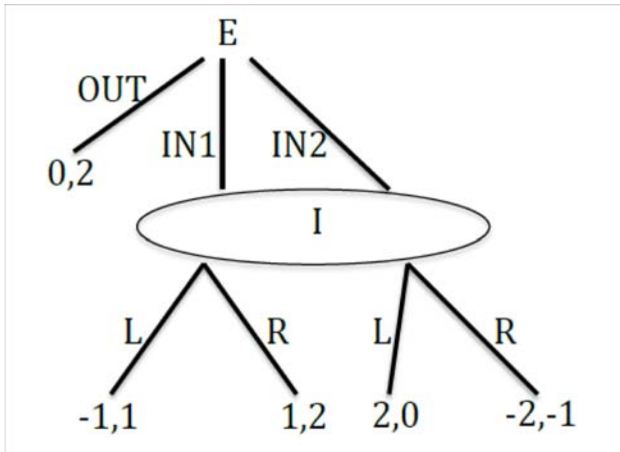
Example 60 *Here's a problem that was on the 2014 midterm: Determine all weak perfect Bayesian-Nash equilibria of the following game.*



Let μ denote the probability that I assigns to being at the left hand node in his information set. We first determine the values of μ that support I's choice of L over R:

$$L : 1 \cdot \mu + 0 \cdot (1 - \mu) = \mu$$
$$R : 2\mu + -1 \cdot (1 - \mu) = 3\mu - 1$$
$$L \text{ is preferred to } R \text{ iff } \mu > 3\mu - 1 \Leftrightarrow 1 > 2\mu \Leftrightarrow \mu < \frac{1}{2}$$

We first try to construct an equilibrium in which I chooses L. This leads E to choose IN2, in which case $\mu = 0$. We have our first WPBNE: E chooses IN2, I chooses L, $\mu = 0$.

We next try to construct an equilibrium in which I chooses R. In this case, E chooses IN1, resulting in $\mu = 1$. We therefore have a second WPBNE: E chooses IN1, I chooses R, $\mu = 1$.

We consider $\mu = \frac{1}{2}$ so that I is indifferent between L and R. Let σ denote the probability that I chooses L. If E is to choose IN1 and IN2 with positive probability, we must have

$$\begin{aligned} IN1 &: -\sigma + (1 - \sigma) = 2 \cdot \sigma + -2 \cdot (1 - \sigma) : IN2 \\ IN1 &: -2\sigma + 1 = 4\sigma - 2 : IN2 \Leftrightarrow \\ \sigma &= \frac{1}{2} \end{aligned}$$

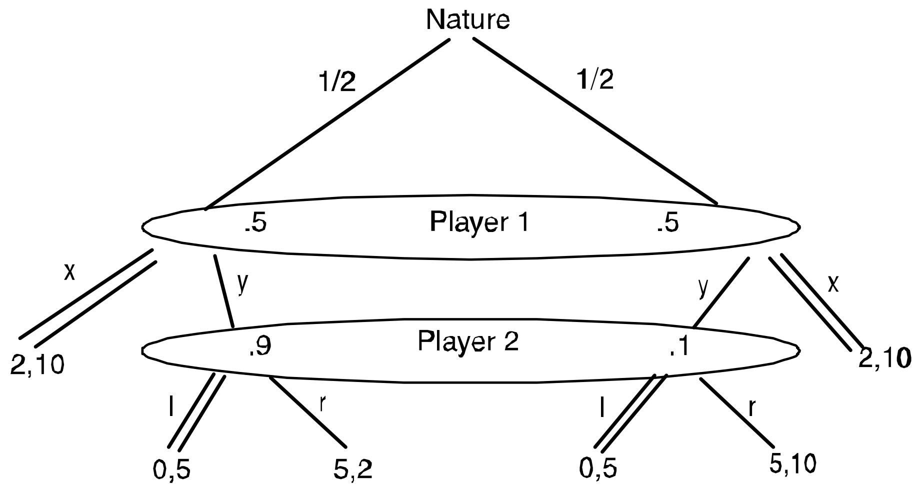
With σ, E is indifferent among OUT, IN1 and IN2, as each gives him an expected payoff of zero. We thus have a family of WPBNE: For any $\varepsilon \in [0, 1]$, E plays OUT with probability $1 - 2\varepsilon$ and IN1 and IN2 each with probability ε. We have $\mu = \frac{1}{2}$, and I plays each of L, R with probability $\frac{1}{2}$.

Are there any more WPBNE? We're still working in the case of $\mu = \frac{1}{2}$. For $\sigma \neq \frac{1}{2}$, one of IN1, IN2 produces a produces a positive expected payoff for E and the other produces a negative expected payoff. I's best response is therefore exactly one of IN1, IN2, which contradicts $\mu = \frac{1}{2}$. We are therefore done.

Problems with the Weak Perfect Bayesian Equilibrium Concept

Off the equilibrium path beliefs – should they be "sensible" in some way?

Example 61 *9.C.4*



The move of Nature determines the payoff to Player 2 along the equilibrium path y,r. This is a game of incomplete information in which neither player knows with certainty the payoffs of 2.

Let's verify that we have a WPBNE:

l vs. r for player 2:

$$5 = (.9)(5) + (.1)(5) > (.9)(2) + (.1)(10) = 2.8$$

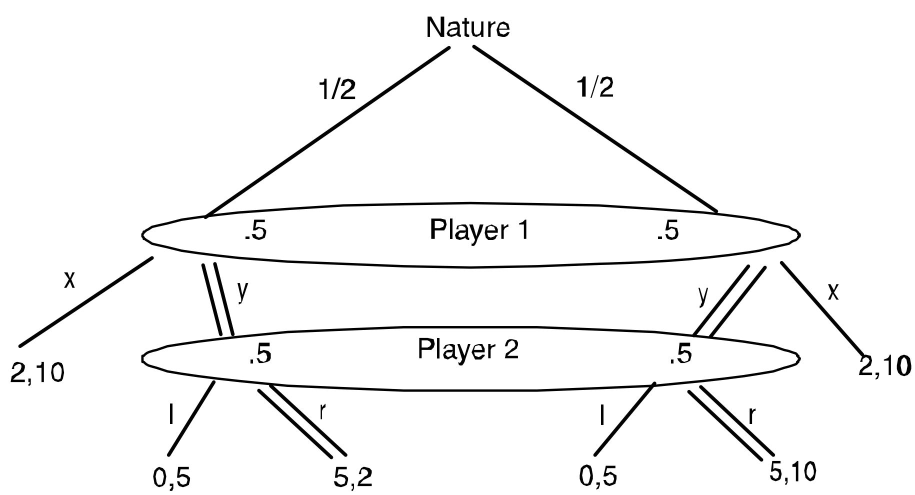
Therefore, l is best for 2 given his beliefs. Player 1 has a choice between 2 from choosing x and 0 from choosing y; x is clearly his best response.

The problem is that WPBNE does not restrict beliefs off the equilibrium path, even though those beliefs are typically important in sustaining equilibrium (i.e.,insuring that players' choices are best responses).

Clearly, we could perturb the beliefs of player 2 without breaking this equilibrium (5 > 2.8).

One might expect in this example that, knowing the move of Nature, 2 should assign equal probability to each of the nodes in his information set (notice that 1 must make the same choice at each of his nodes). This doesn't follow from Bayes Rule, however; 2's information set is reached with probability zero in this equilibrium.

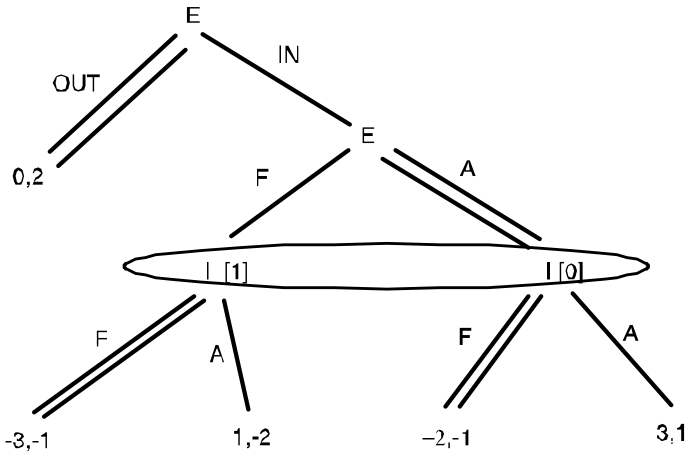
Let's concoct another equilibrium:



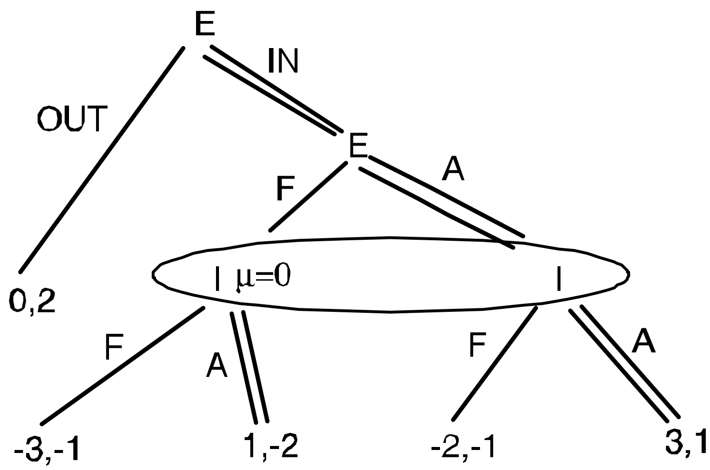
What principle supports this equilibrium over the first one discussed for this game? *We don't just want to be correct in particular examples, we want principles that explain the right answers (that's what game theory is about). Remember that the "weak" in "weak perfect Bayesian" refers to the lack of restrictions on off-the-equilibrium path beliefs. We're headed toward restricting these beliefs in a suitable way.*

Example 62 *9.C.5 A WPBNE need not be subgame perfect.*

If the entrant enters, then each firm simultaneously chooses F or A.



This is a WPBNE that is not SPNE (not a NE in the subgame following entry: I's choice at his information set is not a best response to E's choice of A). The problem is that Firm I's beliefs at its information set are unrestricted because this is off the equilibrium path.



Another WPBNE: I chooses F over A if

$$\begin{aligned} -1(\mu) - 1(1-\mu) &\geq -2(\mu) + 1(1-\mu) \\ -2 &\geq -3(\mu) \\ \frac{2}{3} &\leq \mu \end{aligned}$$

So we'll assume $\mu \leq \frac{2}{3}$. E will therefore choose In and A, and the equilibrium is completed by setting $\mu = 0$.

Strengthening the Weak Perfect Bayesian Solution Concept

Definition 63 *(Kreps and Wilson) A WPBNE (σ, μ) is a **sequential equilibrium** if there exists a sequence of completely mixed strategies $\left(\sigma^k\right)_{k=0}^{\infty}$ such that*

$$\lim_{k\to\infty} \sigma^k = \sigma$$

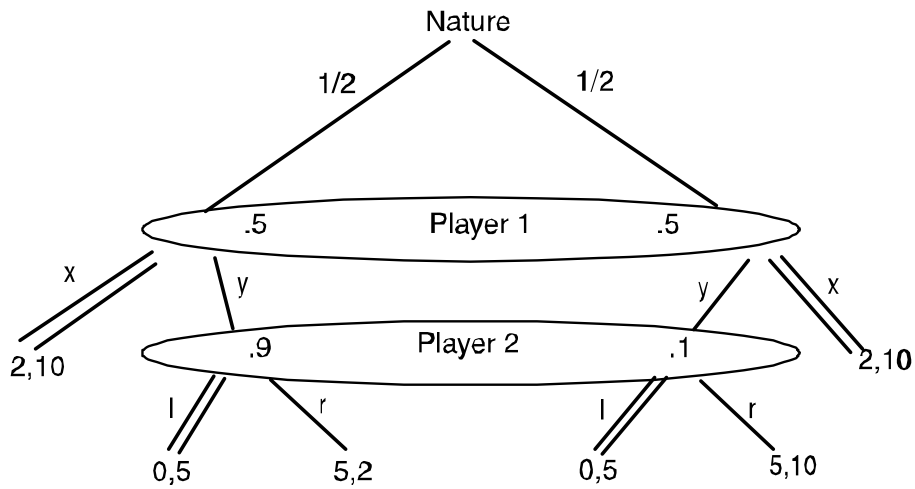
and

$$\lim_{k\to\infty} \mu^k = \mu$$

where $\left(\mu^k\right)_{k=0}^{\infty}$ denotes the beliefs derived from $\left(\sigma^k\right)_{k=0}^{\infty}$ using Bayes Rule.

In other words, we introduce completely mixed strategies so that Bayes Rule surely applies.

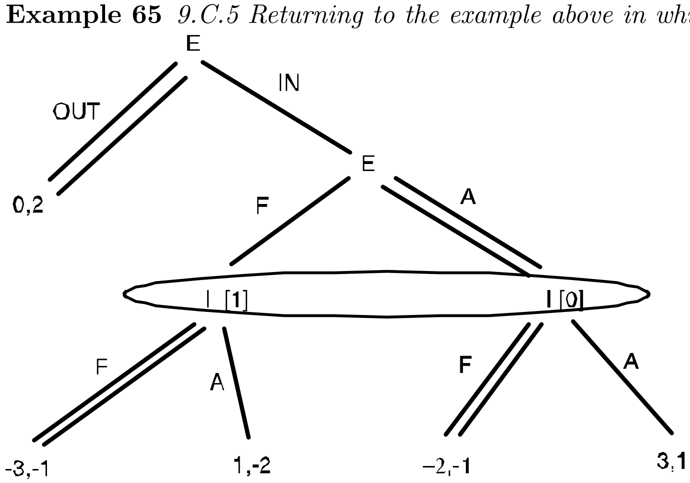
Example 64 *9.C.4 Off the equilibrium path beliefs – should they be "sensible" in some way? Any sequential equilibrium assigns equal probability to each node in each agent's information set.*



Supppose Player 1 plays y with probability $\varepsilon > 0$. Then Player 2 must assign equal probability to each node. Consequently, the above equilibrium can't be sequential.

In a sequential equilibrium, 2 must play r and 1 must play y, with probability .5 assigned to each node in each information set. The above argument determines Player 2's beliefs at his information set. Once these beliefs are determined, it is easy to complete the equilibrium by determining the players' choices.

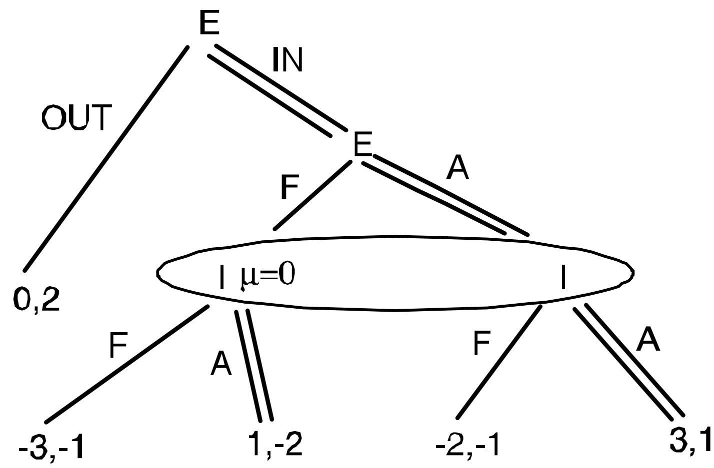
Example 65 *9.C.5 Returning to the example above in which the WPBNE need not be subgame perfect.*



Suppose E chooses In with probability $\delta > 0$ and F with probability $\varepsilon > 0$. Then I must assign probability ε to the left hand node and $1 - \varepsilon$ to the right hand node. In the limit, we have I assigning probability 1 to the right hand node. The above equilibrium is thus not a sequential equilibrium.

Sequential equilibrium is enough to insure that I's beliefs must be consistent with E's strategy after entry. As above,

I chooses A over F if $\mu \leq \frac{4}{5}$.



E will therefore choose In and A, and the equilibrium is completed by setting $\mu = 0$.
Unique sequential eq:
E: (In, Accomodate if in)
I: (Accomodate if E plays "in")

Notice that a sequential equilibrium is necessarily a SPNE:

Proposition 66 *A sequential eq. is necessarily a subgame perfect Nash equilibrium*

Thus, sequential equilibrium strengthens both subgame perfection and weak perfect Bayesian Nash equilibrium

Behavioral motivation for sequential equilibrium? It seems to work, but why is it the right way to refine WPBNE?

9.D. Reasonable Beliefs and Forward Induction

Backward induction (and subgame perfection) models a person who anticipates *future* rational consequences to his actions (i.e., evaluating his choices, he presumes best responses in the future by himself and his opponents, or Nash equilibrium). *Forward induction* concerns the sensibleness of a player's actions and beliefs based upon the *preceding* moves in the game (i.e., a player reasons about what could rationally have happened in the past). This notion is ill-formulated; we aren't going to end up with a definition here that wraps everything up nicely. Instead, we'll identify a set of problems or puzzles that game theorists are still trying to resolve. In this sense, it's a bit like the centipede game: we may have an answer for the game, but it just doesn't seem to be right.

Forward induction addresses the assumption of knowing another player's strategy and knowing that he will stick to it throughout the play of the game. How do players come to know each other's strategy before the game starts? This doesn't model the way play unfolds over time in many situations. This has bothered many of you so far, but it has proven difficult to formalize a theory of games without this assumption.

Nash equilibrium over and above rationalizable: correctness of beliefs about opponents' choices.

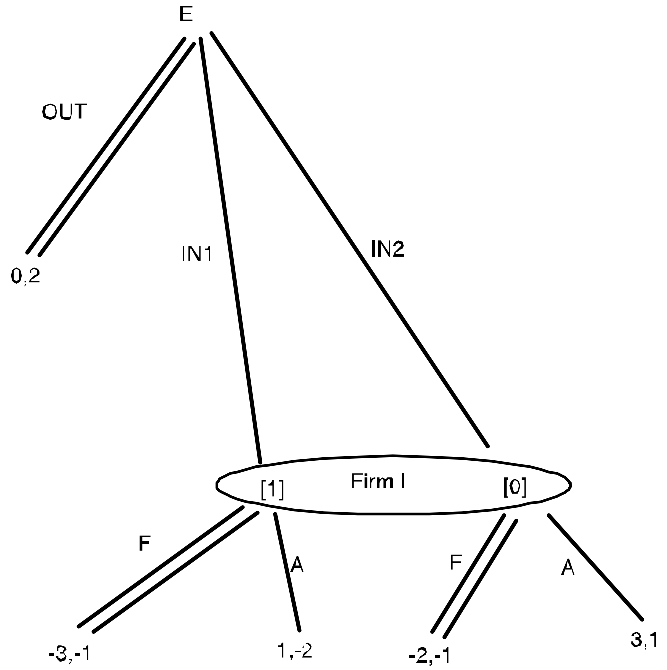
The issue in both of the following examples is off the equilibrium path beliefs, namely I assigning positive probability to E playing a strictly dominated strategy off the equilibrium path.

Example 67 *9.D.1 a*

This is a weak perfect Bayesian equilibrium. In fact, it is a sequential equilibrium. But consider Firm I's beliefs.

If Firm I finds himself at the information set, he is "certain" that he is at the left hand node. This justifies the choice of F. The strategy In2, however, strictly dominates In1 for E. If Firm I finds himself in the position of having to move, shouldn't he presume that E has put him in that situation by choosing In2? If he does, of course, then he would choose A, in which case E does in fact choose In2.

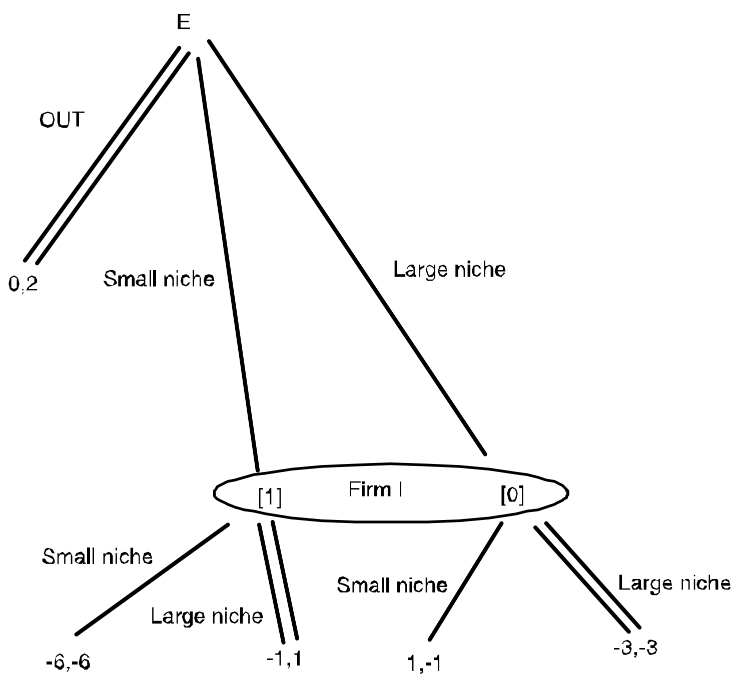
Notice that there is another sequential equilibrium: E chooses In2, Firm I is certain that he's at the right hand node and chooses A. Forward induction thus serves as a principle that helps us to select one sequential *equilibrium as more believable than the other.*



9.D.1b. The Incumbent I here chooses "large niche" because he is certain that he is at the left hand node. This causes the entrant to choose OUT.

Here, "small niche" is strictly dominated for E by OUT. Suppose I finds himself at his information set and has to move. Why would he believe that E chose "small niche"? "Large niche" is not dominated for E by OUT, and so I might conclude that E choose "large niche" in the hope or expectation that I would then choose "small niche".

We have an alternative sequential equilibrium: E chooses "large niche", and I, knowing he's at the right hand node, chooses "small niche". Forward induction perhaps serves as a refinement that helps us to choose this second sequential equilibrium as more reasonable or believable to the outside observer. Firm I acts assuming that E acted rationally to start the game.



With forward induction, a player chooses a move based upon his analysis of the preceding play of the game, and his assumption that all prior moves have been rational. Forward induction may not be so convincing, however, if there is simply a possibility that players make mistakes.

Text: **"Clearly, the issues here, although interesting and important, are also tricky."**

A Digression on Risk Aversion

Question: We've been dealing with mixed strategies and beliefs in games. How do we incorporate risk aversion?

Let's begin by backtracking a moment. If there is no uncertainty in the game, then the possible outcomes needn't have numerical utility values for the players. We can analyze choices simply by assuming that the players have preferences over the outcomes. This is useful (for instance) in political examples in which the outcomes are selections of candidates.

Example 68 *Two voters (1 and 2) will choose among 3 alternatives (A, B, and C) by successively eliminating choices: 1 goes first and eliminates A, B, or C, and 2 then eliminates one of the two remaining choices, which determines the winning choice. We can draw this as an extensive form game, and we can analyze the game by backwards induction if each player has complete and transitive preferences over the choices (e.g., A>B>C). Backwards induction can be complicated by multiple equilibria if either player is indifferent among some outcomes, but strict preferences are not required to analyze the game by backwards induction.*

We need numerical values of utility, however, if we are to calculate expected return. Mixed strategies and nontrivial beliefs at information sets require a utility representation of preferences. Risk aversion is incorporated by the utility assignments to outcomes (specifically, the utility received from money).

When we consider a particular game with numerical payoffs given for the various outcomes, risk aversion has already been incorporated in the numbers assigned to the outcomes. Changing risk preferences, or examining the effect of risk aversion, requires changing the payoffs associated with the outcomes of the game.

Throughout our discussion of game theory, we have assumed that players know the structure of the game that they are playing. If there is complete information, then this effectively means that each player knows the "risk preferences" of his opponents. That is, if there are monetary payoffs in the game and complete information, then each player knows the utility that any other player assigns to the different monetary outcomes of the game. This is a big assumption!

Example 69 *Matching Pennies with a Risk Averse Opponent. We'll assume that player 1 wins when the coins match and 2 wins when they don't. 1 is assumed to be risk neutral and 2 is risk averse. We thus have the following table for the game. The utility function of player 2 determines particular values of ε, δ. Given these values, we could determine the equilibria of the game.*

$1\backslash 2$	H	T
H	$1, -1-\delta$	$-1, 1-\varepsilon$
T	$-1, 1-\varepsilon$	$1, -1-\delta$