Unified Growth Theory

We begin this chapter by combining the Malthus model studied in Chapter 5 with the Solow model studied in Chapter 3 for the purpose of accounting for the key development and growth facts outlined in Chapter 4. After combining the two models and studying its equilibrium properties, we proceed to calibrate it so as to explore to what extent various factors can account for today's huge disparity in international income levels. This is really a question of what factor or set of factors can plausibly account for the fact that some late starters started to grow roughly 250 years after England accomplished this feat.

The combined model studied in this chapter is a derivation of a model put forth by Gary Hansen and Edward C. Prescott in a 2002 article titled *Malthus to Solow*. It is as an example of a very recent advance in the growth literature that goes under the heading of ***Unified Growth Theory***. This literature owes much of its start to the work of Oded Galor and David Weil, both at Brown University, particularly their 2000 paper titled *Population, Technology, and Growth: From Malthusian Stagnation to the Demographic Transition and Beyond*. *Unified Growth Theory* takes a very long historical view of economic development. It seeks to account for the fact that living standards were low and stagnant for most of human history standards but increased dramatically over the last three centuries. In effect, a unified growth model needs to generate an initial period whereby income levels are stagnant followed by a take-off where the economy escapes from the Malthusian era and begins to experience sustained increases in living standards, followed by a modern growth era where the economy appears to be on a balanced growth path with a constant and positive rate of growth of per capita output.

II. The Combined Theory

Let us recall the structures of the Malthus Model and Solow Model. To distinguish the two model variables we will use the subscripts M and S where appropriate. The four equations that describe the Malthus model are

(C-M) $N_t c_t = (1-s)Y_t$

(Y-M) $Y_{Mt} = AK_{Mt}^{\phi} L_{Mt}^{\alpha}[(1+\gamma_m)^t N_{Mt}]^{1-\alpha-\phi}$

(K-M) $K_{t+1} = (1-\delta)K_t + sY_t$

(Pop Gr- M) $N_{t+1} = N_t G(c_t)$

The four equations that describe the Solow Model are:

(C-S) $N_t c_t = (1-s)Y_t$

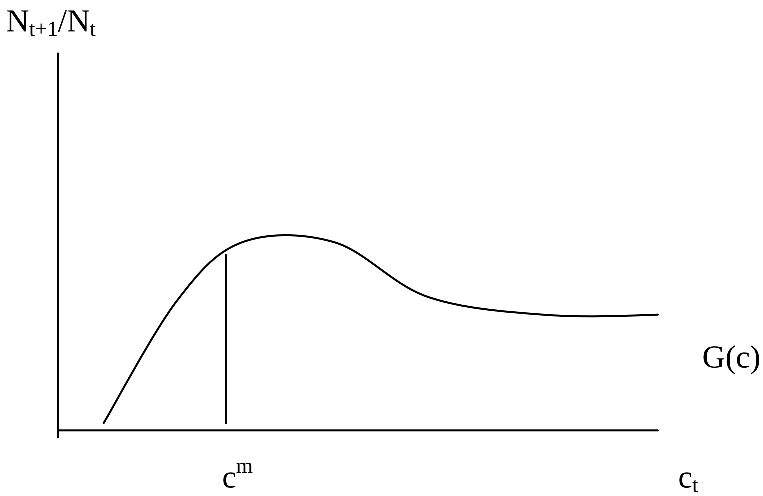
(Y-S) $Y_{St} = AK_{St}^{\theta}[(1+\gamma_S)^t N_{St}]^{1-\theta}$

(K-S) $K_{t+1} = (1-\delta)K_t + sY_t$

(Pop Gr-S) $N_{t+1} = N_t(1+\gamma_N)$

In both models, the behavior that describes aggregate consumption is the same. Thus, in the combined model, it makes sense that the consumption equation be given by (C). A similar logic applies to the capital stock equation (K), and this one will likewise be preserved in the combined model. The production functions, however, are very different. To combine this part of these two models, we simply assume that both production technologies are available for use by firms in the economy in the combined model. This is to say that the economy's total output in any period will be the sum of output produced with the Malthus technology and output produced with the Solow technology. If Y_t denotes total output, then $Y_t=Y_{Mt}+Y_{St}$. Similarly, the total capital will be split between use in the Solow technology and the Malthus technology, and the total population will either work in the Malthus technology or the Solow technology. These equations are $N_t=N_{Mt}+N_{St}$ and $K_t=K_{Mt}+K_{St}$ and they effectively correspond to the labor market and capital market clearing conditions. Notice, that the labor market and capital market clearing conditions implicitly assume that capital and labor are not-specific to the individual technologies.

The population equation is the fourth equation in each model. As can be seen they are different between the models and so we must figure out a way to combine them. For the Malthusian era, the population growth rate function needs to be sufficiently large and must be increasing in consumption. This guarantees a steady state with a constant living standard and an increasing population. For the Solow model, the population growth rate is exogenously given and hence independent of the living standard. This guarantees a balanced growth path with a rising living standard. Since the Malthusian era is assumed to correspond to lower living standards then the modern growth era, a natural way to combine these functions is to have the population growth function be sufficiently large and positively sloped for low living standards and have zero slope for sufficiently large living standards. One such combined function is shown below in Figure 1



The combined model equations are thus:

(C) $N_t c_t = (1-s)Y_t$

(Y_{Mt}) $Y_{Mt} = AK_{Mt}^{\phi} L_{Mt}^{\alpha} [(1+\gamma_M)^t N_{Mt}]^{1-\alpha-\phi}$

(Y_{St}) $Y_{St} = AK_{St}^{\theta} [(1+\gamma_S)^t N_{St}]^{1-\theta}$

(Y) $Y_t = Y_{St} + Y_{Mt}$

(K) $K_{t+1} = (1-\delta)K_t + sY_t$

(K-Mkt) $K_t = K_{St} + K_{Mt}$

(N-Mkt) $N_t = N_{St} + N_{Mt}$

(Pop Gr) $N_{t+1} = N_t G(c_t)$

III. General Equilibrium

We now proceed to study the general equilibrium properties of the model with an eye to determining if the model gives rise to an early period of stagnant living standards followed by a take-off, a so-called industrial revolution, followed by a modern growth era. We begin by showing that there is an initial period whereby only the Malthusian technology is used, and the economy is on its Malthusian Balanced Growth Path provided that TFP in Solow is not very high initially. In the process, we show that there is some date when the Solow technology starts to be used, and that eventually the economy converges to the Solow Balanced Growth Path Equilibrium. Every economy, therefore, will eventually make the transition from Malthus to Solow and hence begin the process of development.

Recall, that in a Malthusian steady state (where there is exogenous technological change), consumption per capita is constant and determined by the population growth function $G(c_t)$ where it equals $(1+\gamma_M)^{\alpha/(1-\alpha-\phi)}$. Additionally, the economy's capital stock and population are given by

$$N_t^M = L(1+\gamma_m)^{t(1-\alpha-\phi)/\varepsilon}\left[\frac{sA}{g+\delta}\left[\frac{sc^M}{(1-s)(g+\delta)}\right]^{\phi-1}\right]^{1/\alpha} \qquad (1)$$

$$K_t^M = \frac{c^M s}{(1-s)(g+\delta)} N_t^M \tag{2}$$

Notice that in the above equations we use the superscript to indicate that these are the Malthusian balanced growth path variables.

Importantly, in the balanced growth path, the rental price of capital and the wage rate are both constant. This follows from the profit maximizing conditions were each factor is paid its marginal product. Namely,

$$w_t = (1-\alpha-\phi)\frac{Y_{mt}}{N_{mt}} \tag{3}$$

$$r_{kt} = \phi \frac{Y_{mt}}{K_{mt}}. \tag{4}$$

Let us suppose that the economy is on this Malthus-only balanced growth path then, and let us consider whether any firm would find it profitable to start producing output with the Solow technology. If it is the case that no firm would want to use the Solow technology, then we have effectively shown that there is an initial equilibrium where only the Malthus technology is used and the economy mimics the Malthusian balanced growth path.

For a firm that is considering using the Solow technology can hire capital and labor at the Malthusian balanced growth path values, w^m and r_k^m. To economize on notation and in anticipation of the calibration component of this chapter, let us write the pure technology component of TFP as Γ_{st} rather than $(1+\gamma_{st})^t$. Then the profits of the firm using Solow are

$$A_s K_{st}^{\theta}(\Gamma_t N_{st})^{1-\theta} - w^m N_{st} - r_k^m K_{st} \qquad (5)$$

Maximizing profits requires differentiating by K_{st} and N_{st} and setting the derivatives equal to zero. These are just the usual marginal cost = marginal product conditions. In particular, they are

$$w_t = (1-\theta)A_s \Gamma_{st}^{1-\theta} K_{st}^{\theta} N_{st}^{-\theta} = (1-\theta)\frac{Y_{st}}{N_{st}} \qquad (6)$$

$$r_t = \theta A_s K_{st}^{\theta-1}(\Gamma_{st} N_{st})^{1-\theta} = \theta \frac{Y_{st}}{K_{st}} \qquad (7)$$

We can use (6) and (7) to solve for the ratio of the rental prices as a function of the labor and capital stocks. This is

$$\frac{r_k}{w} = \frac{\theta}{1-\theta}\frac{N_{st}}{K_{st}} \qquad (8)$$

Equation (8) allows us to solve for the optimal capital input as a function of the labor input. This is

$$K_{st} = \frac{\theta}{1-\theta}\frac{w}{r_k} N_{st}. \qquad (9)$$

We can now substitute for K_{st} in the firm's profits using equation (9).

$$A_s\left(\frac{\theta}{1-\theta}\frac{w}{r_k}\right)^{\theta} N_{st}^{\theta}(\Gamma_{st} N_{st})^{1-\theta} - wN_{st} - r_k \frac{\theta}{1-\theta}\frac{w}{r_k} N_{st} \qquad (10)$$

With a little algebra, this can be rewritten as

$$\left(A_s \left(\frac{\theta}{1-\theta} \frac{w_m}{r_k} \right)^{\theta} \Gamma_{st}^{1-\theta} - w_m - w_m \frac{\theta}{1-\theta} \right) N_{St} \tag{11}$$

The key insight from Equation (11) is that profits are linear in the labor choice. Thus, the profitability of the Solow technology only depends on the sign of the expression in the brackets in Equation (11). The expression in the brackets can be further simplified to

$$A_s \left(\frac{\theta}{1-\theta} \frac{w_m}{r_k} \right)^{\theta} \Gamma_{st}^{1-\theta} - \frac{w_m}{1-\theta} \tag{12}$$

We want to determine the conditions for the expression in (12) to be negative, i.e., .

$$A_s \left(\frac{\theta}{1-\theta} \frac{w_m}{r_k} \right)^{\theta} \Gamma_{st}^{1-\theta} - \frac{w_m}{1-\theta} < 0$$

A necessary and sufficient condition for (12) to be negative is

(Profit <0) $$A_s \Gamma_{st}^{1-\theta} < \left(\frac{r_k}{\theta} \right)^{\theta} \left(\frac{w_m}{1-\theta} \right)^{1-\theta}.$$

If (Profit<0) holds then no firm would want to use the Solow technology. In words, this says that the Solow technology will not be used as long as Solow TFP is sufficiently low.

Equation (Profit <0) has another interpretation, however, that is even more intuitive. To understand this, suppose we were to figure out the cheapest way of producing one unit of output. This is what we define as the per unit production cost. Here we would minimize $r_k K_s + w_m N_S$ subject $Y_S=1$. Solving this cost minimization problem yields the following per unit production cost

$$\left(\frac{r_k}{\theta} \right)^{\theta} \left(\frac{w_m}{1-\theta} \right)^{1-\theta} \frac{1}{A_s} \frac{1}{\Gamma_{st}^{1-\theta}}. \tag{13}$$

This is effectively the expression in Equation (Prof<0) if we divide both sides by the TFP/Technology term on the left hand side. Rewriting (Prof<0) in this way implies that the Solow technology will not be used as long as the cost of producing a unit of output exceeds one unit of output. Recall since the numeraire for our economy is the final good, the per unit production cost is measured in units of output. This makes sense: why would you try to produce one unit of output when it costs more than one unit of output?

To summarize, what we have shown so far is that provided Solow TFP is sufficiently low, the Solow technology will not be used and we have a Malthusian balanced growth path. In effect, we have the first phase of development- a constant living standard and population growth.

The implication of (Profit <0) is even broader. In particular, (Profit < 0) implies that the take-off must eventually occur; an economy will eventually switch to Solow, and hence depart from the Malthusian steady state. The reason for this is straightforward. The right hand side of (Profit <0) is constant as the wage rate and the rental price of capital are constant along the Malthusian balanced growth path. Technology, however, increases in the Solow technology, as it is exogenous. Thus, at some date, T*, the negative profit condition will be violated, and Solow will start to be used. We will refer to this date as the start of an economy's industrial revolution.

What happens after the economy reaches this date, T*? To answer this, it is useful to think what the effect of technological change on the real wage rate is in the Solow model.

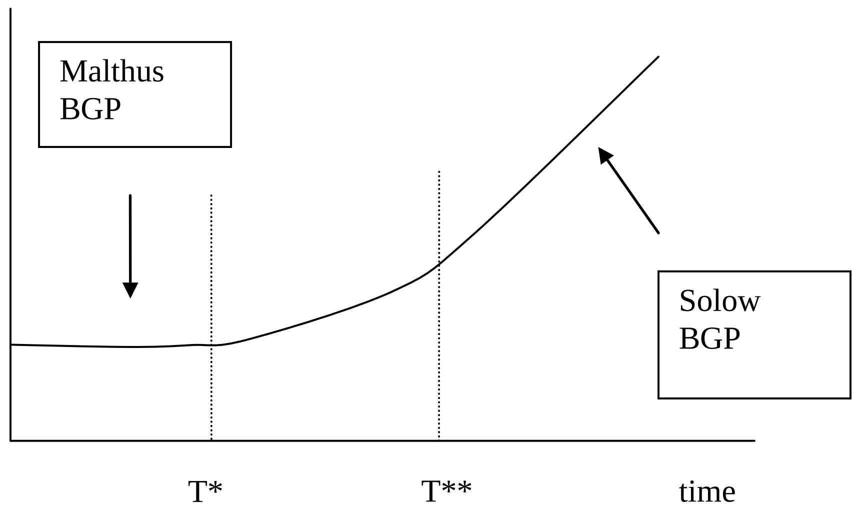
Recall, that in the Solow model, the real wage increases along the balanced growth path. Therefore, in the combined model, once Solow becomes profitable to use, subsequent increases in the technology will cause the real wage rate to increase. Thus, any firm using the Malthusian technology will be faced with paying higher and higher wages over time. Faced with this prospect, they will demand less than the quantity of total labor supplied in the economy in each period. With fewer workers, Malthus firms will not want to hire as much capital either. Consequently, the economy will move more and more resources into the Solow technology over time. In this way, the share of output produced by Malthus firms decreases over time.

Although the economy continues to move resources out of the Malthusian technology, the Malthus technology never disappears. The reason why the Malthus technology never goes out of business is that land has no other value but its use in Malthus production. As its supply is vertical, it must always be fully employed. What this means is that when the price of labor and capital are high, the land rental rate will decrease to a sufficiently low level so that some firms will still want to produce with the Malthus technology.

Despite always being used, its importance in the economy say as measured by the ratio Y_{Mt}/Y_t continues to fall, and in the limit, its importance is essentially zero so that the economy approximates a Solow-only-world, with a balanced growth path. This is the epoch of modern growth. These three phases are shown in Figure 2. T* denotes the period where Solow first becomes profitable to use, and T** is the period where the share of Malthus output in total output is effectively zero. Prior to T*, the economy is on the

Malthusian Balanced Growth Path and after T** the economy is on its Solow Balanced Growth Path.

Figure 2: From Malthus to Solow



In between period T* and period T**, both technologies contribute to total output. One might think that we could easily characterize the equilibrium in this phase of the model economy's development, but there is a set of important equilibrium condition to solve: the split of the entire's economy capital and labor inputs between the two production functions.

Mathematically, this is not a trivial issue. In fact, it is generally not possible to find a closed form solution to this problem. The principle that guides the split of the capital and labor inputs between the two technologies is that the returns must be equated. In particular, the optimal allocations of labor and capital between the technologies must satisfy the following four equations.

$$w_t = (1-\alpha-\phi)A_m(1+\gamma_m)^{t(1-\alpha-\phi)}K_{mt}^{\phi}L_{mt}^{\alpha}N_{mt}^{-\alpha-\phi} \quad (14)$$

$$r_{kt} = \phi A_m K_{mt}^{\phi-1}L_{mt}^{\alpha}[(1+\gamma_m)^t N_{mt}]^{1-\alpha-\phi}. \quad (15)$$

$$w_t = (1-\theta)A_s\Gamma_{st}^{1-\theta}K_{st}^{\theta}N_{st}^{-\theta} \quad (16)$$

$$r_{kt} = \theta A_s K_{st}^{\theta-1}(\Gamma_{st}N_{st})^{1-\theta} \quad (17)$$

In addition, the split must equal the aggregate quantities. Namely,

$$K_{st} + K_{mt} = K_t \quad (18)$$

$$N_{st} + N_{mt} = K_t. \quad (19)$$

Equations (14)-(19) represent six equations in six unknown variables. Thus, the rental prices, and allocations can be solved using these six equations. However, it is not possible to solve explicitly for the optimal allocations in general. The problems at the end of the chapter include one specific case where it is possible.

IV. Calibration

Having characterized the equilibrium path, we proceed to calibrate the model economy with the purpose of answering the question: What accounts for the differences in starting dates across countries. Recall, the England started to experience sustained increases in per capita output around 1750. Some countries such as India and China did not start until 1970s whereas others primarily located in sub-Sahara Africa are only now showing some signs of starting. Can our combined model generate 250 year delays in the start of growth? This is Step # 1 in our calibration procedure. The other steps are as follows:

Step #2 is trivial. Our choice of measuring device is the combined model. We have developed this model because it gives rise to a Malthusian era characterized by population growth and stagnant living standards, a take-off and an eventual convergence to a modern growth era described by the Kaldor facts. It is also consistent with the observation that growth miracles are a recent phenomenon and limited to late starters.

Step #3 involves reorganizing the national income and product accounts so that the actual data can be compared with the data generated by our model. The necessary adjustments are the same as those we made in Chapter 3 when we calibrated the Solow model and in Chapter 5 when we calibrated the Malthus Model. Thus, we can proceed directly to Step#4.

Our strategy in restricting the model parameters is to match the main development observations of the leading country since 1200 AD. Specifically, from 1200 to 1750, we will treat the leader as being on its Malthusian balanced growth path. We date the start of the industrial revolution in the leader, (which corresponds in the model to when the Solow technology becomes profitable to use) to be 1750. Finally, we treat the leader as being on its Solow balanced growth path in the latter part of the 20th century. What this means is that parameters associated with the Malthusian part of the model can be calibrated to the pre-1700 observations, and the parameters associated with the Solow side of the economy calibrated to the 20^{th} Century US growth facts. These parameter values are those that we used in Chapters 3 and 5. For the Malthus side of the economy, the parameter values are

Table 1: Malthusian Technology Calibrated Parameters

parameter	value	Comment/observation
α	¼	Historical estimates of land's share of income
ϕ	1/12	Historical estimates of labor's share of income
A_m	1	Normalization; choice of units
L	1	Normalization; choice of units
Γ_m	.001	Consistent with observed world population growth of .3 percent per year

With respect to the Solow Production function in Chapter 3 we assigned parameters so that it matched the post-1900 steady state observations of the United States and the industrial countries. The parameters and their calibrated values are shown in Table 2.

Table 2.a: Solow Model Calibrated Parameters from Chapter 3

parameter	value	Comment/observation
θ	1/3	Capital's post-1900 share of Income in US
δ	.05	US post-1900 depreciation
A_s	1	Normalization; choice of units
s	.25	US Savings Rate post-1900
γ_s	.02	Post-1900 growth rate of per capita output in US
γ_n	.01	Post-1900 population growth rate in US

There are a couple of issues or re-parameterizations that we need to make on account that we now have a model whereby both the Solow and Malthusian production technologies are available. The first issue relates to the rate of technological change, which determines Γ_{St}. It made sense in the Solow only model of Chapter 3 to assume that $\Gamma_{St} = (1+\gamma_s)^t$, namely that technology increased at a constant rate, because the growth rate of per capita output in the United States in the 20th century was more or less constant at 2 percent per year. However, growth of per capita output in the Industrial leader was not so robust between the time growth started sometime in the 18th century and the time modern economic growth started around 1900. Thus, it is not reasonable to assume that the growth rate of Γ_{St} was either constant or as large over the 1750 to 1900 period. For this reason we assume that the following pattern of the growth rate of Γ_{St}:

$$\Gamma_{St} = \begin{cases} (1+\gamma_m)^{t-1200} & 1200 < t \leq 1750 \\ \Gamma_{s1750}(1.005)^{t-1750} & 1750 < t \leq 1800 \\ \Gamma_{s1800}(1.01)^{t-1800} & 1800 < t \leq 1850 \\ \Gamma_{s1850}(1.015)^{t-1850} & 1850 < t \leq 1900 \\ \Gamma_{s1900}(1.02)^{t-1900} & t > 1900 \end{cases}$$

In words, this is just a piecewise linear function increasing by .5 percentage points between 1750 and 1900, thereafter which it is constant at 2% per year. Given this piecewise linear function, the actual value for Γ_{St} over time are shown in Table 3.

Table 3: Solow Technology Levels

Year	Γ_{st}
1200	1.0
1700	1.64
1750	1.73
1800	2.21
1850	3.64
1900	7.66
1950	20.64
2000	55.96

Another issue in the recalibration is what value to assign for the TFP parameter, A_s, associated with the Solow production function. For the Solow only model, we were free to normalize this to 1. However, in the combined model we cannot do this because we normalized the Malthusian TFP parameter, A_m, to 1. Given the importance of the Solow

TFP parameter and the level of technology in determining the date a country starts to realize increases in per capita income as shown by equation (Profit <0), we can assign a value to A_s so that economic growth in England begins in 1750. This is left for you to do as a homework assignment. Its value is not important for Step 5 of the calibration.

Another issue is what savings rate to use in the combined model. Recall in our study of the Malthus only model we pointed out that savings rates were probably a lot lower in the pre-1700 era. Dean and Cole page 260 and 266 report an investment to GNP ratio in 1688 for England between 6 and 8 percent. We could certainly assign a time varying value for the savings rate, at some constant low value between 1200 and 1700 and then have it increase to 25% by 1900. However, this is more work than we want to do, and for this reason we simply assume that s=.25 for the entire model period.

The last thing that we must do is specify the population growth function and assign parameter values to this function. We have the calibrated first part of the population growth function with a positive slope from Chapter 5. This is

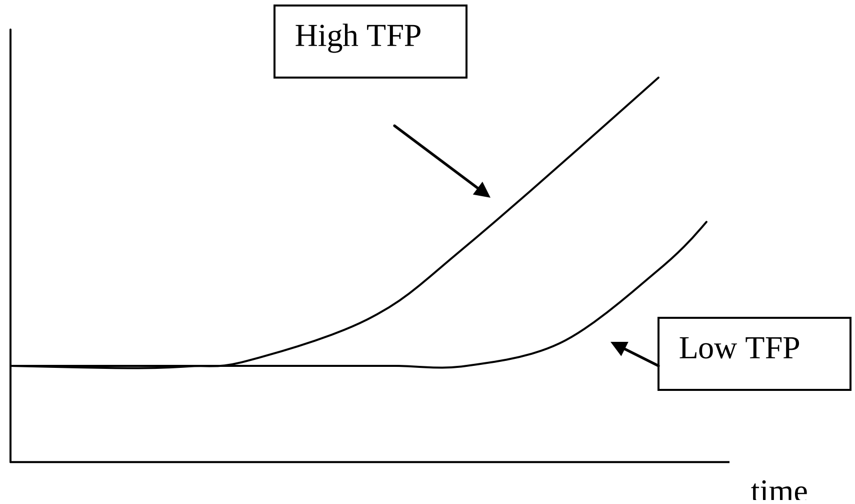
$$\frac{N_{t+1}}{N_t} = .99 + .004 \ln c_t$$

Given the observed population growth rate before 1700 of .3% per year, the above equation allows us to solve for the Malthusian steady state consumption, c^m. In the combined model, we want our growth function to have a slope of zero with N_{t+1}/N_t exactly equal to 1.01 for large enough living standards. The living standard in the industrial leader increased by roughly a factor of 16 from 1750 to 1900, which suggests that the population growth function have a zero slope for $c_t > 16c^m$. For $0 \le c_t \le 2c^m$ we shall assume that the $G(c_t)$ is given by $\frac{N_{t+1}}{N_t} = .99 + .004 \ln c_t$. As $G(c_t)=1.01$ for $c_t > 16c^m$, we only need to specify and parameterize the population growth function for $2c^m < c_t < 16c^m$. In this range we will assume that the $G(c_t)$ function changes linearly from $G(2c^m)=.99+.004\ln(2c^m)$ to its value at $G(16c^m)=1.01$.

V. Differences in Starting dates

We are now ready to answer the question we posed: Can the model give rise to differences in starting dates of the order of 250 years or more? Equation (Profit <0) points to differences in TFP in the Solow technology, A_s, as potential source of these differences in starting dates. The starting date, as you may recall, occurs in the first period when Solow TFP reaches a critical value that corresponds to the cost of producing a unit of output. Obviously, if one country has a high TFP, and the other a low TFP, they will reach their starting dates, with the lower TFP country reaching it later in time. This is represented in Figure 3.

Figure 3: Different Starting Dates



We will now examine the consequences of different values for A_s for starting dates in the calibrated model of the last chapter. In particular, we will determine the difference in TFP between an early and late starter needed to generate a 250 year difference in starting dates. In this analysis, we will assume that the savings rates and the population growth rates of countries are the same. As long as this is the case the right hand side of equation (Profit<0), which defines the critical value when growth begins is the same for both early and late starters. This being the case the difference, the early starter and late starter's starting dates satisfies

$$A_s^{UK}\Gamma_{S1750}^{1-\theta} = A_s^{P}\Gamma_{s2000}^{1-\theta}.$$

Thus, the required difference in TFPs is $\frac{A_s^{UK}}{A_s^{Poor}} = \left(\frac{\Gamma_{s2000}}{\Gamma_{s1750}}\right)^{1-\theta}$.

To determine quantitatively the difference in TFPs, we simply need to plug in the value of the capital share parameter for the Solow production function and the technology level at 1700 and 1950. The calibrated value of the capital share parameter is $\theta = 1/3$. The technology level at various points was determined in Table 3.

It is now straightforward to determine the relative differences in TFP between a 1750 and 200 starter. This is

$$\frac{A_s^{UK}}{A_s^{Poor}} = \left(\frac{\Gamma_{s2000}}{\Gamma_{s1750}} \right)^{1-\theta} = 32.3^{2/3} = 10.27$$

We could also explore how differences in savings rates, particularly with respect to the pre-1700 era would affect starting dates. These will affect the right hand side of the (Profit<0) condition. This is left for you as an assignment in the back of the Chapter. In the next chapter, we will examine other evidence that will suggest that these implied differences in Solow TFP's or implied differences in savings rates offer a plausible explanation for these differences in starting dates. Thus, in a sense we must wait until the next chapter to answer our question.

VI. Other Key Growth and Development Facts:

Before we conclude this chapter, it is informative to examine how this model accounts for some of the other development and growth facts we documented in Chapter 4. Obviously accounting for the huge differences in starting dates is a major test of the theory. In addition, a useful theory of international income level differences must account for why some countries (Latin America in particular) failed to catch-up 100 years after starting modern economic growth. Additionally, it must account for the fact that some countries (the South East Asian Countries) have caught up or eliminated much of their income gap with the industrial leader. Some of the latter countries in the process have experienced a growth miracle. These miracles are relatively recent phenomenon and limited to poor countries at the time their miracles began.

The case of Latin America is easy to explain within the context of the model. The model does not predict any convergence of income levels between late starters or earlier starters. The properties of the model are such that each country will eventually be on its balanced growth path associated with the Solow-only model. In that model, differences in TFP translate into permanent differences in income levels.

The case of many Asian countries necessitates that we allow for increases in these countries' TFPs over time, in particular the A_s component. We know that in the Solow model an increase in TFP causes a transition to a higher balanced growth path. Associated with this transition is a period of rapid growth. This is the interpretation of a growth miracle in the context of the unified growth theory. The model is consistent with the fact that growth miracles are limited to countries that were initially poor at the time their miracles began because a growth miracle requires a large increase in a country's TFP. A large increase in TFP can only occur in a poor country with a currently low efficiency parameter. This rules out a rich country, which by definition has a high TFP

The theory is also consistent with the fact that growth miracles are a relatively recent phenomenon. Growth miracles are a relatively recent phenomenon because, as Figure 1 shows, differences in relative incomes between the low-efficiency and high-efficiency countries widen over time before leveling off. This widening is due to growth in the stock of pure knowledge associated with the modern production function, which the high-efficiency country uses from a very early date. Thus, as one goes back in time, the gap that a low-efficiency country could close by becoming a high-efficiency country becomes smaller and smaller. Obviously, if the gap is less than 50 percent, the low-efficiency country could never double its income in less than a decade. For the same reason, the unified theory is consistent with the fact that late starters have been able to double their incomes in far shorter times compared to early starters. The gap between rich and poor starters has increased over time. There is more knowledge out there now than before. A country that increases its TFP can grow faster than any before it on account that the gap has widened.

VII. Conclusion:

Unified Growth Theory seeks to understand that for most of human history living standards were low and constant. Then starting in the 1700 this all changed. Countries took-off and began to experience sustained increases in their living standards. The combined Malthus to Solow model generates this pattern: initially, there is a period of constant living standards and population growth; next is a transition period whereby per capita output begins to grow; lastly there is a modern growth period whereby per capita

GDP grows at a constant rate. The parameters can be assigned so as to replicate the exact historical experience of England. The theory predicts that Solow TFP between late starters and early starters must differ by a factor of 10 to account for a 250 year difference in starting dates. What remains to be done is to examine whether this difference is plausible. This is the subject of our next chapter.

References

1. Dean, Phyllis and W.A. Cole. 1969. British Economic Growth 1688-1959. New York: Cambridge University Press.
2. David N. Weil & Oded Galor, 2000. "**Population, Technology, and Growth: From Malthusian Stagnation to the Demographic Transition and Beyond**," American Economic Review, American Economic Association, American Economic Association, vol. 90(4), pages 806-828, September.
3. Gary D. Hansen & Edward C. Prescott, 2002. "**Malthus to Solow**," American Economic Review, American Economic Association, American Economic Association, vol. 92(4), pages 1205-1217, September.

PROBLEMS:

1. Complete the following table for the Malthus to Solow model we calibrated in class. For each date, determine the relative TFP of the early starter to late starter that gives rise

Starting Date	A^{1750}/A^{Late}
1750	
1800	
1850	
1900	
1950	
2000	

2. Repeat Question 1, except this time use a capital share θ= .50.

Starting Date	A^{1750}/A^{Late}
1750	
1800	
1850	
1900	
1950	
2000	

3. Suppose that the TFP parameter, A_s, in the Solow production function is the same across countries. What differs across countries is their savings rate, s. For the 1750 starter its savings rate is .25. Find the savings rate for a country that switches to Solow in 2000. Use the calibrated parameters from class. Also, assume the same population growth function holds in the 1750 starter and the 2000 starter. Repeat the calculation assuming that θ=1/2.

4. For the calibrated model parameters, find the English TFP parameter, A_s, that would imply a 1750 take off.

5. For this problem, assume that α+ϕ=θ.

a. Suppose that the economy has reached T* so that Solow becomes profitable to use. Given a population N_t and a total capital stock K_t solve for the optimal allocation of K_{Mt}, K_{St}, N_{St}, and N_{Mt}. (Hint: start with Equations (16) and (17) and solve for $(N_s / N_M)^{\theta} = A_s K_s^{\theta} /(A_m K_m^{\phi} L^{\alpha})$. Next use (16)-(19) to show that

$(N_s / N_M) = (\phi / \theta)(K_s / K_m)$. From here you can solve for the K_m, and once you have done this, you can solve for the remaining variables.

b. Verify that if and that the growth rates of the Malthusian technology and the Solow technology are equal in each period then K_{Mt} will be constant every period after T*, and will decrease if the growth rate of the Solow technology exceeds that of Malthus.