Calibration

I. Introduction

The main purpose of this chapter is to demonstrate the calibration procedure that will serve as the main quantitative tool used in this book. We do so using a very important model in macroeconomics, the *Solow Growth Model*, named after Robert Solow who introduced it in a1956 paper published in the *Quarterly Journal of Economics*. The Solow Growth model is not a truly modern macroeconomic model on account that it assumes a certain behavior on the part of consumers instead of deriving it from utility maximization. Despite this shortcoming, the Solow Growth Model is a useful model to study for the purpose of gaining economic intuition on a number of issues. We start with the description of the Solow Growth model and a study of its equilibrium properties. This is followed by a calibration exercise using the Solow model.

II. The Solow Model

People/Households

Initially, there are N_0 people alive in our model world. We use N_t to denote the number of people in the economy at date t. Typically, we assume that people have a utility function which they try to maximize subject to a budget constraint. In the Solow growth model, this has all been assumed away. The assumption is that people prefer more consumption to less and save a constant fraction *s* of their income. Let c_t denote consumption per person or capita and let Y_t denote total household income. Then

(C) $$N_t c_t = (1-s)Y_t$$

This just says that total consumption is fraction (1-s) of total household income. In this model, the savings rate is treated as a parameter.

Demographics: For the Solow model, we assume that population growth is exogenously determined with a constant rate of increase equal to $\gamma_N \geq 0$. More specifically,

(N) $$N_{t+1} = (1+\gamma_N)N_t$$

Endowments: Each person in the economy is endowed with one unit of time each period which he or she uses to work. In future models, we will allow households to split their time between leisure and work, but in the growth model, things are simplified by abstracting it from the analysis. Additionally, people are endowed with some capital initially. In a narrow sense, capital is defined as the stock of machines and structures in the economy. In a more general sense, capital is something of value which is used to produce a good and service and is not completely used up in the process. The aggregate endowment of capital in the economy is denoted by K_0.

Production Function: The economy produces a single final good using labor and capital. The production function is given by

(Y) $$Y_t = AK_t^{\theta}[(1+\gamma_A)^t N]^{1-\theta}$$

The parameter γ_A is the rate of exogenous technological change. We shall examine both the case where γ_A is strictly positive and the case where its value is zero. The latter case is particularly useful for gaining intuition for the model's properties. It also allows for a graphical depiction of the equilibrium. The letter A is a parameter that reflects the efficiency at which resources are combined to produce output. We call this parameter *Total Factor Productivity (TFP).* Technically, Total Factor Productivity is the entire combined term $A(1+\gamma_A)^{t(1-\theta)}$.

The production function is subject to the law of diminishing returns, as discussed in Chapter 2. It also is subject to constant returns to scale. This is a desirable property because as you will show, it implies zero profits as long as firms are price takers.

Capital Stock – The capital stock evolves according to the following equation:

(K) $$K_{t+1} = (1-\delta)K_t + sY_t$$

In the above equation, δ is the depreciation rate parameter. Namely, δ represents the fraction of machines and structures that wear out between periods. Total savings, sY_t, equals the economy's investment, namely, the purchases of new machines, equipment and structures.

Summary of Solow Growth Model: Aggregate Variables

The Solow Growth Model is completely described by the following four equations.

(C) $N_t c_t = (1-s)Y_t$

(Y) $Y_t = AK_t^{\theta}[(1+\gamma_A)^t N]^{1-\theta}$

(K) $K_{t+1} = (1-\delta)K_t + sY_t$

(N) $N_{t+1} = (1+\gamma_N)N_t$

Solow Model – the Per Capita Variable Representation

For many purposes, it is more interesting and relevant to study the behavior of per capita variables. This is because a change in the per capita variable is more informative of a change in peoples' welfare. For example, we could have an increase in total output with no change in per capita output on account of population growth. Just looking at the increase in total output would not tell you if people are better off or worse off over time.

It is not difficult to transform the Solow model into its per capita representation. We begin by defining $y \equiv Y/N$, $k \equiv K/N$. These are the per capita output and capital. The convention in this book is to denote aggregate variables by upper case letters and per capita variables by lower case letters.

We begin with the consumption equation (C). The per capita representation is trivially found by dividing both sides of the equation by N_t. Using the definition of y, we arrive at

(c) $$c_t = (1-s)y_t$$

We next transform Equation (Y) into its per capita representation. This we do by dividing the aggregate production function by N_t. Exploiting the mathematical properties of exponents, (i.e., $x^{a+b} = x^a x^b$ and $x^a/z^a = (x/z)^a$) allows us to write the per capita production function as

(y) $$y_t = A(1+\gamma_A)^{t(1-\theta)} k_t^{\theta}$$

It is easy to show that the per capita production function has the same shape as the aggregate production function, namely increasing and concave.

The law of motion for the per capita capital stock equation is:

(k) $$(1+n)k_{t+1} = (1-\delta)k_t + sy_t$$

This is obtained by dividing both sides of the equation for the aggregate capital stock given by Equation (K) by N_t . This yields

$$\frac{K_{t+1}}{N_t} = (1-\delta)k_t + sy_t \tag{1}$$

The only trick in getting to (k) is to express the left hand side of the above equation in terms of k_{t+1}. To do this we use the population growth function which can be rewritten as $N_t = (1+\gamma_N)^{-1} N_{t+1}$. Substituting for N_t into the above equation yields

$$(1+\gamma_N)\frac{K_{t+1}}{N_{t+1}} = (1-\delta)k_t + sy_t \tag{2}$$

As K_{t+1}/N_{t+1} is just k_{t+1}, we now arrive at Equation (k). This completes the transformation.

Summary of Solow Growth Model: Per capita variables

The Solow Growth Model is completely described by the following three equations.

(c) $c_t = (1-s)y_t$

(y) $y_t = A(1+\gamma_A)^{(1-\theta)t} k_t^{\theta}$

(k) $(1+\gamma_N)k_{t+1} = (1-\delta)k_t + sy_t$

Because we used the population growth function to derive (k), it is left out from the system as it is reflected in (k).

General Equilibrium

Modern macroeconomics is based on general equilibrium analysis, namely, that all markets must clear simultaneously. There are really three markets in the model: the labor market, the capital rental market, and the goods market.

There is no money in the growth model. As such there are no nominal prices to be determined. The prices in this economy are real prices. Namely, for each good that is traded, its price is expressed in terms of another good in the economy. In this model we will measure the price of each good in the economy in terms of the final good, Y. The payment to labor, denoted by w_t, therefore, is the quantity of the final good paid to a unit of labor. The payment to capital, denoted by r_t, is the quantity of the final good paid to a unit of capital rented by a firm from households. The price of the final good is one. This is trivially since we have chosen to express the prices of all goods in terms of the final good.

In the Solow model, it is trivial to find the market clearing quantities of labor and capital. This is because the supply of labor and the supply of capital are perfectly, inelastic, i.e. vertical. As such, demand is irrelevant for determining equilibrium quantities; labor demand and capital services demand only pin down the equilibrium prices.

Derivation of Labor Demand and Capital Services Demand.

With vertical supplies for capital and labor, equilibrium quantities are trivially determined. However, to determine equilibrium prices we need to determine demand for labor and capital by firms. For this purpose, we study the profit maximization problem of firms.

Profit Maximization of Firms

Profits of the firm are defined as sales less wage payments less capital service payments. More specifically,

(Profits) $$AK_t^{\theta}[(1+\gamma)^t N_t]^{1-\theta} - w_t N_t - r_t K_t$$

Standard microeconomic theory states that profits are maximized at the point where the marginal product of the input equals its marginal cost. The marginal product of labor or capital is just the derivative of the production function with respect to that variable. The marginal cost is the wage rate in the case of labor and the rental rate in the case of capital. Thus, the profit maximizing conditions are:

(LD) $$w_t = (1-\theta)A(1+\gamma)^{t(1-\theta)} K_t^{\theta} N_t^{-\theta} = (1-\theta)\frac{Y_t}{N_t}$$

(KD) $$r_t = \theta A(1+\gamma)^{t(1-\theta)} K_t^{\theta-1} N_t^{1-\theta} = \theta\frac{Y_t}{K_t}$$

To plot the demand for capital, we would hold N_t fixed and trace out the (KD) equation. A similar procedure for plotting the demand for labor would be used, although here we would keep K_t fixed and map out (LD). Were you to do this, you would find that the demand for both inputs would be decreasing and in the limit converge to zero. This is just the law of diminishing returns. This is because for each price we consider, the quantity demanded is the marginal product.

Balanced Growth Path

As long as there is exogenous technological change, the model has a balanced growth path equilibrium rather than a steady state. We begin by showing that there is now positive growth rate in the long-run. Again, we start with the law of motion for the per capita capital stock,

$$(1+\gamma_N)k_{t+1} = (1-\delta)k_t + sy_t \tag{3}$$

and divide both sides by k_t. This yields

$$(1+\gamma_N)\frac{k_{t+1}}{k_t} = (1-\delta) + s\frac{y_t}{k_t}. \tag{4}$$

Now in a steady-state or balanced growth path, $k_{t+1}/k_t = 1+g_k$ where $g_k \geq 0$. We now invoke this condition and substitute this condition for k_{t+1}/k_t in the above equation. This yields

$$(1+\gamma_N)(1+g_k) - (1-\delta) = s\frac{y_t}{k_t} \tag{5}$$

Again, since in a steady state or balanced growth path, the right hand side of this equation is constant, it follows that y and k grow at the same rate, g.

We now use the per capita production function (y) with result that $y_{t+1}/y_t = k_{t+1}/k_t = (1+g)$ to solve for g. The per capita production function is

$$y_t = A(1+\gamma_A)^{t(1-\theta)}k_t^{\theta}.$$

This relation holds for all periods, including t+1 so that

$$y_{t+1} = A(1+\gamma_A)^{(t+1)(1-\theta)} k_{t+1}^{\theta}.$$

Taking the ratio between date t+1 and date t output yields

$$\frac{y_{t+1}}{y_t} = \frac{A(1+\gamma_A)^{(t+1)(1-\theta)} k_{t+1}^{\theta}}{A(1+\gamma_A)^{t(1-\theta)} k_t^{\theta}} = (1+\gamma_A)^{(1-\theta)} \left(\frac{k_{t+1}}{k_t}\right)^{\theta}. \tag{6}$$

As the growth rate of output and capital per person are the same, it follows that

$$1+g = (1+\gamma_A)^{1-\theta}(1+g)^{\theta}. \tag{7}$$

Solving for 1+*g* yields

(BGPR) $$1+g = 1+\gamma_A.$$

Using the profit maximizing conditions that the real wage rate equals the marginal product of labor and the rental price of capital equals the marginal product of capital, we we can solve for the growth rate of the wage and the rental price of capital along the balanced growth path. In particular, since

$$w_t = (1-\theta)\frac{Y_t}{N_t} = (1-\theta)y_t \ ,$$

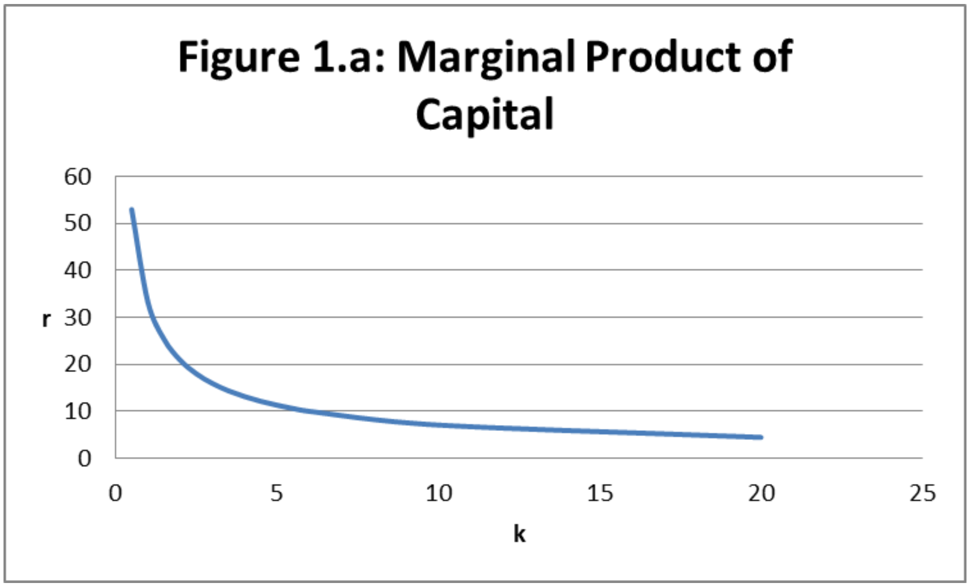
it immediately follows that the real wage grows at the same rate as output. The rental price of capital, in contrast, does not grow. Since y_t and k_t grow at the same rate, the ratio of y to k is constant, and so r must be constant since

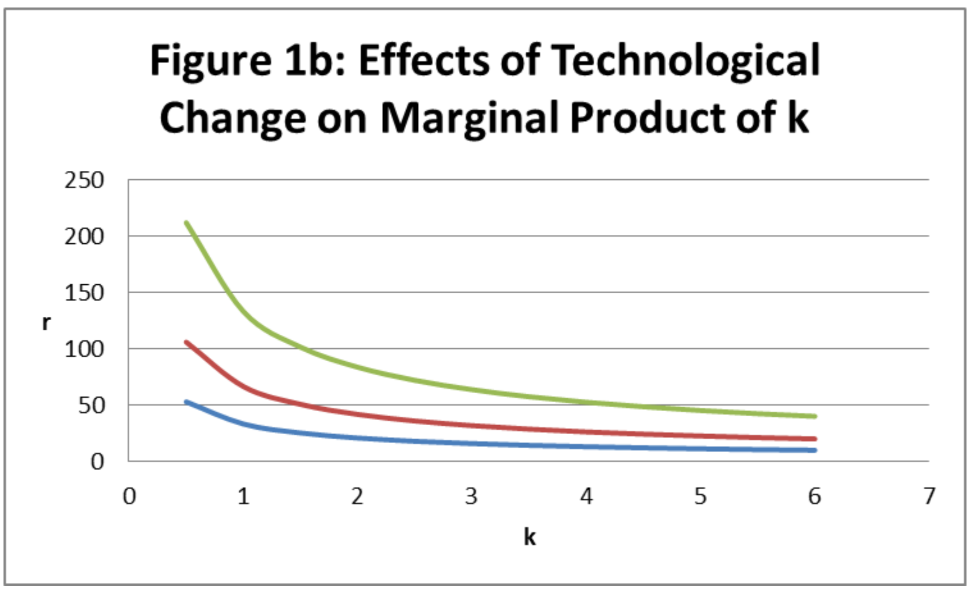
$$r_t = \theta\frac{Y_t}{K_t} = \theta\frac{y_t}{k_t} \ .$$

Notice that if $\gamma_A = 0$ the only solution to Equation (BGPR) is g=0. Thus, when there is no exogenous technological change, we have a steady state where none of the per capita variables change.

The intuition for why there must be exogenous technological change for there to be sustained growth is that technological change effectively negates the law of diminishing returns. In the Solow growth model, the marginal product of capital is shown in Figure 1.a. It decreases and in the limit goes to zero. In the absence of technological change, increasing the capital stock per person is the only way that an increase in per capita output can be achieved. However, adding capital is a movement down the marginal product curve. This explains why you cannot have sustained output per person in this case; each unit of capital you add generates a smaller and smaller increase in output.

As technology increases, the marginal product of capital increases. This is shown as a shift outward in the marginal product curve in Figure 1b. Specifically, in the Solow growth model the marginal product curve of capital shifts out every period by the increase $(1+\gamma_A)$. A firm can continue to add capital over time without experiencing any decline in the marginal product of capital over time. Technological change effectively offsets the law of diminishing returns.





Balanced Growth Paths of Per capita variables

Having solved for the economy's rate of growth, we now solve for the levels of the equilibrium prices and quantities along the balanced growth path. We start by solving for the per capita capital stock. We begin by using the equation for the per capita capital stock and the result that in the balanced growth path $k_{t+1}^{bg} = (1+\gamma_A)k_t^{bg}$. In the algebra that follows we shall drop the superscript from the notation as it is understood that all variables are their balanced growth path counterparts. Using this result, we can rewrite the per capita capital stock equation (k) along the balanced growth path as

$$(1+\gamma_N)(1+\gamma_A)k_t = (1-\delta)k_t + sA(1+\gamma_A)^{t(1-\theta)}k_t^{\theta}. \tag{8}$$

With some algebra, the above equation becomes

$$[(1+\gamma_N)(1+\gamma_A) - (1-\delta)] = sA(1+\gamma_A)^{t(1-\theta)}k_t^{\theta-1} \tag{9}$$

We can now solve for k_t along the balanced growth path. This is

(k-BGP) $$k_t^{bg} = (1+\gamma_A)^t\left[\frac{sA}{(1+\gamma_N)(1+\gamma_A)-(1-\delta)}\right]^{1/(1-\theta)}.$$

It is easy to see from (k-BGP0 how the per capita capital stock depends on the parameters of the model. For example, a country with a higher TFP, a higher savings rate, or a lower population growth rate will have a higher balanced growth path capital stock. Although a lower depreciation rate will have the same effect on the balanced growth path capital stock, this is a factor that economists do not emphasize as an important difference across countries.

For the sake of better understanding the mechanics behind the Solow model it is useful to shut down exogenous technological change. When $\gamma_A = 0$, the steady state capital stock can be shown graphically. Rearranging terms in Equation (9) yields,

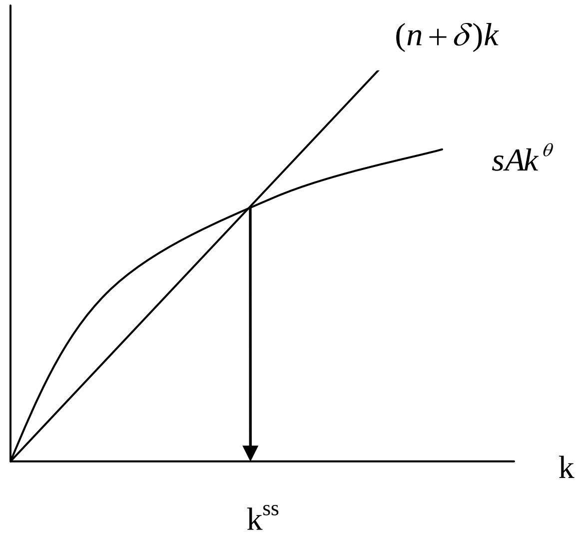
$$(\gamma_N + \delta)k = sAk^{\theta}. \tag{10}$$

The steady state capital stock must satisfy this equation. The left hand side of this equation represents the amount of savings that must be done to keep the per capita capital stock fixed at some given level k. Intuitively, to keep the per capita capital stock at k, we must replace the amount that wears out, δk and in addition give each new-born of which there is number γ_N , k units of capital. The right hand side is the actual savings done in the economy. The steady state capital stock is, thus, the capital stock for which actual savings equals the amount needed to keep the per capita capital stock from changing.

The solution can be shown graphically by plotting the actual savings curve and the needed savings curve. The needed savings curve is just a straight line through the origin with slope $(\gamma_N+\delta)$. The actual savings curve mimics the shape of the per capita production function, namely, increasing and concave. Because the savings rate, s, is less than one, the actual savings curve lies everywhere below the per capita production function.

The steady state capital stock is shown graphically as the intersection of the two curves. With the graph, one can easily compare steady state capital stocks associated with different parameters. For example, a higher population growth rate or depreciation rate shift the needed savings curve to the left, so that the new intersection point with the actual savings curve will lie to the left of the original intersection point.

Figure 2: Determination of Steady State Capital Stock

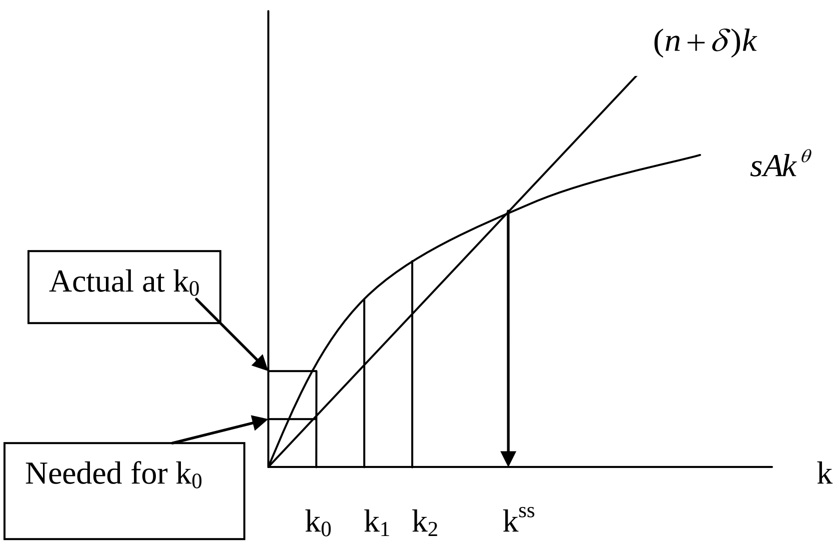


Transitional Dynamics

What if the economy failed to start with the balanced growth path quantity of capital? A natural question is whether it would converge to the balanced growth path levels? The answer is yes. In what follows, we show convergence is implied in the case where γ_A =0. We do this case because it is analytically simpler but develops the intuition that lies behind the case where γ_A >0. We leave the transitional dynamics for this case to the home problems at the chapter's emd.

In the case γ_A =0, we can show convergence will occur either graphically or analytically. For the graphical exposition, we use the *needed savings* and *actual savings* curves that we used to graphically solve for the steady state capital stock (Figure 2). They are redrawn in Figure 3. Looking at Figure 3, we see that if the economy starts out at $k_0 < k^{ss}$, then actual savings exceeds the needed savings to keep the capital stock per person fixed at k_0. It follows that tomorrow's capital stock, k_1, must be larger than k_0. What happens at k_1? In this case, actual savings again exceeds the needed savings to keep the capital stock fixed at k_1, so that the capital stock must increase. Generalizing, as long as the capital stock is below the steady state level, it is the case that actual savings is greater than the needed savings, so the per capita capital stock must increase. As a result, the capital stock must converge to the steady state level.

Figure 3: Convergence to Steady State



The reverse holds in the case in which $k_0 > k^{ss}$. We still have convergence, but from above. We do not show this, but it is easy to do.

Analytically, we can show convergence by using the law of motion for the capital stock per capita given by Equation (k) where we have substituted in for the per capita production function and set $\gamma_A = 0$. This is

$$(1+\gamma_N)k_{t+1} = (1-\delta)k_t + sAk_t^{\theta}$$

Our first step is to divide both sides by (1+ γ_N). This yields

$$k_{t+1} = \frac{1-\delta}{1+n} k_t + \frac{sA}{1+n} k_t^{\theta} \tag{11}$$

Equation (11) expresses k_{t+1} as a function $g(k_t)$ given by the right hand side of the expression, i.e.,

$$g(k_t) = \frac{1-\delta}{1+n} k_t + \frac{sA}{1+n} k_t^{\theta} \quad . \tag{12}$$

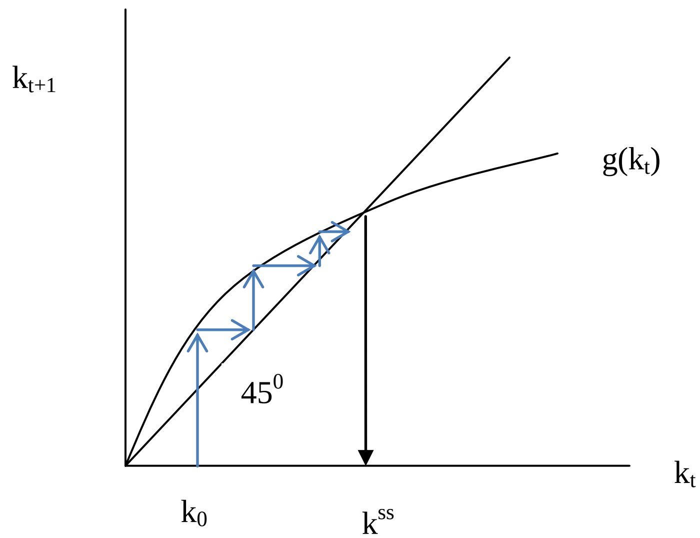
What are the properties of this function? Toward this goal, we take the first and second derivatives of $g(k_t)$. The first derivative is $g'(k_t) = \frac{1-\delta}{1+n} + \frac{sA}{1+n}\theta k_{t-1}^{\theta-1}$. Given that $\theta < 1$, it follows that g has a positive slope. The second derivative is equal to $g''(k_t) = \frac{sA}{1+n}\theta(\theta-1)k_t^{\theta-2}$. The second derivate is negative given our assumption that $0 < \theta < 1$. A negative second derivative is the property of a concave curve, i.e., given the curve increases, it bends inward, or more rigorously, that if you draw a line between any two points on the curve, the point on the line will lie below the point on the curve. Furthermore, we have g(0) = 0. Thus, the function g is a strictly increasing, strictly concave function that originates from the origin.

Now a steady state is a capital stock where $k_{t+1}=k_t$. Graphically, points where the value

Reinforcement: If you understand this, try changing the shape of the curve so that it cuts the 45^0 line from below. How many steady states are there? Is it globally stable or instable?

on the x-axis equals the value on the y-axis are represented by a straight line from the origin with a slope equal to 1, or a 45^0 degree line out of the origin. The function $g(k_t)$ and the 45^0 line are shown graphically in Figure 4. Due to the concavity, the function $g(k)$ cuts the 45^0 from above which implies that the economy converges to the steady state starting with any initial capital stock. You can see this by starting with some arbitrary k_0, and then finding k_1 from the $g(k_0)$ function. Now use the 45^0 line to show k_1 on the horizontal axis, and then find k_2. If you keep on doing this, you will see the economy approaches k^{ss}. We call this convergence property "Global stability", on account that it does not matter where you start. You always end up at k^{ss}.

Figure 4: Transitional Dynamics



III. Calibration

Having completely characterized the equilibrium properties of the Solow Growth Model, we are now in a position to go through the calibration procedure. Recall the five steps of the calibration procedure:

1. Pose a Question
2. Choose a good measuring device
3. Define Consistent Measures
4. Assign Parameters
5. Comparison of Model Predictions to Data

We will go through each step thoroughly in this section. In doing so, we will need to take a detour into the *National Income and Product Accounts*.

The calibration is done within the context of understanding the huge differences in living standards that currently exist across countries. As we shall document more formally in the next chapter, today's differences in living standards are huge. On a per person basis, the average person living in the richest countries in the world is about 50 times richer than the average person living in the poorest countries of the world.

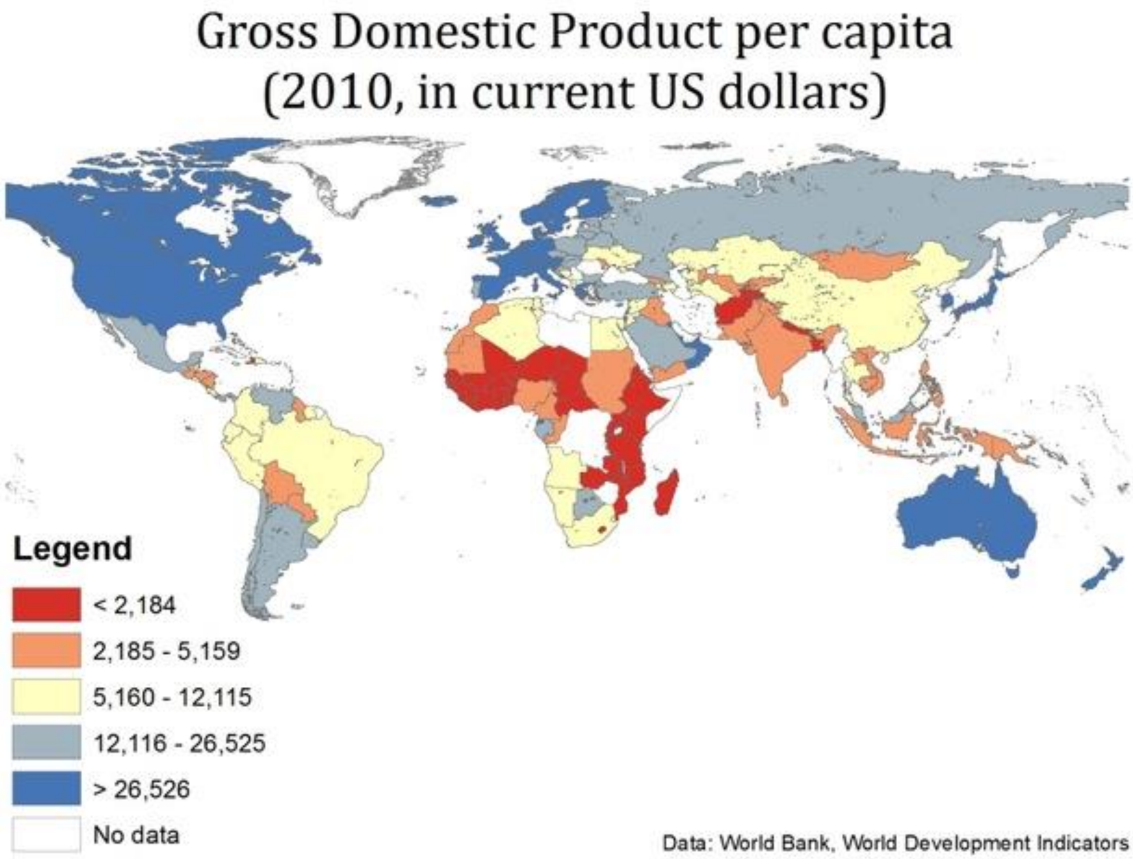
For reasons that will become apparent in the next chapter, the Solow model is not a particularly good model for answering the question, why some countries are so much richer than others. Nevertheless, it is a good starting point and a great way to see what the calibration procedure involves, especially steps (3) and (4) of the process. These steps will be the same for a large number of questions for which we use the Solow model to and later the Neoclassical Growth model to answer. Thus, in future Chapters, we will be able to skip over these steps.

Step 1: Pose a Question

As we shall document in depth in the next chapter, there are huge differences in living standards across countries today. The following figure taken from the World Bank displays the amount of variation in living standards across countries measured by each country's Gross Domestic Product per capita. The average citizen living in the richest countries (indicated by the color blue) is roughly 10 times better off than their counterparts living in the poorest countries (represented on the map by a red color). Among the poorest of the poor and the richest of the rich, these differences are even larger, being closer to a factor of 50.

An important question is what accounts for these huge differences. This is the question that we address in this illustration (and in the next part of the book). More specifically, here we shall ask the question of whether differences in the savings rate can account for the factor 10 differences in per capita GDP between the richest and poorest countries of the world. This is step 1 in the calibration procedure.

Figure 5: International Income Differences



Step 2: Choose a Good Measuring Device

The next step is to choose a model, our so called measuring device to test our theory. We shall choose the Solow model. Why is the Solow model a good measuring device for a host of macro related questions? The reason is that its balanced growth path is both qualitatively and quantitatively consistent with the experience of the US economy over the 20th Century. This makes the model attractive for analyzing business cycles, international income differences, and for evaluating the effects of alternative government policies.

Recall, that in the first part of this chapter, we solved for the balanced growth path equilibrium of the Solow Growth Model and showed that

$$\frac{y_{t+1}}{y_t}=\frac{k_{t+1}}{k_t}=\frac{c_{t+1}}{c_t}=\frac{w_{t+1}}{w_t}=1+\gamma_A \text{ and } \frac{r_{t+1}}{r_t}=1.$$

An implication of these findings, is that capital's share of income and labor's share or income are each constant over time. Capital's share of income is defined as $r_t K_t / Y_t$ and labor's share of income equals $w_t N_t / Y_t$. Moreover, these two shares are simply the coefficients on each respective input in the production process. This follows from (LD) and (KD). To see this, we take the profit maximization conditions

(LD) $$w_t = (1-\theta)\frac{Y_t}{N_t} \ .$$

(KD) $$r_t = \theta\frac{Y_t}{K_t} = \theta\frac{y_t}{k_t}$$

Multiplying both sides of (LD) by N_t/Y_t yields and multiply both side of (KD) K_t/Y_t to arrive at

(LS) $$\frac{w_t N_t}{Y_t} = 1-\theta .$$

(KS) $$\frac{r_t K_t}{Y_t} = \theta$$

Thus, not only are the capital and labor shares of income invariant over time, but they are equal to their respective exponents in the production function. These relations will be particularly useful when we assign values of the parameters in Step 4 of the calibration procedure.

As we said, the Solow model is a good measuring device because it accounts for the long run performance of the US economy. The characteristics of the long-run performance of the US economy are referred to as the *Kaldor Growth Facts,* after Nicholas Kaldor (1957). The stylized growth facts of the US economy are as follows:

- (Real) per capita GDP has grown at roughly 2 percent per year on average
- Real wage growth of roughly 2 percent per year on average
- A constant real return to capital
- A constant consumption to output, and investment to output ratio.
- A constant capital share of income equal to roughly 1/3 and a constant labor share of income equal to 2/3
- A constant capital to GDP ratio of 2.5 on a yearly basis.

Figure 6 plots per capita GDP for the US economy over the 20th Century. As can be seen, the slope of the path is essentially constant. For sure, there is the below average growth period associated with the *Great Depression* that spanned the 1929 - 1940 period, and there is well above average growth following World War II. But other than those periods, the average annual growth rates for any consecutive 20 year period are pretty

Calculating Average Annual Growth Rates and Graphing Variables that Grow

Calculating Average Annual Growth Rates and Graphing Variables that Grow. Often, one will hear that GDP or some other variable grew at an average annual rate of x% over some period. How is an average annual growth rate calculated? There are two ways to calculate an average annual growth rate, one of which assumes that the variable of interest grows geometrically and the other that assumes it grows exponentially. In the former, the variable of interest evolves according to $y_{t_2} = (1+g)^{t_2 - t_1} y_{t_1}$, and in the latter $y_{t_2} = e^{g(t_2 - t_1)} y_{t_1}$. To determine the average annual growth rate in both instance, applies the the natural logarithm to both sides. Exploiting the property of logs, on arrives at

$$\log y_{t_2} = (t_2 - t_1)\log(1+g) + \log y_{t_1}$$

and $\log y_{t_2} = (t_2 - t_1)g + \log y_{t_1}$. The next step is to solve for g. In the case of geometric growth, one typically makes use of the property of logs that log(1+x)≅x, i.e., that the log of a variable plus 1 is approximately equal to the variable. If we make this assumption, then the solution to g in each case is $g = \dfrac{\log(y_{t_2}) - \log(y_{t_1})}{t_2 - t_1}$.

An important insight from this analysis is that the natural log of output is linear in time, with a slope equal to the average annual rate of growth, and an intercept equal to the log of the starting variable. This explains the phenomenon of platting the logarithm of a variable, rather than a variable against time, as the slope of the latter corresponds to the growth rate. The figure in this chapter makes use of the logarithm of base 2, not the natural logarithm, but the implications are the same for interpreting the slop of the plot.

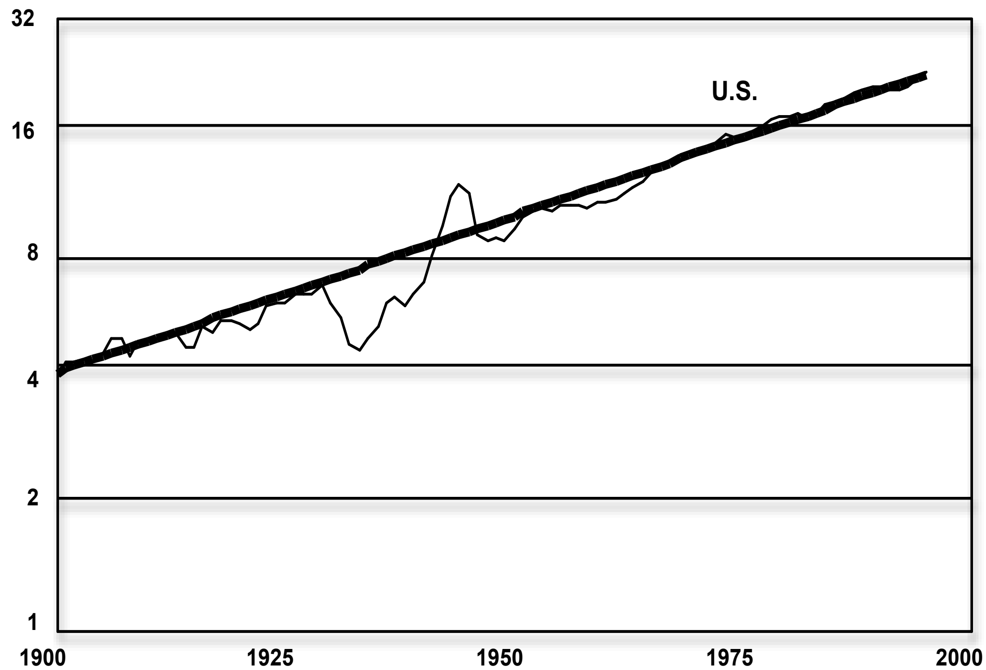
much the same. Thus, it appears that the US has been on a balanced growth path, at least with respect to total output.

The balanced growth path equilibrium of the Solow model matches all of these US growth observations. The constant growth of per capita GDP and the constant growth of the real wage are just what the model predicts. The model likewise predicts a constant real rental price of capital, and it predicts constant labor and capital share of incomes. Since it also predicts that per capita GDP and the capital stock grow at the same rate, the model predicts a constant capital to output ratio. For the same reason, it also predicts that the consumption to output ratio is constant.

The fact that the balanced growth path of the Solow growth model matches the US long run growth facts is really not surprising. Robert Solow (1956) purposely chose the functional forms for the model so that it would match these facts.

Now in using the Solow model as our measuring device, we are going to proceed by comparing the balanced growth path levels of per capita GDPs of countries that differ in their savings rates. As we shall see, whereas the US looks like it is on its balanced growth path, this is not the case for most countries. In this sense, the calibration exercise we are going through is deficient.

Figure 6 Per Capita GDP of US Economy



Step 3: Defining Consistent Measures

The Solow Growth model is abstract, and thus does not exactly match up with the data available in the *National Income and Product Accounts (NIPA)*. This necessitates that we reorganize the NIPA data so that it matches the income, expenditure, and input categories in the Solow model.

Expenditure reorganization

In the model, expenditures of output are either in the form of *Consumption* (C) or *Investment* (sY). In the NIPA, the uses/expenditures of output are Personal Consumption Expenditures, C, Gross Private Investment, I, Government Expenditures, G, and Net Exports, Nx, which is the difference between a nation's exports and imports. The sum of these expenditures, C + I + G+ Nx, equal GDP.

The first thing we must do in the reorganization is move the government expenditure category and the Net Exports category in the NIPA to either model C or model sY. Some government expenditures are classified in the national accounts as public consumption, and thus, are in the nature of consumption goods. Naturally, these should be moved into C. The rest of the government expenditure goods are classified in the national accounts as public investment. Naturally, these goods should be moved into sY.

The economy is closed. Thus, in the model there are no exports or imports. The question is what to do with the Net Export category. To the extent that a country is a net importer, it is borrowing from the rest of the world. This means that foreigners are accumulating claims to future output. As investment is made for the purpose of increasing future output, the most logical adjustment is to reduce investment by the amount of Net Exports (if it is negative) and increase investment in the case the country is a net exporter. The expenditure reclassifications are summarized in the following table.

Table 1: Expenditure Reclassification

	NIPA	Theory
Consumption	C	C
Investment	I	sY
Net Exports	Nx	sY
Government • Public consumption • Public Investment	G	 C sY

With this reorganization, the ratio of Consumption to GDP is roughly 75 percent and the ratio of investment to GDP is roughly 25 percent for the United States over the post war period. These numbers are based on the estimates contained in the *Economic Report of the President*.

Input Reorganization

The Solow growth model has only two inputs, capital and labor, whereas in reality there is a third input or factor of production, land. The most natural adjustment is to add the value of land to the capital stock input. In the United States, the value of land is roughly 5 percent of GDP. To the value of land, we add the value of equipment and machines, non-residential structures, residential structures, and the stock of government capital. Estimates of the value of these capital categories are reported by the *Bureau of Labor*

Statistics. For the post World War II period, the value of these capital categories and land totals approximately 2.75 times GDP.

Income Reorganization

The national income accounts report estimates for the income categories listed in the first column of the following table. In the model, there are just two types of income: capital income and labor income. Thus we need to allocate the income categories in the NIPA into labor income, w_tN_t in the model, and capital income, r_tK_t in the model. These reallocations are shown in the second column of the table.

Table 2: Income Reclassification

Category	Allocation
Rental Income	Capital Income
Proprietor's Income	Ambiguous
Corporate Profits	Capital Income
Capital Consumption Allowance	Capital Income
Taxes on Production and Imports	Ambiguous
Business Transfers	Capital Income
Wages, Salaries and Other Compensation	Labor Income
Net Interest Income	Capital Income
Profits of Government Enterprises	Capital Income

Most firms own their own capital. As such there is no explicit payment to capital in the NIPA. Nevertheless, as we shall show in Chapter 9, the implicit rental payment to capital equals the real interest rate plus the depreciation rate. This explains why the Capital consumption Allowance category and the Net interest income are allocated to capital income. Rental income can be thought of as payment to both the structure and to the land, and so it too belongs in the capital income category.

There are no profits in the Solow growth model as the economy is competitive. In the National accounts there is the category of corporate profits. These we assign to capital income, as the idea is that they are residual payments to capital, owned by the firm. For a similar reason, *Government Profits Less Subsidies* is allocated to capital's income. Business transfers are likewise allocated to capital as they are for the most part bad debt held by a business. *Proprietor's Income*, in contrast, is a bit trickier as it represents both a payment for the labor of the proprietor and the implicit return to the capital he/she employs. For this reason, it is ambiguous as to whether it represents the payment to labor or capital. *Wages, Salaries and other compensation* is clearly a payment to labor, and so thus is appropriately allocated to labor income. *Taxes on Production and Imports* are like *Proprietor's Income* in that it is not clear how much of it represents capital income and labor income.[1] These taxes are in the form of sales taxes, excise taxes, and property taxes in the United States. A value added tax (VAT), is similarly an example of such a tax.

There is no way to know for sure how much of *Proprietor's Income* and *Taxes on Production and Imports* belong in the capital income category and the labor income category. A number of treatments are available. There is one extreme of assigning all of

[1] In the past, this category was called Indirect Business Taxes.

the income in these categories to labor, and there is the other extreme of assigning all of the income in these categories to capital. An intermediate assumption is to assume that the allocation of these ambiguous components to capital and labor is the same as the share of the total income category to GDP. The estimates for capital's share of income that are obtained range from around 25 percent to 40 percent in the United States. We will use a middle value of 1/3rd.

Step 4: Assigning Parameter Values of the Model

There are 6 parameters that we need to assign values to. These are: the depreciation rate, δ; the exogenous growth rate of technology, γ_A ; the exogenous growth rate of population, γ_N; the exponent on capital in the production function, θ; TFP, A; and the savings rate, s.

Assigning the value to γ_A

According to the model, the growth rate of per capita GDP along the balanced growth path is equal to the rate of exogenous technological change, γ_A . For the United States, per capita GDP grew at an average annual rate of 2 percent per year. This means that the parameter, γ_A , needs to be set to .02 for the model economy to match quantitatively the long-run growth rate of the U.S. economy in the 20th century.

Assigning a value to γ_N.

For the population growth rate, the assignment of the exogenous rate of population growth is trivial. The population growth parameter, γ_N, is just that- the rate of growth of the economy's population. As population growth in the U.S. economy is 1 percent per year, γ_N is obviously assigned this value so that the model matches the prediction.

Solow Residuals, Growth Accounting and the Productivity Slowdown

Robert Solow's contribution to growth theory goes beyond his 1956 paper. In 1957, Solow published a paper in the *Review of Economics and Statistics* that introduced the practice of growth accounting and led to the term "Solow Residuals". What Solow did in his 1957 was impute TFP in the production function. More specifically, taking the production function for aggregate output we can solve for output per unit of composite input, i.e. $\frac{Y_t}{K_t^{\theta} N_t^{1-\theta}} = A(1+\gamma)^{(1-\theta)t} \equiv A_t$. The right hand side of this equation is referred to as the Solow Residual or TFP. Notice, that if we use the data on GDP, capital stock, and labor input, we can solve for the Solow Residual, A_t. This is effectively what Solow did in 1957, although he did the exercise in growth rates and not levels. In growth accounting, we taking the natural log of both sides of the aggregate production function, and differentiate with respect to time. Assuming all variables including TFP are continuous fu, one can solve for the growth rate of TFP, namely, $g_A = g_Y - \theta g_K - (1-\theta) g_K$ using the value for θ=1/3. Thus, the growth rate of TFP is the growth of GDP less the contribution of growth from capital input less the contribution of growth from labor inputs. What Solow found was that approximately half of the growth in US output over the 1909-1949 period was the result of TFP growth. Researchers have subsequently applied this procedure to the second half to the 20th Century and were surprised to find that between 1973 and 1995 the growth rate of TFP was cut in half.

Assigning a value for θ

The assignment of the exponent on capital in the production function is straightforward because it equals capital's share of income as seen from Equation (KS). In the US economy, capital's share of income is estimated to be roughly 1/3rd . For this reason, θ is set equal to 1/3rd.

Assigning a value for s

The savings rate, s, is the fraction of output that is not consumed. It is the amount that is invested. In reorganizing the national income and product accounts, we reduced the expenditure categories from C + I + G + Nx to C+sY. The empirical counterpart of the savings rate in the model is thus, sY/GDP. As the ratio of sY/GDP for the US economy in the postwar period is 25 percent, we set s equal to this number.

Assigning a value for δ

The depreciation rate can be assigned using the law of motion for the capital stock, the assigned values for s, γ_N and γ, and the observation for the capital to GDP ratio in the United States. Recall, the capital stock (per capita) law of motion equation is

$$(1+\gamma_N)k_{t+1} = (1-\delta)k_t + s y_t \ .$$

Along the balanced growth path, $k_{t+1}=(1+\gamma_A)k_t$. Consequently,

$$(1+\gamma_N)(1+\gamma_A)k_t = (1-\delta)k_t + s y_t \ .$$

If we divide both sides by k_t, we arrive at

$$(1+\gamma_N)(1+\gamma_A) = (1-\delta) + s\frac{y_t}{k_t} \ .$$

Plugging in n=.01, s=.25, γ_A = .02, y_t/k_t=1/2.75 = .36, we solve for δ = .06. This is for a period of a year. In general, the depreciation rate depends on the period that we are studying.

Assigning a value to A

We normalize the value of A to 1. Its value is not important to the calibration. The reason for this is that we can always choose the units in which output are measured. For instance, if we were looking at a product such as milk, we could define the measurement unit to be a liter. Alternatively, we could define the unit of measurement to be a centiliter, of which there are 100 in a liter. If we were to define the unit of measurement to be a centiliter the TFP parameter in that case would have to be 100 times larger than its value in the case we measured milk in liters. Whether we choose the liter of centiliter unit, is unimportant as in either case we have the same amount of milk being produced. In this sense, it does not matter what the value of A is, and hence we opt for the simplest of values, 1.

Step 5: Testing the Theory

Having parameterized the model to match the US growth facts, we now turn to the question we started with, namely, how much of the factor 10 difference in per capita GDP we observe between rich and poor countries is the result of differences in savings rates? To start, we shall answer this question assuming that model economies only differ in their savings rate, and have the same population growth rates, TFPs, depreciation rates and exogenous rates of technological change. Population growth rates surely differ and we could explore how big of an effect on relative balanced growth path levels of per capita GDP these have. In fact, this is left as an assignment at the end of the chapter. TFP is another possible difference, and indeed, one which we shall consider in the next part of the book.

The depreciation rate, and the capital share parameter, and the exogenous rate of technological change are assumed to be the same across countries. The main reason for this is that they are given by the laws of physics and the like, and therefore the same across countries. Technology, being exogenous, effectively falls from the sky, and hence there is no reason for it to differ across countries in the context of the model. Empirically, the share of income that is paid to capital does not differ systematically across countries. This was pointed out by Douglas Gollin in a 2002 paper published in the *Journal of Political Economy*.

Given the parameterization and the assumptions of what differs across countries, we are now able to answer our question of whether differences in savings rates account for the differences in international income levels. To do this, we return to equation (k-BGP),

$$k_t^{bg} = (1+\gamma)^t \left[\frac{sA}{(1+n)(1+\gamma)-(1-\delta)} \right]^{1/(1-\theta)}$$

Substituting the balanced growth path per capita capital stock into the per capita production function we arrive at

$$\text{(y-BGP)} \qquad y_t^{bg} = (1+\gamma)^{(1-\theta)t} A k_t^{\theta} = (1+\gamma)^t A^{1/(1-\theta)} \left[\frac{s}{(1+n)(1+\gamma)-(1-\delta)} \right]^{\theta/(1-\theta)} .$$

Under the assumption that countries only differ in s, the ratio of country i and country j's per capita output along the balanced growth path is simplifies to

$$\frac{y_t^i}{y_t^j} = \left[\frac{s^i}{s^j} \right]^{\theta/(1-\theta)} .$$

Now in the calibration, $\theta = 1/3$, so the above expression is

$$\frac{y_t^i}{y_t^j} = \left[\frac{s^i}{s^j} \right]^{1/2} .$$

We are now ready to answer the question we posed. To answer the question of whether savings rate differences account for the observed differences in international income differences, we could go to an independent data source such as the *International Financial Statistics* compiled by the IMF and retrieve the savings rate for each country and then plug them into the above equation. Instead, we will take a slightly different approach and answer the inverse question, namely, how much would savings rates have to differ across countries to account for the factor 10 difference observed between rich and poor countries. In effect, we solve for s^i/s^j such that

$$\frac{y_t^i}{y_t^j} = 10 = \left[\frac{s^i}{s^j}\right]^{1/2}.$$

To do this we simply take both sides of the above equation and raise it to the power 2.

$$10^2 = \left[\frac{s^i}{s^j}\right]^{2/2}.$$

This yields the solution that s^i/s^j=100. In words, the rich country must have a savings rate that is 100 times greater than the savings rate in the poor countries.

Without knowing anything about the data, you would probably guess that these required differences in savings rates are implausible. They are indeed implausible. The fact of the matter is that in the postwar period, the fraction of output saved does not vary systematically with a country's level of output. The following table taken from the IMF makes this point.

Table 3: Fraction of GDP Invested: 1966–93

	Industrialized	Developing	Africa
1966	22.7	17.6	19.0
1970	23.7	17.5	22.9
1975	21.6	25.5	29.2
1980	23.2	25.5	28.0
1985	21.3	22.3	20.3
1990	21.5	24.3	19.6
1993	19.4	23.3	18.8

Source: International Monetary Fund (1994)

IV. Conclusion and Discussion

This chapter started by presenting the Solow Growth Model and characterizing its equilibrium. We saw that in order for there to be sustained long-run growth of per capita output, the Solow model requires that we assume increases in technology or TFP. The need to assume exogenous technological change to generate sustained growth is due to the marginal product of capital decreasing and going to zero as a worker is given more and more machines. If TFP increases at a constant, non-zero rate, there is a balanced growth path where per capita output, consumption, capital and the real wage all grow at the growth rate of TFP. Any differences in savings rates or levels of TFP will lead to different income levels across countries, but not long-run growth rates across countries. A country that begins with a capital stock below its balanced growth stock will, however, converge to its balanced growth path.

After studying the properties of the Solow Growth model, we used it to illustrate the calibration procedure. Specifically, it served as our measuring device to determine how much of the difference in international incomes could be attributed to differences in savings rates. The conclusion of the exercise is not much; the model implies that savings rates have to be 100 times greater in the rich countries than in the poor. The data shows that the differences are extremely small.

In retrospect, given the observation that savings rates do not differ a systematic way across countries, this entire exercise was not worth the effort in that the findings are rather obvious. The real point of this exercise, however, was to familiarize the reader with the calibration steps using the most straightforward example possible. In the next part of the book, we will look at other factors that may differ across countries and to what extent they help account for these large differences in living standards. Additionally, we will examine whether a view of international income differences as differences in balanced growth paths is useful interpretation of the data. As we will show in the next chapter, Chapter 4, it is not.

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Solow, Robert. 1956. Contribution to the Theory of Economic Growth. The Quarterly Journal of Economics, 70 (1): 65-94.

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Problems:

1. Suppose an earthquake destroys part of a nation's capital stock, but does not kill off any people. Use the Solow model without technological change to describe the effect of this event on the country's total output and its per capita output over time. Assume prior to this event the economy was on its steady state path.

2. Suppose a war destroys part of a nation's population but not its capital stock, (say on account of a neutron bomb being deployed). Use the Solow model without technological change to show the effect of this event on the country's total output and per capita output over time. Assume prior to this event the economy was on its steady state path.

3. Assume a Solow growth model economy with no exogenous technological change is initially at a steady state. Suppose there is a permanent decrease in the population growth rate, say on account of the spread of AIDS – a major current problem in Sub-Sahara Africa. Show graphically the path of the economy's capital per capita and output per capita over time following this event.

4. Take the Solow model without technological change. Assume there is a government that taxes consumer's income at the tax rate τ. The government uses the tax receipts to buy some of output. Assume that individuals save a fixed fraction, s, of their after tax income so $S_t = s(1-\tau)Y_t$. Show graphically how an increase in government expenditures will affect the steady state capital stock of an economy. Solve algebraically for the steady state capital stock.

5. Again assume that technology does not change. Now assume that the government imposes a lump-sum tax, T_t, and just uses the receipts to buy goods (so that $N_t\, g = T_t$ where g would be the government expenditures per person). With a lump-sum tax, consumers have an after tax disposable income equal to Y-T, i.e., $N_t c_t = (1-s)(Y_t - T_t)$. Consequently, saving $S_t = s(Y_t - T_t)$. Show graphically that there are 2 steady states. Also, show graphically what the transitional dynamics for this economy are. Namely, suppose you start to the left of the low steady state capital stock, what happens? What about to the right?

6. Suppose total National Savings, S_t, is $S_t = sY_t - hK_t$. The extra term $-hK_t$ reflects the idea that when wealth (as measured by the capital stock) is higher, savings is lower, (i.e., wealthier people have less need to save for the future). Assume that production is given by the Solow model without technological change. Solve algebraically for the steady state capital stock. Show graphically how the steady state capital stock compares for an economy with h= 0 and another with h>0.

7. Go to the Penn World Tables 7.1. For each country calculate its average annual rate of population growth from 1960-2000. (If either the 1960 or 2000 value is missing, just drop the country from the analysis). Use this calculated population growth rate with each country's 2000 investment share of output (denoted by ci) with Equation (y-BGP) to determine the income of each country relative to the US level. (So the US will always be the country whose BGP per capita output you solve for in the analysis). Plot the relative incomes predicted by the model against the actual 2000 relative income differences reported in the PWT and given by the variable y. Assume that all countries have the same values for γ_A, δ, θ, and A.

8. Suppose that the capital share is 2/3. Assume that the only difference between rich and poor countries is their savings rate. What would be the implied difference in the savings rate of rich and poor countries to account for a factor 10 difference in per capita output? A factor 50?

9. Take the Solow Growth Model but alter the Equation for (Y) such that technological change is Hicks-Nuetral rather than labor augmenting. In particular, assume $Y_t = A(1+\gamma_A)^t K_t^{\theta} N_t^{1-\theta}$. Find the growth rates of per capita variables along the Balanced Growth Path. Is this model consistent with the Kaldor Growth Facts?

10. Take the Solow Growth Model but alter the Equation for (Y) such that technological change is Capital-Augmenting rather than labor augmenting. In particular, assume $Y_t = A[(1+\gamma_A)^t K_t]^{\theta} N_t^{1-\theta}$. Find the growth rates of per capita variables along the balanced Growth Path. Is this model consistent with the Kaldor Growth Facts?

11. For the Solow Model with $\gamma_A > 0$, find the function $g(\hat{k}_t)$ where $\hat{k}_t = \dfrac{k_t}{(1+\gamma_A)^t}$ and show graphically that the economy converges to a balanced growth rate.