8.E Games of Incomplete Information: Bayesian Nash Equilibrium

games of *complete information* vs. games of *incomplete information*: The issue is incomplete knowledge of one's opponent's preferences over the possible outcomes of the game.

formulated by Harsanyi, corecipient with Nash and Selten of the first Nobel Prize in economics awarded in the field of game theory

This is a mathematically rigorous way of modeling the idea that players make decisions using their beliefs about the preferences of each other. The beliefs are modeled here using probability theory, and as with Nash equilibrium, it is typically necessary to assume that the beliefs of each player about the others are correct. How beliefs turn out to be correct, or whether or not beliefs are consistent with probability theory, are both legitimate concerns for the theory of games of incomplete information. We know, however, that people make decisions in situations of uncertainty using their beliefs, and this is a logically consistent way of modeling this kind of interaction among economic agents.

Bayesian Hypothesis: Whatever an agent doesn't know for certain, he has complete, probabilistic beliefs about

Ex.: Auctions

represent as game of imperfect information with a move of nature

incomplete vs. imperfect information

Example 28 *The DA's Brother (continued)*

There are two types of player 2: with probability μ, the game is

	$1\backslash 2$	DC	C
type I for player 2:	DC	$0,-2$	$-10,-1$
	C	$-1,-10$	$-5,-5$

and with probability $1-\mu$, the game is

	$1\backslash 2$	DC	C
type II for player 2:	DC	$0,-2$	$-10,-7$
	C	$-1,-10$	$-5,-11$

Type II represents a psychic penalty for confessing (i.e., player 2 hates being a "rat"). Notice that the issue is the payoffs to player 2. Player 2 is assumed to know his type while player 1 knows the probabilities of the two different games. In other games, it will be important to also assume that player 2 knows the beliefs of player 1 so that he can think about how player 1 considers his options.

Player 2's dominant strategy is to choose C if he is of type I and DC if he is type II. Player 1 therefore evaluates his choices as follows:

$$C : \mu(-5) + (1-\mu)(-1) = -1 - 4\mu$$

$$DC : \mu(-10) + (1-\mu)(0) = -10\mu$$

The equilibrium therefore depends on the value of μ: player 1 will choose C if

$$\begin{aligned} -1 - 4\mu &> -10\mu \\ 1 &< 6\mu \\ \mu &> \frac{1}{6} \end{aligned}$$

and player 1 will choose DC if

$$\mu < \frac{1}{6}.$$

We have 2 "candidate" equilibria:
1: C
2: C,DC
and
1:DC
2: C,DC

Which is actually an equilibrium depends on the value of μ. In the case of $\mu = 1/6$, each is an equilibrium

This is an example of a pure strategy Bayesian Nash equilibrium ("pure strategy" because there is no randomization in the choice of moves). It would typically be computed and discussed without reference to the extensive form representation.

General Framework

$u_i(s_i, s_{-i}, \theta_i)$ where $\theta_i \in \Theta_i$

- it is not necessary to assume that Θ_i is finite
- private value assumption, for now: $u_i(s_i, s_{-i}, \theta_i)$ instead of $u_i(s_i, s_{-i}, \theta)$

$F(\theta_1, \ldots, \theta_n)$
definition: *type* of player $i = \theta_i$
pure strategy: $s_i : \Theta_i \to S_i$
pure strategy Bayesian Nash equilibrium
$\Theta = \prod_{i=1}^{n} \Theta_i$ *type space*, or set of *states* of the world
mixed strategy: probability distribution over functions (the pure strategies); not used very often
assumption that the conditional expected payoffs exist and can be computed (conditions on F and on s_i)
interpretation of each type of a player as a distinct player; in the above example, each type of player 2 has a dominant strategy

Example 29 *This problem appeared on the 2014 midterm. Consider the following two-player game of incomplete information:*

1/2	*L*	*R*
T	θ_1, θ_2	$1, \frac{1}{2}$
B	$\frac{1}{2}, 0$	$0, -1$

It is common knowledge among the two players that player 1's type θ_1 and player 2's type θ_2 are independently drawn from the uniform distribution on $[0,1]$. Derive a pure strategy Bayesian-Nash equilibrium in this game.

It is clear that player 1 should choose T if $\theta_1 > \frac{1}{2}$. We conjecture a strategy for him as follows:

$$\begin{aligned} T \text{ if } \theta_1 &> \theta_1^* \\ B \text{ if } \theta_1 &< \theta_1^* \end{aligned}$$

Similarly, We conjecture the following strategy for player 2:

$$\begin{aligned} L \text{ if } \theta_2 &> \theta_2^* \\ R \text{ if } \theta_2 &< \theta_2^* \end{aligned}$$

We solve for θ_1^* and θ_2^* by noting that each player should be indifferent between his two choices at this value of his type:

$$\begin{aligned} 1 \quad &: \quad \theta_1^* (1 - \theta_2^*) + 1 \cdot \theta_2^* = \frac{1 - \theta_2^*}{2} \\ &\Leftrightarrow \quad \theta_1^* - \theta_1^* \theta_2^* = \frac{1}{2} - \frac{3\theta_2^*}{2} \end{aligned}$$

and

$$\begin{aligned} 2 \quad &: \quad \theta_2^* (1 - \theta_1^*) + 0 \cdot \theta_1^* = \frac{1}{2} (1 - \theta_1^*) - 1 \cdot \theta_1^* \\ &\Leftrightarrow \quad \theta_2^* - \theta_1^* \theta_2^* = \frac{1}{2} - \frac{3\theta_1^*}{2} \end{aligned}$$

Subtracting the two equations produces

$$\begin{aligned}\theta_1^* - \theta_2^* &= -\frac{3\theta_2^*}{2} + \frac{3\theta_1^*}{2} \Leftrightarrow \\ 0 &= -\frac{\theta_2^*}{2} + \frac{\theta_1^*}{2} \Leftrightarrow \\ \theta_1^* &= \theta_2^*\end{aligned}$$

Substituting into the equation for player 1 produces

$$\theta_1^* - \theta_1^{*2} = \frac{1}{2} - \frac{3\theta_1^*}{2} \Leftrightarrow$$
$$0 = 2\theta_1^{*2} - 5\theta_1^* + 1$$
$$\theta_1^* = \frac{5 \pm \sqrt{25-8}}{4} = \frac{5 \pm \sqrt{17}}{4}$$

The only solution that is meaningful is

$$\frac{5-\sqrt{17}}{4} \approx 0.104$$

Example 30 *8.E.2. Two firms jointly share their research outputs. They might be envisioned as divisions of the same firm. Each firm can independently choose to spend* $c \in (0,1)$ *to develop the "zigger", a device that is then made available to the other firm. Firm* i*'s type is* θ_i*, which is believed by firm* $-i$ *to be independently drawn from the uniform distribution on* $[0,1]$.

$c \in (0,1)$*: This assumption insures that a firm may want to provide the zigger on its own, which makes the model interesting. If* $c = 0$*, then both firms surely want to provide the zigger, and if* $c \geq 1$*, then neither firm would ever choose to provide the zigger.*

value of the zigger to firm i*:* θ_i^2

payoff if the zigger is not provided: 0

payoff if it builds the zigger: $\theta_i^2 - c$

payoff if it does not build the zigger but firm $-i$ *does:* θ_i^2

$s_i : [0,1] \to \{yes, no\}$

Let p_{-i} *denote the probability that firm* $-i$ *produces the zigger, given its strategy* s_{-i}

firm i*'s utility given its decision and its type* θ_i*:*

$$yes : \theta_i^2 - c$$

$$no : p_{-i}\theta_i^2$$

Firm i *should thus provide the zigger only if*

$$\theta_i^2 - c \geq p_{-i}\theta_i^2$$
$$(1-p_{-i})\theta_i^2 \geq c$$
$$\theta_i \geq \sqrt{\frac{c}{1-p_{-i}}}$$

Firm i *thus uses a cutoff strategy. A similar analysis applies to firm* $-i$*. Let* $\widehat{\theta}_i$ *be the cutoff point for firm* i*. We have:*

$$p_i = 1 - \widehat{\theta}_i = 1 - \sqrt{\frac{c}{1-p_{-i}}}$$
$$= 1 - \sqrt{\frac{c}{\widehat{\theta}_{-i}}}$$

Therefore

$$\begin{aligned}\widehat{\theta}_i &= \sqrt{\frac{c}{\widehat{\theta}_{-i}}} \\ \widehat{\theta}_i^2\widehat{\theta}_{-i} &= c\end{aligned}$$

and symmetrically,

$$c = \widehat{\theta}_i\widehat{\theta}_{-i}^2$$

Canceling,

$$\begin{aligned}\widehat{\theta}_i^2\widehat{\theta}_{-i} &= c = \widehat{\theta}_i\widehat{\theta}_{-i}^2 \\ \widehat{\theta}_i &= \widehat{\theta}_{-i}\end{aligned}$$

i.e., the only equilibrium is symmetric. Substituting into an equation above implies

$$\widehat{\theta}_i = \widehat{\theta}_{-i} = c^{\frac{1}{3}}$$

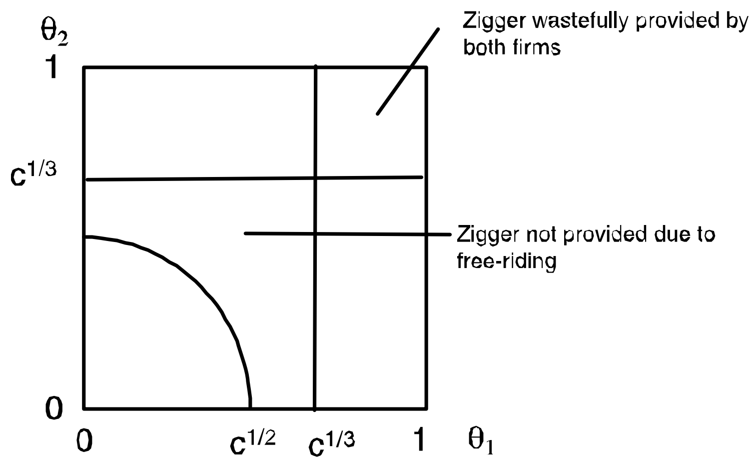
Where is the cost of free riding? The zigger should be provided by one of the two firms if

$$\theta_1^2 + \theta_2^2 \geq c.$$

Graph and compare $\theta_1^2 + \theta_2^2 = c$, $\widehat{\theta}_i = c^{\frac{1}{3}}$,$\widehat{\theta}_{-i} = c$.

Note: Because $c \in (0,1)$,

$$c^{\frac{1}{2}} < c^{\frac{1}{3}}$$



January 29, 2014

Example 31 *Derivation of an Equilibrium in the First Price Sealed Bid Auction with Uniformly Distributed Private Values*

n bidders, with the valuation v_i of bidder i independently drawn from the uniform distibution on $[0,1]$

$\theta_i = v_i$*: bidder i's type θ_i is his valuation v_i, which he knows privately.*

We wish to solve for a common rule $b : [0,1] \to [0,1]$ *that defines a Bayesian Nash equilibrium* ($b(v_i) = s_i(v_i)$)

Assumption: b is strictly increasing and differentiable (the answer we come up with will have this property)

Condition for equilibrium: for each $v_i \in [0,1]$, $b(v_i) = x$ *maximizes*

$$U_i(v_i, x) = (v_i - x)\left(b^{-1}(x)\right)^{n-1}$$

Note: the assumption that b is increasing is used in this formula.

Two distinct ways of deriving a formula for $b(v_i)$ *are presented below. The first is straightforward as an approach, the second is more clever and ultimately simpler.*

FOC:

$$-\left(b^{-1}(x)\right)^{n-1} + (v_i - x)\left[(n-1)\left(b^{-1}(x)\right)^{n-2}\left(\frac{1}{b'\left(b^{-1}(x)\right)}\right)\right] = 0$$

Satisfied at $b(v_i) = x$*:*

$$-\left(b^{-1}(b(v_i))\right)^{n-1} + (v_i - b(v_i))\left[(n-1)\left(b^{-1}(b(v_i))\right)^{n-2}\left(\frac{1}{b'\left(b^{-1}(b(v_i))\right)}\right)\right] = 0$$

$$-(v_i)^{n-1} + (v_i - b(v_i))\left[(n-1)(v_i)^{n-2}\left(\frac{1}{b'(v_i)}\right)\right] = 0$$

$$(v_i - b(v_i))\left[(n-1)(v_i)^{n-2}\right] = v_i^{n-1} b'(v_i)$$

$$(n-1)(v_i)^{n-1} = v_i^{n-1} b'(v_i) + b(v_i)(n-1)(v_i)^{n-2}$$

$$\left(\frac{n-1}{n}\right)\frac{d}{dv_i}(v_i)^n = \frac{d}{dv_i}\left[v_i^{n-1} b(v_i)\right]$$

$$\left(\frac{n-1}{n}\right)(v_i)^n = v_i^{n-1} b(v_i) + k$$

Argue: $k = 0$ *by substituting* $v_i = 0$ *into the equation. Reducing, we then obtain*

$$b(v_i) = \frac{(n-1)}{n} v_i.$$