Finite Repetition of a Game

stage game: the game that is played repeatedly

n stage game: repetition of the game n times, with payoffs accumulating. For now, we won't address discounting.

Example 51 *Example from Friedman's book (p. 78-79), "Game Theory with Applications to Economics". Notice the large number of strategies and the increase in the number of Nash equilibria with repetition.*

Theorem 52 *If a game has a unique Nash equilibrium, then its finite repetition has a unique SPNE*

Proof by induction on the number n of repetitions. Consider the n stage game.

1. The case of $n = 1$ is trivial.
2. In the nth repetition, consider each $n - 1$ stage game that follows as a subgame determined by the outcome of the first play of the game. Subgame perfection requires a subgame perfect NE in each subgame, and the induction hypothesis therefore uniquely characterizes the play of the game in stages 2 through n (regardless of the outcome of stage 1).
3. The play of the game in stage 1 does not alter the play of the game in all subsequent stages. Consequently, the assumption of Nash equilibrium for the n-stage game requires the play of the unique Nash equilibrium in the first stage.

Notice that the result concerns not only the equilibrium path, but also the specification of the strategies off the equilibrium path. As we'll see in the example from Friedman's book that follows below, there can be other Nash equilibria that are not subgame perfect and that share the same equilibrium path as the unique subgame perfect equilibrium.

Note also the independence of history that is described by the theorem. Long-term relationships ought to be different, but modeling an enduring relationship as a finite repetition of a game may fail produce interesting or different results. This is one of the reasons that so much of dynamic game theory focuses on infinitely repeated games.

Example 53 *prisoner's dilemma (p. 236)*

1/2	DC	C
DC	$-2, -2$	$-10, -1$
C	$-1, -10$	$-5, -5$

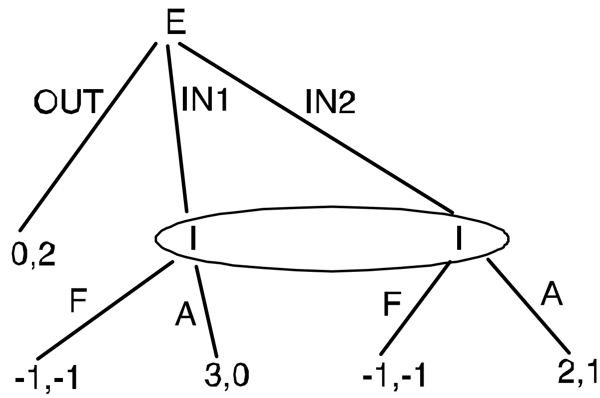
Example 54 *Battle of the Sexes*

1/2	B	F
B	$2, 1$	$0, 0$
F	$0, 0$	$1, 2$

Consider playing this 5 *times. Note the multiplicity of subgame perfect Nash equilibria.*

9.C. Beliefs and Sequential Rationality

Example 55 *Entrant has two strategies in addition to entry and the incumbent does not observe which one has been selected*



Two pure strategy Nash equilibria:

Out, Fight if entry occurs

In1, Accomodate if entry occurs

There is only one subgame of the game (the entire game itself), and so subgame perfection is ineffective as a refinement.

Bayesian Hypothesis: *A person has probabilistic beliefs about anything and everything that he does not know.*

sequential rationality given system of beliefs

Definition 56 ***weak perfect Bayesian equilibrium****: actions consistent with beliefs, and beliefs consistent with actions (whereever the actions restrict the beliefs)*

weak: any beliefs can be assigned at an information set that is reached with zero probability

It is important to understand the following perspective of game theory: we include all possible outcomes and behavior as "equilibrium" behavior unless we have some principle for ruling it out. Here, for instance, we allow a player to "rationalize" his choices with any beliefs that he may possibly concoct, so long as the beliefs are consistent with the equilibrium behavior in the game. Unless we have a principled argument for ruling out certain behaviors, we'll keep them around as possibilities.

Example 57 *Returning to the above example, let μ denote the probability that the Incumbent assigns to being at the left hand node. WPBE means that, conditional on choosing In1 or In2, μ is the probability that E chooses In1, and $1-\mu$ is the probability that E chooses In2.*

Expected payoff of I from choosing F: -1

Expected payoff of I from choosing A: $0+(1-\mu)=1-\mu$

It is not possible to assign beliefs for I that would induce him to choose F. We thus have a principle that rules out "Out, Fight if entry occurs" as an equilibrium.

A weak perfect Bayesian Nash equilibrium:

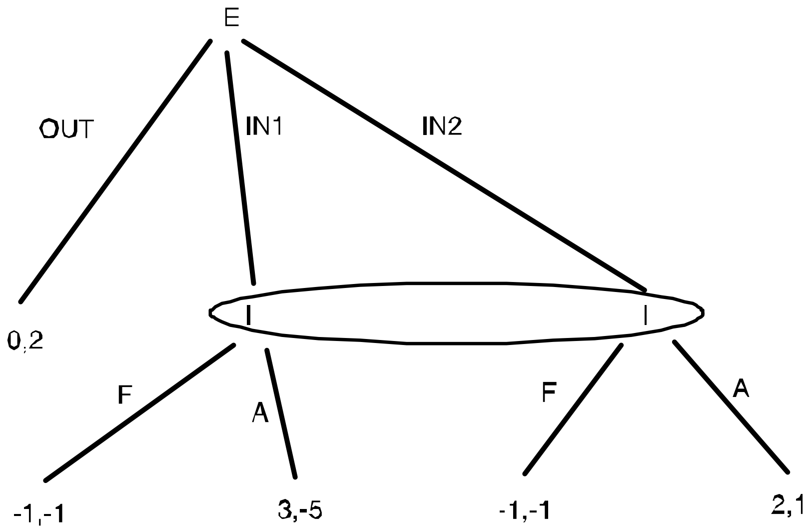
E: In1

I: A and $\mu=1$.

Notice how the strategies pin down the beliefs.

Note: It is typically assumed in game theory that a player knows his opponents' strategies in a Nash equilibrium. With this knowledge, he can verify that deviations are not beneficial. How he gains this knowledge of his opponents' strategies is not well understood (recall our discussion of rationalizability, and the assumption of correct beliefs about one's opponents' actions in Nash equilibrium). *With weak perfect Bayesian Nash equilibrium, we require that the player's beliefs at any information set be consistent with his knowledge of his opponents' strategies and his own strategy.*

Example 58 *Let's change the above example to make it more interesting. The game above had the property that A dominated F for I. The payoff of 0 for I has been changed to -5 so that I might plausibly choose either A or F:*



Expected payoff of I from choosing F: -1

Expected payoff of I from choosing A: $-5\mu + (1-\mu) = 1 - 6\mu$

When could I choose F? If μ *satisfies*

$$-1 \geq 1 - 6\mu$$
$$6\mu \geq 2$$
$$\mu \geq \frac{1}{3}$$

The weak perfect Bayesian Nash equilibrium from before is no longer an equilibrium:

E: In1

I: A and $\mu = 1$.

With these beliefs, I would choose F over A.

Let's be more careful. If $\mu > \frac{1}{3}$, *then I chooses F. If I chooses F, the E should choose Out. This defines a weak perfect Bayesian Nash equilibrium. It will also be an equilibrium if* $\mu = 1/3$.

If $\mu < \frac{1}{3}$, *then I chooses A. If I chooses A, the E should choose In1. Consistency of beliefs with actions would then require that* $\mu = 1$, *which is a contradiction.*

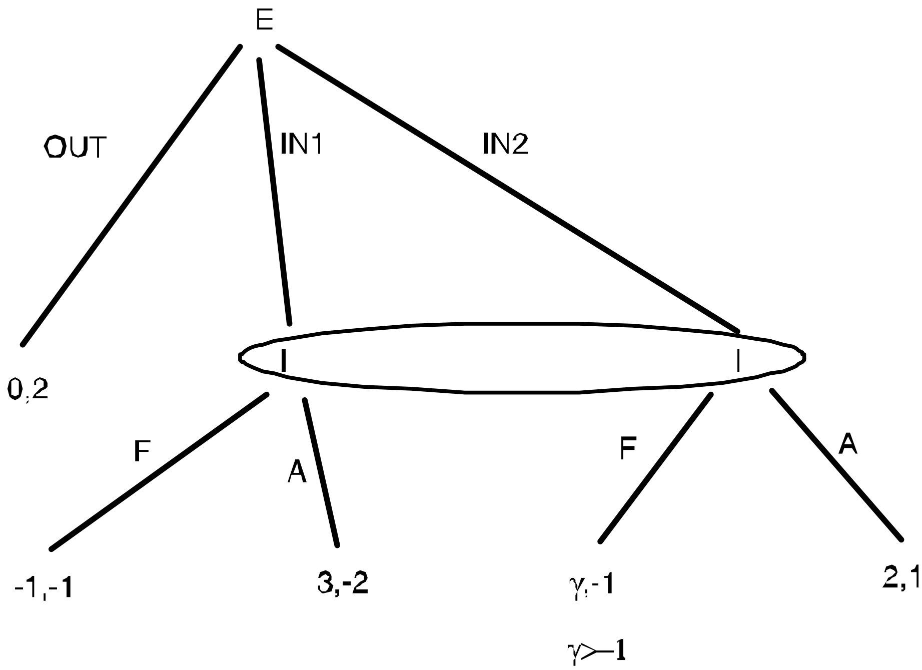
Thus, there is only one weak perfect Bayesian-Nash equilibrium:

E: Out

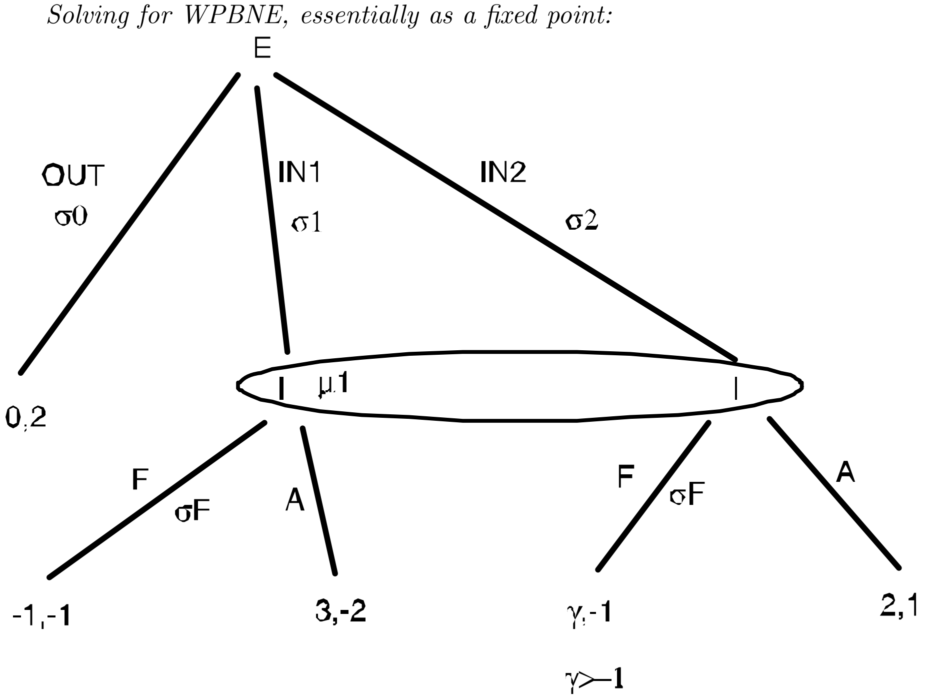
I: Fight, with $\mu \geq 1/3$.

As with subgame perfect equilibrium, our refinement is very much dependent on behavior off the equilibrium path. In this example, it is beliefs off the equilibrium path that sustain the behavior on the equilibrium path as rational.

Example 59 *9.C.3. This example is not simplified by the identification of a dominant strategy for some player. The good news is that the analysis is mostly arithmetic. If* $\gamma < -1$ *in the game below, then IN1 strictly dominates IN2 for player E, which makes the game less interesting.*



Solving for WPBNE, essentially as a fixed point:



- *I chooses F if*

$$\mu_1 \geq \frac{2}{3}$$

  expected payoff from choosing F$= -1$

  expected payoff from choosing A: $-2\mu_1 + (1-\mu_1) = 1 - 3\mu_1$

  F$\geq$*A iff* $-1 \geq 1 - 3\mu_1 \Leftrightarrow \mu_1 \geq \frac{2}{3}$

- *Suppose* $\mu_1 < \frac{2}{3}$. *Then firm I chooses A with probability 1, and so* $\sigma_1 = 1$. *This contradicts* $\mu_1 < \frac{2}{3}$. *We conclude that* $\mu_1 \geq \frac{2}{3}$ *in a WPBNE*

- *Suppose* $\mu_1 > \frac{2}{3}$. *Firm I must therefore choose F with probability 1.*
- *If* $\gamma > 0$, *then E chooses In2, and so* $\sigma_2 = 1$. *This contradicts* $\mu_1 > \frac{2}{3}$.
- *If* $\gamma < 0$, *then E chooses Out, and we have our first WPBNE in this example.*
- *If* $\gamma = 0$, *then E is indifferent between Out and In2. E would choose In1 with probability* 0, *however, and so he must choose In2 with probability 0 also in order to support I's beliefs at its information set. And so we are back at the WPBNE that we've just described.*
- ***Summary to this point:*** *We have completed the analysis of the case of* $\gamma > 0$ *and we hereafter restrict attention to the case of* $\gamma \in (-1, 0]$. *For* $\gamma \in (-1, 0]$, *we've derived one WPBNE: E chooses Out, and I chooses F because* $\mu_1 > \frac{2}{3}$. *We assume that* $\mu_1 = \frac{2}{3}$ *and* $\sigma_1 = 2\sigma_2$ *throughout the remainder of the example. The condition is required for I's beliefs to be consistent with E's strategy. Note that* $\sigma_1 = \sigma_2 = 0$ *remains as a possibility.*
- *Assuming that E chooses In1 or In2 with positive probability, Firm I must randomize between F and A to make E indifferent between In1 and In2 (this is a property of mixed strategy equilibrium):*

E's expected payoff from playing In1: $\sigma_F(-1) + (1 - \sigma_F)(3) = -4\sigma_F + 3$
E's expected payoff from playing In2: $\sigma_F(\gamma) + (1 - \sigma_F)(2) = (\gamma - 2)\sigma_F + 2$

$$-4\sigma_F + 3 = (\gamma - 2)\sigma_F + 2$$

$$1 = (\gamma + 2)\sigma_F$$

$$\sigma_F = \frac{1}{\gamma + 2}$$

With I's randomization, E's payoff from playing either In1 or In2 is

$$\begin{aligned} -4\sigma_F + 3 &= \frac{-4}{\gamma + 2} + 3 \\ &= \frac{-4 + 3\gamma + 6}{\gamma + 2} \\ &= \frac{3\gamma + 2}{\gamma + 2} \end{aligned}$$

We now have 3 cases:

1. $\gamma > \frac{-2}{3}$: *This expression is positive. E would therefore never choose Out (i.e.,* $\sigma_0 = 0$*). The WPBNE equilibrium is completed by setting* $\sigma_1 = \frac{2}{3}$ *and* $\sigma_2 = \frac{1}{3}$, *with* σ_F *as above and* $\mu_1 = 2/3$.