+N\frac{\mathbf{v}_E}{\mu}e_s\right)\frac{\partial y_s}{\partial E}+t_w w\frac{\partial L}{\partial E}}{1-N\sum_{s=1}^m e_s\frac{\partial y_s}{\partial E}}\right]e_s\frac{\partial y_s}{\partial p_j}. \tag{A14}$$

Substituting for the last expression on the right-hand side of (A14) in terms of Γ, as defined by (13), one gets

$$\frac{\partial \mathcal{L}}{\partial t_j} = (\mu - \alpha)\, x_j + \mu \left[\sum_{i=1}^{n} t_i \frac{\partial x_i}{\partial p_j} + t_w w \frac{\partial L}{\partial p_j} + \sum_{s=1}^{m} \left(T_s + N \frac{\mathbf{v}_E}{\mu} e_s \right) \frac{\partial y_s}{\partial p_j} \right] + \mu \sum_{s=1}^{m} \Gamma e_s \frac{\partial y_s}{\partial p_j}. \tag{A15}$$

Rearranging equation (A15) results in (14).

To prove (15), (16), and (17), one can use the same procedure as for the proof of (14). Thus substitute the expressions for $dE/d\tau_k, dE/dt_w$, and dE/dM from (A10)–(A12) into the first-order conditions (A2)–(A4) and simplify, using Roy's identity, and following the same steps.

Part (ii) Decompose the various partial derivatives of goods demands and labor supply function in equations (14)–(16) via the Slutsky equation. Rearranging the terms yields

$$(\mu - \alpha)\, x_j + \mu \left\{ \sum_{i=1}^{n} t_i \left(\frac{\partial \widetilde{x}_i}{\partial p_j} \right) + t_w w \left(\frac{\partial \widetilde{L}}{\partial p_j} \right) + \sum_{s=1}^{m} \left[T_s + \left(N \frac{\mathbf{v}_E}{\mu} + \Gamma \right) e_s \right] \left(\frac{\partial \widetilde{y}_s}{\partial p_j} \right) \right\} =$$

$$\mu x_j \left\{ \sum_{i=1}^{n} t_i \left(\frac{\partial x_i}{\partial M} \right) + t_w w \left(\frac{\partial L}{\partial M} \right) + \sum_{s=1}^{m} \left[T_s + \left(N \frac{\mathbf{v}_E}{\mu} + \Gamma \right) e_s \right] \left(\frac{\partial y_s}{\partial M} \right) \right\},$$

$$(\mu - \alpha)\, y_k + \mu \left\{ \sum_{i=1}^{n} t_i \left(\frac{\partial \widetilde{x}_i}{\partial q_k} \right) + t_w w \left(\frac{\partial \widetilde{L}}{\partial q_k} \right) + \sum_{s=1}^{m} \left[T_s + \left(N \frac{\mathbf{v}_E}{\mu} + \Gamma \right) e_s \right] \left(\frac{\partial \widetilde{y}_s}{\partial q_k} \right) \right\} =$$

$$\mu y_k \left\{ \sum_{i=1}^{n} t_i \left(\frac{\partial x_i}{\partial M} \right) + t_w w \left(\frac{\partial L}{\partial M} \right) + \sum_{s=1}^{m} \left[T_s + \left(N \frac{\mathbf{v}_E}{\mu} + \Gamma \right) e_s \right] \left(\frac{\partial y_s}{\partial M} \right) \right\},$$

$$(\mu - \alpha)\, (-L) + \mu \left\{ \sum_{i=1}^{n} t_i \left(\frac{\partial \widetilde{x}_i}{\partial w_n} \right) + t_w w \left(\frac{\partial \widetilde{L}}{\partial w_n} \right) + \sum_{s=1}^{m} \left[T_s + \left(N \frac{\mathbf{v}_E}{\mu} + \Gamma \right) e_s \right] \left(\frac{\partial \widetilde{y}_s}{\partial w_n} \right) \right\} =$$

$$\mu\, (-L) \left\{ \sum_{i=1}^{n} t_i \left(\frac{\partial x_i}{\partial M} \right) + t_w w \left(\frac{\partial L}{\partial M} \right) + \sum_{s=1}^{m} \left[T_s + \left(N \frac{\mathbf{v}_E}{\mu} + \Gamma \right) e_s \right] \left(\frac{\partial y_s}{\partial M} \right) \right\}.$$

Using (17) in above yields equations (18)–(20).

Proof of Lemma 2: To derive (25), substitute $(\theta e_s + \tau_s)$ for T_s in the Lagrangian expression (12) and differentiate it with respect to θ to get,

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \theta} =& \sum_{k=1}^{m} \frac{\partial \mathbf{v}}{\partial q_k} \frac{\partial q_k}{\partial \theta} + \mu \left[\sum_{i=1}^{n} t_i \sum_{k=1}^{m} \frac{\partial x_i}{\partial q_k} \frac{\partial q_k}{\partial \theta} + \sum_{s=1}^{m} \left(\theta e_s + \tau_s \right) \sum_{k=1}^{m} \frac{\partial y_s}{\partial q_k} \frac{\partial q_k}{\partial \theta} + t_w w \sum_{k=1}^{m} \frac{\partial L}{\partial q_k} \frac{\partial q_k}{\partial \theta} \right] \\
&+ \mu \sum_{s=1}^{m} \left(e_s + \theta \frac{\partial e_s}{\partial \theta} \right) y_s + \left\{ \mathbf{v}_E + \mu \left[\sum_{i=1}^{n} t_i \frac{\partial x_i}{\partial E} + \sum_{s=1}^{m} \left(\theta e_s + \tau_s \right) \frac{\partial y_s}{\partial E} + t_w w \frac{\partial L}{\partial E} \right] \right\} \frac{dE}{d\theta}.
\end{aligned} \tag{A16}
$$

Next, recall that

$$
q_k = C_k \left(e_k \right) + \theta e_k + \tau_k. \tag{A17}
$$

Differentiating this equation with respect to θ yields

$$
\frac{\partial q_k}{\partial \theta} = C_k' \left(e_k \right) \frac{\partial e_k}{\partial \theta} + e_k + \theta \frac{\partial e_k}{\partial \theta} = e_k. \tag{A18}
$$

Substitute from (A18) into (A16) to get

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \theta} =& \sum_{k=1}^{m} \frac{\partial \mathbf{v}}{\partial q_k} e_k + \mu \left[\sum_{i=1}^{n} t_i \sum_{k=1}^{m} \frac{\partial x_i}{\partial q_k} e_k + \sum_{s=1}^{m} \left(\theta e_s + \tau_s \right) \sum_{k=1}^{m} \frac{\partial y_s}{\partial q_k} e_k + t_w w \sum_{k=1}^{m} \frac{\partial L}{\partial q_k} e_k \right] \\
&+ \mu \sum_{s=1}^{m} \left(e_s + \theta \frac{\partial e_s}{\partial \theta} \right) y_s + \left\{ \mathbf{v}_E + \mu \left[\sum_{i=1}^{n} t_i \frac{\partial x_i}{\partial E} + \sum_{s=1}^{m} \left(\theta e_s + \tau_s \right) \frac{\partial y_s}{\partial E} + t_w w \frac{\partial L}{\partial E} \right] \right\} \frac{dE}{d\theta}.
\end{aligned} \tag{A19}
$$

To derive an expression for $dE/d\theta$, differentiating equation (2) with respect to θ and make use of (A18). This yields,

$$
\begin{aligned}
\frac{dE}{d\theta} &= N \sum_{s=1}^{m} \left[\frac{\partial e_s}{\partial \theta} y_s + e_s \left(\sum_{k=1}^{m} \frac{\partial y_s}{\partial q_k} \frac{\partial q_k}{\partial \theta} + \frac{\partial y_s}{\partial E} \frac{dE}{d\theta} \right) \right] \\
&= N \sum_{s=1}^{m} \left[\frac{\partial e_s}{\partial \theta} y_s + e_s \sum_{k=1}^{m} \frac{\partial y_s}{\partial q_k} e_k \right] + N \frac{dE}{d\theta} \sum_{s=1}^{m} e_s \frac{\partial y_s}{\partial E}.
\end{aligned}
$$

Solving for $dE/d\theta$ then results in,

$$
\frac{dE}{d\theta} = \frac{N \sum_{s=1}^{m} \left(\frac{\partial e_s}{\partial \theta} y_s + e_s \sum_{k=1}^{m} \frac{\partial y_s}{\partial q_k} e_k \right)}{1 - N \sum_{s=1}^{m} e_s \frac{\partial y_s}{\partial E}}. \tag{A20}
$$

Now substitute for $dE/d\theta$ from (A20) into (A19) to get

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \theta} =& \sum_{k=1}^{m} -\alpha y_k e_k + \mu \sum_{s=1}^{m} \left(e_s + \theta \frac{\partial e_s}{\partial \theta} \right) y_s + \\
& \mu \left[\sum_{i=1}^{n} t_i \sum_{k=1}^{m} \frac{\partial x_i}{\partial q_k} e_k + \sum_{s=1}^{m} (\theta e_s + \tau_s) \sum_{k=1}^{m} \frac{\partial y_s}{\partial q_k} e_k + t_w w \sum_{k=1}^{m} \frac{\partial L}{\partial q_k} e_k \right] + \\
& \left\{ \mathbf{v}_E + \mu \left[\sum_{i=1}^{n} t_i \frac{\partial x_i}{\partial E} + \sum_{s=1}^{m} (\theta e_s + \tau_s) \frac{\partial y_s}{\partial E} + t_w w \frac{\partial L}{\partial E} \right] \right\} \frac{N \sum_{s=1}^{m} \left(\frac{\partial e_s}{\partial \theta} y_s + e_s \sum_{k=1}^{m} \frac{\partial y_s}{\partial q_k} e_k \right)}{1 - N \sum_{s=1}^{m} e_s \frac{\partial y_s}{\partial E}} \\
=& \sum_{k=1}^{m} -\alpha y_k e_k + \mu \sum_{s=1}^{m} \left(e_s + \theta \frac{\partial e_s}{\partial \theta} \right) y_s + \\
& \mu \left[\sum_{i=1}^{n} t_i \sum_{k=1}^{m} \frac{\partial x_i}{\partial q_k} e_k + t_w w \sum_{k=1}^{m} \frac{\partial L}{\partial q_k} e_k \right] + \mu \sum_{s=1}^{m} (\theta e_s + \tau_s) \sum_{k=1}^{m} \frac{\partial y_s}{\partial q_k} e_k + \\
& \left\{ \frac{N \left[\mathbf{v}_E + \mu \left(\sum_{i=1}^{n} t_i \frac{\partial x_i}{\partial E} + \sum_{s=1}^{m} (\theta e_s + \tau_s) \frac{\partial y_s}{\partial E} + t_w w \frac{\partial L}{\partial E} \right) \right]}{1 - N \sum_{s=1}^{m} e_s \frac{\partial y_s}{\partial E}} \right\} \sum_{s=1}^{m} \left(\frac{\partial e_s}{\partial \theta} y_s + e_s \sum_{k=1}^{m} \frac{\partial y_s}{\partial q_k} e_k \right) \\
=& \sum_{k=1}^{m} -\alpha y_k e_k + \mu \sum_{s=1}^{m} e_s y_s + \mu \left[\sum_{i=1}^{n} t_i \sum_{k=1}^{m} \frac{\partial x_i}{\partial q_k} e_k + t_w w \sum_{k=1}^{m} \frac{\partial L}{\partial q_k} e_k \right] + \\
& \mu \theta \sum_{s=1}^{m} \left(e_s \sum_{k=1}^{m} \frac{\partial y_s}{\partial q_k} e_k + \frac{\partial e_s}{\partial \theta} y_s \right) + \mu \sum_{s=1}^{m} \tau_s \sum_{k=1}^{m} \frac{\partial y_s}{\partial q_k} e_k + \\
& \left\{ \frac{N \left[\mathbf{v}_E + \mu \left(\sum_{i=1}^{n} t_i \frac{\partial x_i}{\partial E} + \sum_{s=1}^{m} (\theta e_s + \tau_s) \frac{\partial y_s}{\partial E} + t_w w \frac{\partial L}{\partial E} \right) \right]}{1 - N \sum_{s=1}^{m} e_s \frac{\partial y_s}{\partial E}} \right\} \sum_{s=1}^{m} \left(\frac{\partial e_s}{\partial \theta} y_s + e_s \sum_{k=1}^{m} \frac{\partial y_s}{\partial q_k} e_k \right) \\
=& (\mu - \alpha) \sum_{s=1}^{m} e_s y_s + \mu \left[\sum_{i=1}^{n} t_i \sum_{k=1}^{m} \frac{\partial x_i}{\partial q_k} e_k + t_w w \sum_{k=1}^{m} \frac{\partial L}{\partial q_k} e_k \right] + \mu \sum_{s=1}^{m} \tau_s \sum_{k=1}^{m} \frac{\partial y_s}{\partial q_k} e_k + \\
& \left\{ \frac{N \left[\mathbf{v}_E + \mu \left(\sum_{i=1}^{n} t_i \frac{\partial x_i}{\partial E} + \sum_{s=1}^{m} (\theta e_s + \tau_s) \frac{\partial y_s}{\partial E} + t_w w \frac{\partial L}{\partial E} \right) \right]}{1 - N \sum_{s=1}^{m} e_s \frac{\partial y_s}{\partial E}} + \mu \theta \right\} \times \\
& \sum_{s=1}^{m} \left(\frac{\partial e_s}{\partial \theta} y_s + e_s \sum_{k=1}^{m} \frac{\partial y_s}{\partial q_k} e_k \right)
\end{aligned}$$

$$
= (\mu - \alpha) \sum_{s=1}^{m} e_s y_s + \mu \left[\sum_{i=1}^{n} t_i \sum_{k=1}^{m} \frac{\partial x_i}{\partial q_k} e_k + t_w w \sum_{k=1}^{m} \frac{\partial L}{\partial q_k} e_k \right] + \mu \sum_{s=1}^{m} \tau_s \sum_{k=1}^{m} \frac{\partial y_s}{\partial q_k} e_k +
$$

$$
\left\{ \frac{N \left[\mathbf{v}_E + \mu \left(\sum_{i=1}^{n} t_i \frac{\partial x_i}{\partial E} + \sum_{s=1}^{m} \tau_s \frac{\partial y_s}{\partial E} + t_w w \frac{\partial L}{\partial E} \right) \right] + \mu\theta}{1 - N \sum_{s=1}^{m} e_s \frac{\partial y_s}{\partial E}} \right\} \sum_{s=1}^{m} \left(\frac{\partial e_s}{\partial \theta} y_s + e_s \sum_{k=1}^{m} \frac{\partial y_s}{\partial q_k} e_k \right) \tag{A21}
$$

But the bracketed expression on the right-hand side can be simplified as

$$
\begin{aligned}
&\frac{N \left[\mathbf{v}_E + \mu \left(\sum_{i=1}^{n} t_i \frac{\partial x_i}{\partial E} + \sum_{s=1}^{m} \tau_s \frac{\partial y_s}{\partial E} + t_w w \frac{\partial L}{\partial E} \right) \right] + \mu\theta}{1 - N \sum_{s=1}^{m} e_s \frac{\partial y_s}{\partial E}} = \\
&\mu \frac{N \left[\sum_{i=1}^{n} t_i \frac{\partial x_i}{\partial E} + \sum_{s=1}^{m} \left[\tau_s + \left(\theta + N \frac{\mathbf{v}_E}{\mu} \right) e_s \right] \frac{\partial y_s}{\partial E} + t_w w \frac{\partial L}{\partial E} \right]}{1 - N \sum_{s=1}^{m} e_s \frac{\partial y_s}{\partial E}} + \\
&\mu \frac{-N \left(\theta + N \frac{\mathbf{v}_E}{\mu} \right) \sum_{s=1}^{m} e_s \frac{\partial y_s}{\partial E} + \theta + N \frac{\mathbf{v}_E}{\mu}}{1 - N \sum_{s=1}^{m} e_s \frac{\partial y_s}{\partial E}} = \\
&\mu\Gamma + \mu \frac{-N \left(\theta + N \frac{\mathbf{v}_E}{\mu} \right) \sum_{s=1}^{m} e_s \frac{\partial y_s}{\partial E} + \left(\theta + N \frac{\mathbf{v}_E}{\mu} \right)}{1 - N \sum_{s=1}^{m} e_s \frac{\partial y_s}{\partial E}} = \\
&\mu\Gamma + \mu \left(\theta + N \frac{\mathbf{v}_E}{\mu} \right) \frac{-N \sum_{s=1}^{m} e_s \frac{\partial y_s}{\partial E} + 1}{1 - N \sum_{s=1}^{m} e_s \frac{\partial y_s}{\partial E}} = \\
&\mu \left(\theta + N \frac{\mathbf{v}_E}{\mu} + \Gamma \right).
\end{aligned}
$$

Substituting in (A21) yields,

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \theta} =& (\mu - \alpha) \sum_{s=1}^{m} e_s y_s + \mu \left[\sum_{i=1}^{n} t_i \sum_{k=1}^{m} \frac{\partial x_i}{\partial q_k} e_k + t_w w \sum_{k=1}^{m} \frac{\partial L}{\partial q_k} e_k \right] + \mu \sum_{s=1}^{m} \tau_s \sum_{k=1}^{m} \frac{\partial y_s}{\partial q_k} e_k + \\
& \mu \left[\theta + N \frac{\mathbf{v}_E}{\mu} + \Gamma \right] \sum_{s=1}^{m} \left(\frac{\partial e_s}{\partial \theta} y_s + e_s \sum_{k=1}^{m} \frac{\partial y_s}{\partial q_k} e_k \right) \\
=& (\mu - \alpha) \sum_{s=1}^{m} e_s y_s + \mu \left[\sum_{i=1}^{n} t_i \sum_{k=1}^{m} \frac{\partial x_i}{\partial q_k} e_k + t_w w \sum_{k=1}^{m} \frac{\partial L}{\partial q_k} e_k \right] + \\
& \mu \sum_{s=1}^{m} \sum_{k=1}^{m} \left\{ \left[\theta + N \frac{\mathbf{v}_E}{\mu} + \Gamma \right] e_s + \tau_s \right\} \frac{\partial y_s}{\partial q_k} e_k + \mu \left[\theta + N \frac{\mathbf{v}_E}{\mu} + \Gamma \right] \sum_{s=1}^{m} \frac{\partial e_s}{\partial \theta} y_s
\end{aligned}
$$

$$= (\mu - \alpha) \sum_{s=1}^{m} e_s y_s + \mu \left(\theta + N \frac{\mathbf{v}_E}{\mu} + \Gamma \right) \sum_{s=1}^{m} \frac{\partial e_s}{\partial \theta} y_s +$$

$$\mu \left\{ \sum_{i=1}^{n} t_i \sum_{k=1}^{m} \frac{\partial x_i}{\partial q_k} e_k + t_w w \sum_{k=1}^{m} \frac{\partial L}{\partial q_k} e_k + \sum_{s=1}^{m} \sum_{k=1}^{m} \left[\left(\theta + N \frac{\mathbf{v}_E}{\mu} + \Gamma \right) e_s + \tau_s \right] \frac{\partial y_s}{\partial q_k} e_k \right\}$$

$$= (\mu - \alpha) \sum_{s=1}^{m} e_s y_s + \mu \left(\theta + N \frac{\mathbf{v}_E}{\mu} + \Gamma \right) \sum_{s=1}^{m} \frac{\partial e_s}{\partial \theta} y_s +$$

$$\sum_{k=1}^{m} e_k \mu \left\{ \sum_{i=1}^{n} t_i \frac{\partial x_i}{\partial q_k} + t_w w \frac{\partial L}{\partial q_k} + \sum_{s=1}^{m} \left[\tau_s + \left(\theta + N \frac{\mathbf{v}_E}{\mu} + \Gamma \right) e_s \right] \frac{\partial y_s}{\partial q_k} \right\}. \quad \text{(A22)}$$

Finally, recall from the first-order condition with respect τ_k that

$$\mu \left\{ \sum_{i=1}^{n} t_i \frac{\partial x_i}{\partial q_k} + t_w w \frac{\partial L}{\partial q_k} + \sum_{s=1}^{m} \left[\tau_s + \left(\theta + N \frac{\mathbf{v}_E}{\mu} + \Gamma \right) e_s \right] \frac{\partial y_s}{\partial q_k} \right\} = -(\mu - \alpha) y_k$$

Substituting this expression in (A22) then simplifies it to,

$$\frac{\partial \mathcal{L}}{\partial \theta} = -(\mu - \alpha) \sum_{k=1}^{m} e_k y_k + (\mu - \alpha) \sum_{s=1}^{m} e_s y_s + \mu \left(\theta + N \frac{\mathbf{v}_E}{\mu} + \Gamma \right) \sum_{s=1}^{m} \frac{\partial e_s}{\partial \theta} y_s,$$

which simplifies to the expression given in (25).

Proof of (38): Differentiate equations (36)–(37) with respect to M, p, and q to get, when M and t_w are set equal to zero,

$$\begin{aligned}
\frac{\partial x}{\partial M} &= \frac{p^{-\rho}}{1 + p^{1-\rho} + q^{1-\rho}}, \\
\frac{\partial x}{\partial p} &= -p^{-1-\rho} \frac{\rho \left(1 + q^{1-\rho}\right) + p^{1-\rho}}{\left(1 + p^{1-\rho} + q^{1-\rho}\right)^2}, \\
\frac{\partial x}{\partial q} &= -(1 - \rho) \frac{p^{-\rho} q^{-\rho}}{\left(1 + p^{1-\rho} + q^{1-\rho}\right)^2}, \\
\frac{\partial y}{\partial M} &= \frac{q^{-\rho}}{1 + p^{1-\rho} + q^{1-\rho}}, \\
\frac{\partial y}{\partial p} &= -(1 - \rho) \frac{p^{-\rho} q^{-\rho}}{\left(1 + p^{1-\rho} + q^{1-\rho}\right)^2}, \\
\frac{\partial y}{\partial q} &= -q^{-1-\rho} \frac{\rho \left(1 + p^{1-\rho}\right) + q^{1-\rho}}{\left(1 + p^{1-\rho} + q^{1-\rho}\right)^2}.
\end{aligned}$$

Based on these equations and using the Slutsky equation, derive the derivatives of the compensated demand functions $\widetilde{x}$ and $\widetilde{y}$ with respect to prices p and q. Consequently, one can write the Slutsky matrix as

$$\begin{aligned}
\widetilde{\Delta}_w &= \begin{pmatrix} \frac{\partial \widetilde{x}}{\partial p} & \frac{\partial \widetilde{x}}{\partial q} \\ \frac{\partial \widetilde{y}}{\partial p} & \frac{\partial \widetilde{y}}{\partial q} \end{pmatrix} \\
&= \begin{pmatrix} -p^{-1-\rho} \frac{\rho\left(1+q^{1-\rho}\right)+2p^{1-\rho}}{(1+p^{1-\rho}+q^{1-\rho})^2} & (\rho-2)\frac{p^{-\rho}q^{-\rho}}{(1+p^{1-\rho}+q^{1-\rho})^2} \\ (\rho-2)\frac{p^{-\rho}q^{-\rho}}{(1+p^{1-\rho}+q^{1-\rho})^2} & -q^{-1-\rho}\frac{\rho\left(1+p^{1-\rho}\right)+2q^{1-\rho}}{(1+p^{1-\rho}+q^{1-\rho})^2} \end{pmatrix}.
\end{aligned}$$

Simplifying yields

$$\begin{aligned}
\widetilde{\Delta}_w^{-1} &= \frac{1+p^{1-\rho}+q^{1-\rho}}{\rho\left[\rho+2\left(p^{1-\rho}+q^{1-\rho}\right)\right](pq)^{-1-\rho}} \times \\
&\quad \begin{pmatrix} -q^{-1-\rho}\left[\rho\left(1+p^{1-\rho}\right)+2q^{1-\rho}\right] & -(\rho-2)\,p^{-\rho}q^{-\rho} \\ -(\rho-2)\,p^{-\rho}q^{-\rho} & -p^{-1-\rho}\left[\rho\left(1+q^{1-\rho}\right)+2p^{1-\rho}\right] \end{pmatrix}.
\end{aligned}$$

Now setting $w_n = 1$ and $M = 0$ in equations (36)–(37) results in,

$$\begin{aligned}
x &= \frac{p^{-\rho}}{1+p^{1-\rho}+q^{1-\rho}}, \\
y &= \frac{q^{-\rho}}{1+p^{1-\rho}+q^{1-\rho}}.
\end{aligned}$$

It follows from the above expressions for $\widetilde{\Delta}_w^{-1}, x, y, \partial x/\partial M$, and $\partial y/\partial M$ that, after a bit of algebraic manipulation,

$$\begin{aligned}
\widetilde{\Delta}_w^{-1}\begin{pmatrix} x \\ y \end{pmatrix} &= \frac{-\left(1+p^{1-\rho}+q^{1-\rho}\right)}{\rho+2\left(p^{1-\rho}+q^{1-\rho}\right)}\begin{pmatrix} p \\ q \end{pmatrix}, \\
\Psi_w &= t\left(\frac{\partial x}{\partial M}\right)+\left[\tau+\left(N\frac{\mathbf{v}_E}{\mu}\right)e\right]\left(\frac{\partial y}{\partial M}\right) \\
&= \frac{tp^{-\rho}+\left(\tau+N\frac{\mathbf{v}_E}{\mu}\right)q^{-\rho}}{1+p^{1-\rho}+q^{1-\rho}}.
\end{aligned}$$

where e is set equal to one. Substituting these values in formula (32) leads to (38).

Proof of (44)– (45): With the introduction of elasticities, one can rewrite the equations for t_i and τ_s as

$$\begin{aligned}
\frac{t_i}{p_i} &= \left(1-\frac{\alpha}{\mu}\right)\frac{1}{\varepsilon_i^x}, \quad i=1,2,\ldots,n,\\
\frac{\tau_s}{q_s} &= \left(1-\frac{\alpha}{\mu}\right)\frac{1}{\varepsilon_s^y}+N\frac{-\mathbf{v}_E}{\mu}\frac{e_s}{q_s}, \quad s=1,2,\ldots,m.
\end{aligned}$$

Substituting $1+t_i$ for p_i and $1+\tau_s$ for q_s and rearranging the terms,

$$\begin{aligned}
\frac{1}{1+t_i} &= 1-\left(1-\frac{\alpha}{\mu}\right)\frac{1}{\varepsilon_i^x}, \quad i=1,2,\ldots,n,\\
\frac{1}{1+\tau_s} &= \frac{1-\left(1-\frac{\alpha}{\mu}\right)\frac{1}{\varepsilon_s^y}}{1+N\frac{-\mathbf{v}_E}{\mu}e_s}, \quad s=1,2,\ldots,m.
\end{aligned}$$

Or

$$\begin{aligned}
t_i &= \frac{\left(1-\frac{\alpha}{\mu}\right)\frac{1}{\varepsilon_i^x}}{1-\left(1-\frac{\alpha}{\mu}\right)\frac{1}{\varepsilon_i^x}}, \quad i=1,2,\ldots,n,\\
\tau_k &= \frac{\left(1-\frac{\alpha}{\mu}\right)\frac{1}{\varepsilon_s^y}+N\frac{-\mathbf{v}_E}{\mu}e_s}{1-\left(1-\frac{\alpha}{\mu}\right)\frac{1}{\varepsilon_s^y}}, \quad s=1,2,\ldots,m.
\end{aligned}$$

Equations (44)– (45) follow immediately from these.

Proof of (49)– (51): The equations for t_i, τ_s and t_w in terms of elasticities are

$$\begin{aligned}
\frac{t_i}{p_i} &= \left(1-\frac{\alpha}{\mu}\right)\frac{1}{\varepsilon_i^x}, \quad i=2,3,\ldots,n,\\
\frac{\tau_s}{q_s} &= \left(1-\frac{\alpha}{\mu}\right)\frac{1}{\varepsilon_s^y}-N\frac{\mathbf{v}_E}{\mu}\frac{e_s}{q_s}, \quad s=1,2,\ldots,m,\\
\frac{t_w}{1-t_w} &= \left(1-\frac{\alpha}{\mu}\right)\frac{1}{\varepsilon^L}.
\end{aligned}$$

Substituting $1+t_i$ for p_i and $1+\tau_s$ for q_s and rearranging the terms

$$\begin{aligned}
\frac{1}{1+t_i} &= 1-\left(1-\frac{\alpha}{\mu}\right)\frac{1}{\varepsilon_i^x}, \quad i=2,3,\ldots,n,\\
\frac{1}{1+\tau_s} &= \frac{1-\left(1-\frac{\alpha}{\mu}\right)\frac{1}{\varepsilon_s^y}}{1+N\frac{-\mathbf{v}_E}{\mu}e_s}, \quad s=1,2,\ldots,m,\\
\frac{1}{1-t_w} &= 1+\left(1-\frac{\alpha}{\mu}\right)\frac{1}{\varepsilon^L}.
\end{aligned}$$

Or

$$
\begin{aligned}
t_i &= \frac{\left(1-\frac{\alpha}{\mu}\right)\frac{1}{\varepsilon_i^x}}{1-\left(1-\frac{\alpha}{\mu}\right)\frac{1}{\varepsilon_i^x}}, \quad i=2,3,\ldots,n, \\
\tau_s &= \frac{\left(1-\frac{\alpha}{\mu}\right)\frac{1}{\varepsilon_s^y}+N\frac{-\mathbf{v}_E}{\mu}e_s}{1-\left(1-\frac{\alpha}{\mu}\right)\frac{1}{\varepsilon_s^y}}, \quad s=1,2,\ldots,m, \\
t_w &= \frac{\left(1-\frac{\alpha}{\mu}\right)\frac{1}{\varepsilon^L}}{1+\left(1-\frac{\alpha}{\mu}\right)\frac{1}{\varepsilon^L}}.
\end{aligned}
$$

Equations (49)–(51) follow immediately from these.

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