Overlapping Generations and Pension Systems

I. Introduction

In the previous chapter we studied the consumption-savings decision in the simplest of all frameworks- a two period model. Although useful for gaining intuition, this model is rather limited in addressing interesting macro questions. One clear problem with this construct is that the world does not end in two periods.

In applied dynamic economic theory, there are two standard ways to analyze the consumption-savings decision. They are

1. The infinitely lived families with parents caring about their children's utility.
2. The long but finite-lived people who leave their children no bequests.

For some issues, such as assessing the quantitative consequences of the American pay-as-you-go social security system, it matters which abstraction is used. For others, such as assessing the consequences of technology shocks, public finance shocks or oil shocks for business cycle fluctuations, for all practical purposes it does not matter. The infinitely lived construct is by far the simpler to analyze.

This chapter uses the long but finite lived construct. The next chapter uses the infinitely-lived family construct. The model that is studied in this chapter is the Overlapping Generations Model. The model that is studied in the next chapter is the Neoclassical Growth Model. Effectively, the Neoclassical Growth Model is just the Solow Model but where savings is determined by the utility maximizing choices of households.

II. The Overlapping Generations Model

The overlapping generations model was developed by Paul Samuelson in 1957. In this model, people not only live for a finite number of years or periods, but in any period, there are people of different ages or generations alive.

The simplest of OLG models assumes that people live two periods. Thus, at any point in time, there are some old people and young people alive. Let N_t be the number of people born in period t. Thus, these people are young in period t and old in period t+1. In the simplest OLG model, which will be the basis of the analysis in this chapter, there are N_t young people and N_{t-1} people alive in period t. In what follows, we will assume that N_t=N=1 for all t=1,2,3,4. Note that in period t=1, the first period of our model, that

Each generation enjoys consumption when they are young, c_y and consumption when they are old, c_{old}. Eventually, we will assume that each generation t values leisure when young and old. For now, however, we will deal with the simpler case where utility does not depend on leisure, namely

$$U(c_y, c_{old}) = \log c_y + \log c_{old} \quad (1)$$

Notice that we have implicitly set the discount factor $\beta = 1$. This just simplifies the algebra. None of the results would change if we were to allow for $\beta < 1$.

III. ENDOWMENT ECONOMIES

We begin with a very simple world with no production. Instead, each person starts each period of their life with a certain quantity of the consumption good. We call this an endowment economy. For the present purpose, let us assume that each consumer is endowed with e_y units of the consumption good when young and e_{old} units of the consumption good when old.

We will first solve the equilibrium for this economy when there is no government and hence no government policies. We will see that if people are better endowed when young, the equilibrium will not be very desirable for households. The reason for this is rather straightforward. Households want to have relatively similar consumption over their lifetime. If they are better endowed when young, then the desire for smooth consumption means that they want to save. The problem in this economy is that there are no borrowers. The only other people alive in the period are old. You as a young agent would never lend to an old person because sadly they die at the end of the period and so are not around to pay off the loans.

Model 1. No Government Policy- AUTARKY.

In this first economy, there is no government policy. Given the disparities in endowments between their two life periods, each young person would ideally like to lend so as to smooth out consumption. But can they do this? Here is the beauty or trick of the Overlapping Generations model: Who are they going to lend to? The only possible agents are the old people alive in the period. But this would be a stupid idea to lend to the old. Why? Because they won't be around in period t+1 to pay back the loans. Hence, the only equilibrium for this economy is one where everyone eats their endowment, namely $(c_{1t},c_{2t}) = (e_y,e_{old})$. Thus, there is no trade, and for this reason, the economy is referred to as one being under Autarky.

Formally, a competitive equilibrium must solve the following set of conditions

- Utility maximization. c_y, c_{old}, and s must maximize utility given by Equation (1) subject to his budget constrain when young and his budget constrain when old, namely,

$$c_y + s \le e_y \tag{2}$$

  And

$$c_{old} \le e_{old} + s(1+r) \tag{3}$$

- Market Clearing
  - $$N_t c_{t,y} + N_{t-1} c_{t-1,old} = N_t e_{t,y} + N_{t-1} e_{t-1,old} \tag{4}$$
  - $$N_t s_t = 0 \tag{5}$$

The budget constraints of the household in the Overlapping Generations model is no different than what we studied in the previous chapter on savings. The household when young determines how much of its endowment when young to save, and when old eats its entire old endowment and saving and interest. Note that in every period, the goods market clears as the bond market or savings-loans market clears. Because only young period agents can borrow or lend in this world, the total savings or borrowing of the young must equal zero in equilibrium.

To formally, characterize the equilibrium, we start with the bond market clearing condition. It is obvious, that s*=0 in equilibrium. Using this result and the household's budget constraint when young we arrive that $c_y = e_y$. Using again the equilibrium result that s*=0 together with the household's second period budget constraint yields the result that $c_{old}=e_{old}$. This means that there is only one other thing to solve for in the equilibrium, the real interest rate, r. It is very easy to determine the equilibrium interest rate for this economy. We simply use the utility optimizing condition that

$$\frac{\partial U / \partial c_{old}}{\partial U / \partial c_y} = \frac{c_y}{c_{old}} = \frac{e_y}{e_{old}} = \frac{1}{1+r}\,. \tag{6}$$

Thus, the implication that people eat their endowments in each period is that the real interest rate will be negative whenever $e_y > e_{old}$.

Why is the real interest rate in the equilibrium negative? Recall, that as people are better endowed when young versus old, they really would like to save. However, there are not borrowers in the economy. In an equilibrium, people have to be unwilling to change their allocations at the given prices. The real interest rate must be negative in equilibrium.

This is necessary because in equilibrium there is no borrowing or lending. Given the strong desire to lend by the young, the interest rate must be sufficiently low to dissuade them from borrowing. In this case, it must be negative so that for each $1 saved they would have say $.50 in their bank account when they are old.

Properties of the Equilibrium – Pareto Optimality.

Suppose for the sake of illustration that $e_y=7$ and $e_{old}=1$. Then this would be what households eat in each period, and the real interest rate would be $r=e_{old}/e_y-1=-.85$. The household would obtain utility over its lifetime equal to ln7+ln 1= 1.94.

Now suppose that a planner took 3 units of the good away from each young and gave it to an old person. The $c_y=c_{old}=4$ and the utility over the household's lifetime is ln4+ln 4= 2.77. Every generation would be happier with this allocation including the old people who are alive in the first period of the model. The allocation with $c_y=c_{old}=4$is clearly feasible, as the resource constraint is that the sum of young and old consumption cannot exceed 8 units of output in any period.

Economists have an expression when the competitive equilibrium allocation can be dominated in the above sense. Specifically, they say the Competitive Equilibrium is Pareto Inefficient when there exists an alternative allocation which is feasible which (1) makes at least one household strictly better off and (2) makes no one worse off. In the case that no such allocation exists, we say the Competitive Equilibrium is Pareto Optimal.

The autarkic equilibrium in the above examples is Pareto Inefficient. Given everyone 4 to eat when young and 4 when old Pareto dominates c_y=7 and c_{old}=1. In general, whenever the real interest rate is negative, the competitive equilibrium will be inefficient in the above sense.[1] The inefficiency in the allocation determined by prices means that government policy can be desirable. We now explore this idea by considering two types of pension systems: a pay-as-you-go system and a fully funded system.

Government Policy

We next consider two type of government policies that go under the heading of pensions systems. These are a Pay-as-you-Go/Defined Benefits system, where pension or social security taxes collected in the period are paid out to the retirees in the period, and where the payout to retirees is independent of what taxes they paid when young. The second is a Fully Funded System with Defined Contributions. The key feature of this system is that the amount the government pays out to a retiree is equal to his contribution to a government retirement account that pays an interest rate. The definition of a fully funded system is where the funds you contribute when you are young to the pension system are the ones that are used to pay out your benefits when old. As you shall see, this is a bit of a subtle issue with the policy we study.

A Pay-as-you-Go/Defined Benefit Pension System

[1] Actually, the negative interest rate implies the competitive equilibrium is not Pareto Optimal when there is no population growth. With population growth, the necessary and sufficient condition for the Competitive Equilibrium to be Pareto Optimal is that the interest rate has to be greater or equal to the population growth rate.

We next introduce a transfer/tax system whereby the government imposes a lump-sum tax on the young and gives those receipts back to the old alive in the period in the form of a lump-sum transfer. The government lives forever. The government's budget constraint in each period is thus,

$$0 = N_t Tx - N_{t-1} Tr \tag{7}$$

Where Tx is the lump-sum tax on the young in period t and Tr_t are the transfers on the old alive in period t. The budget constraints of a generation t consumer in period t and period t+1 are

$$c_y + s = e_y - Tx_t \tag{8}$$

$$c_{old} = e_{old} + Tr_{t+1} + (1+r)s \tag{9}$$

An equilibrium must satisfy the following set of conditions

- Utility maximization. c_y,c_{old}, and s must maximize utility given by Equation (1) subject to his budget constraint when young and his budget constrain when old, Equations (8) and (9)

- Government Budget Constraint given by Equation (7)
- Market Clearing
    - $N_t c_{t,y} + N_{t-1} c_{t-1,old} = N_t e_{t,y} + N_{t-1} e_{t-1,old}$
    - $N_t s_t = 0$

Notice that the market clearing conditions are unchanged by this type of goverenmte policy. In the equilibrium, there will not be any borrowing or lending for the same reason as before. Hence, in each period people eat their after tax endowment.

Again for the purpose of illustration, let us assume that e_y=7 and e_{old}=1. Further, let us assume that the government taxes from each young agent 3 units of output and gives those 3 units of output to the old agent. To fully solve out the competitive equilibrium, we again start with the implication that s*=0. Then going to the consumer's budget constraint when young and using the amount of the tax, Tx=3, we conclude that c_y=4. Next, we use the result that s*=0, and Tr=3 together with the budget constraint when old to conclude that c_{old}= 4. Finally, we use the utility maximizing condition given by (6) to conclude that (1+r)=1.

People are surely happier in this world than in the previous world where (c_y,c_{old}) = (7,1). In fact, we could easily determine the difference in welfare by solving for the factor we must scale consumption up in both periods. Namely, we want to find λ such that

$$\log[(1+\lambda)7]+\log[1(1+\lambda)]=\log(4)+\log(4)\,.$$

Pension System#2 Fully Funded/Defined Contributions

The government policy is one which now gives young households the option to save in the form of a retirement account found. In doing so, the government effectively issues debt. In this way the government fills a crucial void in the economy- becoming a borrower in a world short on borrows.

A critical part of the government policy is what how it uses the funds that young agents put in their retirement accounts. The assumption here is the government takes these loans and uses them to pay off the debt obligations it has with the old agents alive in the period, namely, pays for the pensions of the old. This implies the government's budget constraint is

$$B_t = (1 + r_{t-1})B_{t-1} \tag{10}$$

This is actually the constraint in all periods except for the first period of the model. In the first period, there are N_0 old people alive, who have not been part of the fully funded system and have no retirement accounts. Thus, the government has no pension obligations to them. The question still remains as to what the government should do with the savings it takes in from the young born in period 1. What we assume is that the government makes a lump sum transfer to the old alive in period 1 equal to the amount of its borrowings. Namely, in t=1

$$B_1 = -N_0 Tr_0 \tag{11}$$

Formally, a competitive equilibrium must solve the following set of conditions

- Utility maximization. c_y,c_{old}, and s must maximize utility given by Equation (1) subject to his budget constrain when young and his budget constrain when old, namely,

  $$c_y + s \le e_y \tag{12}$$

  And

  $$c_{old} \le e_{old} + s(1 + r) \tag{13}$$

- Government Budget Constraints given by Equations (10) and (11)
- Market Clearing
  - $$N_t c_{t,y} + N_{t-1} c_{t-1,old} = N_t e_{t,y} + N_{t-1} e_{t-1,old} \tag{14}$$
  - $$N_t s_t + B_t = 0 \tag{15}$$

The key condition to note is the bond market clearing condition, which is now the sum of private borrowing/lending and the government borrowing/lending.

Again, to crystalize the results, we return to solving the competitive equilibrium for our numerical example where e_y=7 and e_{old}=1. On the government side, we assume that the Fully Funded pension system is associated with government borrowing 3 units from each young agen, i.e. B_t=-3. In the first period, each old agent that is alive therefore receives a transfer of Tr=+3.

We now can solve for the competitive equilibrium. Again we begin with the bond market. For the current policy, as we now assume that the government borrows 3 units of the good from the young, equilibrium implies that s*=3. Next we use this result with the budget constraint of the young and conclude that c_y*= 7 – s* = 4.

Solving for c_{old} is a bit tricky in this case, because the right hand side of the budget constraint of the household when old depends not only on the savings but the real interest rate. The real interest rate must satisfy the utility maximizing conditions, that

$$\frac{c_{old}}{c_y} = 1 + r$$

which is best rewritten as

$$\frac{c_{old}}{1+r} = c_y \,. \tag{16}$$

To proceed, we make use of an earlier result that the household's two budget constraints yield the following intertemporal budget constraint is

$$c_y + \frac{c_{old}}{1+r} = e_y + \frac{e_{old}}{1+r} \tag{17}$$

Using (16) with (17) implies

$$2c_y = e_y + \frac{e_{old}}{1+r} \tag{18}$$

Now using the result that c_y=4, we have

$$1 = \frac{1}{1+r} \tag{19}$$

so that r*=0. It follows from (16) that c_y=c_{old}=4.

Summary of Findings

Notice the importance of government in this world short of borrowers. The government can make people better off by implementing a pay as you go type pension system or the fully funded system. Moreover, the results are the same in both cases. This equivalence of equilibrium results is a consequence of this being an endowment economy. Recall, there is no production, and people do not value leisure. As such the government policies do not have any distortionary effects. We next compare these two alternative social security systems in a world with production and with leisure.

Before introduction production in the economy, note a surprising result in the case of the fully funded system. As we have specified the policy, the government has a deficit in the first period. (It pays out a transfer of 3 to each old person, and pays for it by borrowing.) In all future periods, the government rolls over its debt. Thus, we have a situation where a government deficit is not met with a future surplus. This may certainly seem a contradiction of the result we established in the previous chapter which introduced the government budget constraint. The seemingly contradictory result is an artifact of the real interest rate under autarky being negative. In this case, the government can roll over its debt each period without seeing the debt explode. This strange result was first shown by Robert Barro in a paper he published in the 1994 Journal of Political titled "Are Government Bonds Net Wealth?"

III. PRODUCTION ECONOMIES

We continue to maintain the assumption that individuals live for two periods with an equal number born every period. Now instead of being endowed with goods when young and old, we assume that they are endowed with time, which they can allocate between market production and non-market activities. We shall assume that they have 1 unit of time that they can split between work and leisure.

The utility of a person born in period t depends on their consumption when young, consumption when old, as well as their leisure when young and leisure when old. We specifically assume that lifetime utility is given by:

$$U(c_y, l_y, c_{old}, l_{old}) = \ln c_y + \alpha_y l_y + \beta[\ln c_{old} + \alpha_{old} l_{old}] \tag{20}$$

We will set β=1 as in the pure exchange model so as to simplify the algebra. The α parameters determine how much you value leisure when young and old. The utility function given by (20) is slightly different from our chapter on work hour differences where we were had leisure in the utility and taking a logarithmic form. The switch to this functional form with utility being a linear function of leisure simplifies the algebra and allows us to consider the case where the household fully retires when old.

The model is only interesting if we assume that people value leisure more when old compared to young, i.e. $\alpha_{old} > \alpha_y$. One way to think of this is that the flipside of leisure, namely working, requires a lot more effort when old. As we shall show, the assumption that there is greater disutility of working when old is important to obtaining the result that the equilibrium under autarky is not Pareto Optimal on account that the economy is short on borrowers. Intuitively, if it is less painful to work when young, people would like to work a lot when young and not much when old. With a big income in their first period, they would like to save so they could enjoy a high level of consumption when old while not working much. This effectively gives us the same situation in the pure exchange economy of a world short on borrowers.

To complete the model, we need to describe the production side of the economy. We will continue with the assumption we made in the chapter on work hour differences that production is linear in total hours. Specifically,

$$C_t^S = AH_t \tag{21}$$

<u>Model #1: Autarky</u>

Now that we have added leisure, it is best to begin with a study of the world where there is no government. Just as before, there will be no trade between generations. This is a production economy, so solving for the equilibrium requires a little bit more work. To do this, we go to the utility maximizing problem of a young consumer. The consumer of generation t will choose (c_y,l_y,h_y, s,c_{old},l_{old},h_{old}) to maximize utility subject to its young period and old period budget constraints

$$c_y + s = w_t h_y$$

$$c_{old} = w_{t+1} h_{old} + (1+r)s$$

and the time use constraints

$$l_y + h_y = 1$$

$$l_{old} + h_{old} = 1$$

We now have three utility maximizing conditions, one for the MRSC =MRSE between c_y and l_y, one for the MRSC =MRSE between c_{old} and l_{old}, and the last for the MRSC =MRSE between c_y and c_{old}. These are

$$\alpha_y c_y = w_t$$

$$\alpha_{old} c_{old} = w_{t+1}$$

$$\frac{\partial U / \partial c_{old}}{\partial U / \partial c_y} = \frac{c_y}{c_{old}} = \frac{1}{1+r}.$$

In addition to utility maximization, an equilibrium must satisfy profit maximization. Given the assumption of the linear production function, the profit maximizing condition is simply $w_t = A_t$. For the production economy, there is also a labor market clearing condition in addition to the goods market and bond market.

An equilibrium must therefore satisfy the following set of conditions

- Utility maximization.
- Profit maximization
- Market Clearing

- $N_t c_{t,y} + N_{t-1} c_{t-1,old} = C^s = A_t H_t$
  - $N_t s_t = 0$
  - $N_t h_y + N_{t-1} h_{old} = H_t$

Now we can solve for the equilibrium. We start with the utility maximizing conditions related to consumption and leisure when young and when old. The MRSC=MRSE conditions imply $c_y=A_t/\alpha_y$ and $c_{old}=A_{t+1}/\alpha_{old}$. Assuming that TFP does not grow, i.e. $A_t=A$, we have that $(1+r)=\alpha_y/\alpha_{old}$. Notice that these are the equilibrium solution as they are expressed as functions of the parameters. Moreover, that the condition for a negative real interest rate in equilibrium is $\alpha_y<\alpha_{old}$.

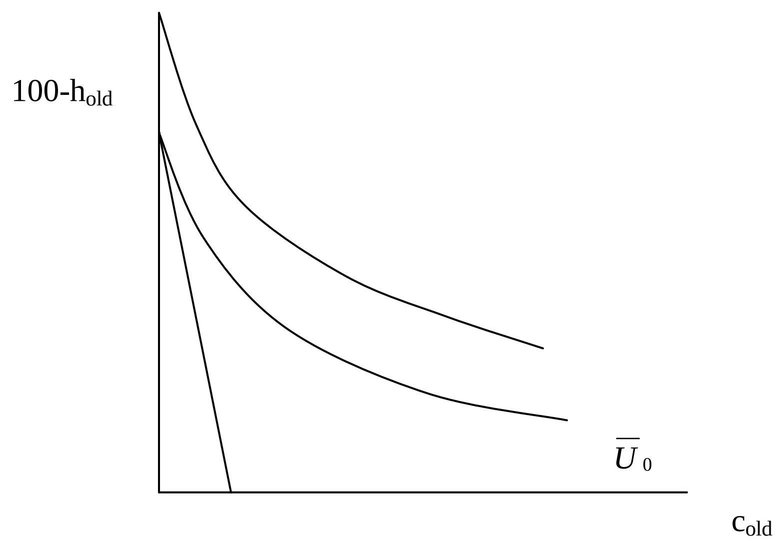
The next step is to solve for h_y and h_{old}. Here we use the result that s*=0 in equilibrium as there are no borrowers in this economy. This result with the results, $c_y=A/\alpha_y$ and w=A in the young period budget constraint allows us to solve for $h_y=1/\alpha_y$. Applying the same logic to the old period budget constraint allows us to solve $h_{old}=1/\alpha_{old}$.

An important point is that even if the household hates to work when old, i.e. α_{old}/α_y, is 1.0 or more, the household must work when old. He must eat when old, and there is no other way to do this but to work, regardless of how unpleasant the task is.

<u>Pension Systems</u>

Now we will reintroduce a government into this economy. In doing so, we shall assume that $\alpha_y< \alpha_{old}$ so that the autarkic equilibrium is not Pareto Optimal. Additionally, we consider only those government policies which guarantee equal consumption for the household when he is young and old. Given the preferences, this implies that the equilibrium interest rate must be zero. Moreover we will have the pension systems guarantee that the household will retire when old. The critical implication of this is the condition that MRSC=MRSE between c_{old} and l_{old} no longer holds. As the household fully retires, he is on a corner with respect to his optimizing conditions and at a corner the slope of the budget constraint need not equal the slope of the indifference curve. This is shown graphically below

Figure 1: Corner Solution



To consolidate on space and redundancy, we will consider a very general specification of the government policy. We will specify the household’s budget constraints when young and old under this general specification of policy, as well as the government’s budget constraint. We will then specialize the government policy so as to consider both the pay-as-you go system and fully funded system.

The government policies will consist of a tax rate on labor income that is the same for both young and old, (τ) borrowing/lending (B), and lump-sum Transfers to old (Tr). These transfers represent the social security benefits that the government pays to old in the economy.

The budget constraints of a person born in period t are

$$c_y + s = (1-\tau)Ah_y$$

$$c_{old} = (1-\tau)Ah_{old} + s + Tr$$

Notice that we have used the profit maximizing result that $w_t=A_t=A$. We can solve for s in the old period budget constraint to arrive at the intertemporal budget constraint, namely, (assuming that r=0)

$$c_y + c_{old} = (1-\tau)h_y + (1-\tau)h_{old} + Tr$$

The government's budget constraint in each period is

$$B_{t+1} = \tau Ah_y + \tau Ah_{old} - Tr + B_t$$

Again, here we are imposing the assumption that r=0. This will be valid by restricting government policies to those that guarantee that consumption is the same in both periods of a person's life.

An equilibrium must satisfy

- Utility maximization
- Profit Maximization
- Government Budget Constraint
- Market Clearing

- $N_t c_{t,y} + N_{t-1} c_{t-1,old} = C^s = A_t H_t$
- $N_t s_t + B_t = 0$
- $N_t h_y + N_{t-1} h_{old} = H_t$

Having described the economy and equilibrium conditions for a general government policy, we specialize the model to consider the pay-as-you go system and the fully funded system.

The Pay as You Go/Defined Benefits System

There is no government debt in this system, i.e. $B_{t+1}=B_t = 0$. The government simply taxes the labor of the young at rate $\tau > 0$, and hands the receipts $Tr=\tau A h_y$ to the old aliven the period in the form of a transfer. Recall, that the key is that the tax rate and social security benefits are such that they guaranteed that $c_y = c_{old}$. To fully characterize the equilibrium, we use the result that s*=0. Then from the young period budget constraint we have that $c_y=(1-\tau)Ah_y$. From the old period budget constraint, $c_{old}= Tr=\tau A h_y$. Using these two results together with $c_y = c_{old}$, we can see that the tax rate $\tau=1/2$. We only need to solve for h_y to complete the solution. Here we use that the MRSC=MRSE condition between young consumption and leisure is the same as in the autarkic world except that the price of leisure is $(1-\tau)w$. It follows that $c_y=(1-\tau)/\alpha_y$. Since $\tau=1/2$, $c_y = \frac{A}{2\alpha_y} = c_{old}$. Given this tax rate and given that $c_y=(1-\tau)Ah_y$, it follows that $h_y = \frac{1}{\alpha_y}$.

Fully Funded/Defined Contribution System

There are no taxes on the young and no transfers to the old (except for an initial old). Instead the government issues debt and roles it over every period. (The alternative Again, the debt is exactly the amount that ensures that consumption is equal in both periods of a person's life. Since there are no taxes on young income, the MRSC=MRSE condition is the same as in autarky, namely, that $c_y = \frac{A}{\alpha_y}$. This is also c_{old}. Using this result with the intertemporal budget constrain, we can solve for h_y. This is $2c_y = Ah_y$, which implies that $h_y = \frac{2}{\alpha_y}$. Young consumption is thus exactly half of young income so that savings $s = \frac{A}{\alpha_y}$, which is the amount of government debt.

Summary and Conclusion

The following table summarizes the equilibrium prices and quantities between autarky and the two government pension systems

	c_y	c_{old}	h_y	h_{old}	w	r	C^s	B	B/C^s
Autarky	A/α_y	A/α_{old}	$1/\alpha_y$	$1/\alpha_{old}$	A	$1-\alpha_{old}/\alpha_y$	$A/\alpha_y + A/\alpha_{old}$	0	0
Pay as you go	$.5A/\alpha_y$	$.5A/\alpha_y$	$1/\alpha_y$	0	A	0	A/α_y	0	0
Fully funded	A/α_y	A/α_y	$2/\alpha_y$	0	A	0	$2A/\alpha_y$	A/α_y	1/2

It is not difficult to show that the utility of each household under the fully funded system is greater than under autarky. For the pay-as-you go system, we need some additional restrictions on the parameters.

The big difference between the two programs is on the amount of work hours by the young, being twice as large in the case of the fully funded system. The reason for this harkens back to our study of the distortionary effects of taxes on labor choices. There we found that taxes on labor income reduces the supply of labor when the tax receipts are returned to the household as a lump-sum tax. This pure wealth effect is what drives the lower young work hours in the case of the pay-as-you go system compared to the fully funded system. The timing of this transfer is not important, as only intertemporal wealth matters. This transfer by making a young household feel wealthy, means that they want to enjoy more leisure when they are young.

The last comment on these policies relates to the size of the government debt in the fully funded system. As the table shows, government debt is 50 percent of GDP in every period. That is a large debt to GDP ratio, and yet there is no risk that the debt will not be paid off in our model. This suggests that it might be somewhat incorrect to call the high debt to GDP ratios in some countries a problem.

Problems

1. Consider the following endowment Overlapping Generations model where people live two periods. Suppose people are endowed with 1-ε units of the good when young and ε units when old. Assume $\varepsilon < 1/2$ Each generation has the same number of people. Preferences are $\ln(c_y) + \beta \ln(c_{old})$ where $\beta<1$

a. Solve out the equilibrium where the government implements a pay as you go social security system with lump sum taxes so that agents consume 1/2 when young and 1/2 when old.

b. Can the government implement a fully funded system that gives the same consumption for young and old as in the pay as you go? Show what happens to government debt if the Government borrows 1 from the first young generation.

c. What should be the government's pay as you go systembe in order for there to be a fully funded system where the government could roll over its debt every period with no change in borrowing every period.

2. Consider the Overlapping Generations model with production studied in the chapter. Namely, Each generation has the same number of people. Preferences are $\ln(c_y) + \alpha_y[1-h_y] + \ln(c_{old}) + \alpha_{old}[1-h_{old}]$. Each agent is endowed with one unit of time in both the first period and second period of his life. Production is given by : $c^s = Ah^d$.

a. Normalize A=1 and calibrate the value of α_y so that in the equilibrium with the pay as you go system young people work .25. (This is the second row in the Summary and Conclusion Table).

b. Suppose $\alpha_{old} = 2\alpha_y$. Verify that people are better off in the pay as you go system compared to autarky?

c. By how much do you have to scale up c_y and c_{old} in the case of no government policy so that people have the same utility as in the pay as you go equilibrium.

3. Use the same Overlapping Generations model with production studied in the chapter with $\alpha_{old} > \alpha_y$. Consider a slightly different pay as you go system. Specifically, suppose the government imposes a tax on young consumption to fund the transfers to the old alive in the period. Suppose the tax on consumption and transfers to the old are such that (1) old people do not work and (2) people enjoy the same amount of consumption when young and when old. Solve for the equilibrium.

Appendix: Overlapping Generations Model- Money

The overlapping generations model has been a workhorse in the study of monetary economics. The reason is related to the results we established before, namely, that autarky is inefficient when the economy is short of borrowers. Young agents want to transfer their income to the second period of their life, but cannot do so if bonds are the only type of assets people can hold. Money provides a second type of asset which can be used to transfer income across periods. It, however, has one big advantage over bonds- namely, that it can be effectively redeemed in the next period. In contrast, a bond cannot. If a young person bought a bond from an old person, they cannot redeem if for goods.

The simplest monetary version of the OG model does not include any leisure in the utility with only young agents endowed with the good so that the endowment pattern is (e_y ,0). Preferences are just

$$U(c_y, c_{old}) = \ln c_y + \ln c_{old}$$

The biggest change is the household's budget constraints. Let m_t^d be the money bought by a generation t household, and let q_t denote the price of money. (Question: can you tell what the relation is between the price of money and the price level?)

When young the household's budget constraint is

$$c_y + s_t + q_t m_t^d = e_y$$

When old, the budget constraint is

$$c_{old} = s_t(1+r) + q_{t+1} m_t^d$$

We can solve for the saving (which is in terms of bonds/loans) in the old period budget constraint and substitute it into the first period budget constraint arrive at the intertemporal budget constraint

$$c_y + \frac{c_{old}}{1+r} = e_y + m_t^d \left(\frac{q_{t+1}}{1+r} - q_t \right)$$

The first thing to establish is that the return to holding money and the return to holding bonds must be equal when money is held by households. The return to saving in the form of a bond is 1+r. Alternatively, you could take 1 unit of output and buy $1/q_t$ units of money. Next period you can sell the money for goods at price q_{t+1}. Hence the return to money is q_{t+1}/q_t. Now if the return to bonds is greater than the return to money, you would never hold money. If you hold money, it must have at least as great a return as bonds. In fact, it must have the same return. Otherwise you could borrow a unit of output, use it to buy money, earn the real return on money, pay of your loan and make a profit. This is called an arbitrage opportunity, and by engaging in this buy low, sell high type of behavior you could make infinite profits. Everyone would try to do this, and so we could never have this happening in equilibrium.

Thus, if people hold money then

$$1+r=\frac{q_{t+1}}{q_t}.$$

Solving for the Equilibrium

If there is no government, then savings (specifically bonds) must be zero. Thus, all savings done by the household must be in the form of money. This means we can drop bonds/savings from the household budget constraints. The budget constrains are thus

$$c_y=e_y-q_t m_t^d$$

and

$$c_{old}=q_{t+1}m_t^d$$

To solve for the equilibrium, we can substitute the two budget constraints directly into the utility function to get

$$U(c_y,c_{old})=\ln c_y+\ln c_{old}=\ln(e_y-q_t m_t^d)+\ln(q_{t+1}m_t^d)$$

This is a simple unconstrained maximization problem over the choice of m_t^d. Taking the derivate and setting his equal to zero implies $\frac{q_t}{e_y-q_t m_t^d}=\frac{1}{m_t^d}$.

With some algebra, we arrive at the money demand equation.

$m_t^d=\frac{1}{2}\frac{1}{q_t}e_y$. Alternatively, the price of money is

$$q_t = \frac{2m_t^d}{e_y}$$

Now in equilibrium money supply equals money demand. If the initial old start by holding M units of money. Also, e_y is total output in the economy, which we shall denote by Y. Then the above equation implies that $q_t = \frac{2M}{Y}$.

Now, returning to the question I posed above, what is the relation between the price of money and the price level? The correct answer is $P_t = \frac{1}{q_t}$. Thus the equation we have above is just a simple version of the quantity theory of money. PM= VY where V is the velocity of money. The equilibrium result is just the quantity theory of money.