Bilateral bargaining and the k-double auction.

Example 35 ***Bilateral bargaining.*** *Here's a model that has become a classic object of study.*[1] *A seller has an indivisible item that he can produce and sell to a buyer. The cost of producing the item for the seller is c and the value of the item to the buyer is v. Utility is quasilinear in the value/cost and money: the utility of the seller is $p_c - c$ if he produces the good for the buyer and receives a payment of p_c, while the utility of the buyer is $v - p_b$ if he buys the item from the seller and pays p_b. The utility of a seller is p_c^* if he receives a payment of p_c^* when he doesn't trade, and the utility of the buyer is $-p_b^*$ if he makes a payment of p_b^* when he doesn't trade.*

In the interest of generality at this point, we allow the possibility for now that $p_b \neq p_c$ and $p_b^* \neq p_c^*$; this would occur (for instance), if trade occurs through a broker or intermediary. Also, all payments/charges can be either positive or negative.

It is arguable that the essence of bargaining as a problem of interest in economics is the uncertainty concerning the terms that the other party will accept. In the interest of modeling this uncertainty, we will assume that v and c are independently drawn from the uniform distribution on $[0,1]$. Later, we can consider other distributions over other intervals. Each trader privately knows his value/cost and the distribution from which they are independently drawn.

Consider the following one-parameter family of rules for bargaining. The buyer submits a bid b as the seller submits an offer s. Trade occurs iff $b \geq 0$, in which case the buyer pays the seller $kb + (1-k)s$. No money changes hands if trade does not occur. Here, $k \in [0,1]$ is a fixed parameter or weight. These are the rules of the ***k-double auction*** *(where "double" refers to the fact that both the buyer and the seller act strategically). A strategy for the seller is thus a function $S : [0,1] \to \mathbb{R}$ and a strategy for the buyer is a function $B : [0,1] \to \mathbb{R}$.*

Exercise 36 *Consider first the case of $k = 1$. In this case, the price is the buyer's bid whenever trade occurs. This is the* ***buyer's bid double auction****.*

1. *Prove that $S(c) = c$ is the unique dominant strategy for the seller (i.e., the offer of c is the unique dominant strategy for the seller with cost c).*
2. *Assuming that the seller uses this strategy, what is the buyer's expected utility when v is his value and he bids b?*
3. *By maximizing your answer to 2., solve for the buyer's equilibrium strategy $B(v)$.*
4. *Efficiency requires that trade should occur whenever $v \geq c$. Is the equilibrium that you have solved for efficient?*

Exercise 37 *Consider next the case of $k = 0$. Derive the results for this procedure that parallel the results in the previous exercise.*

Exercise 38 *Consider next the case of $k = 1/2$.*

1. *Verify that $S(c) = 1$, $B(v) = 0$ is a Bayesian Nash equilibrium. This is a "no-trade" equilibrium.*
2. *Verify that the following strategies define a Bayesian-Nash equilibrium:*

$$S(c) = \begin{cases} 1/2 \text{ if } c < 1/2 \\ 1 \text{ if } c \geq 1/2 \end{cases},$$

$$B(v) = \begin{cases} 0 \text{ if } v < 1/2 \\ 1/2 \text{ if } v \geq 1/2 \end{cases}.$$

[1] The game considered here originated in Chatterjee and Samuelson, "Bargaining Under Incomplete Information" *Operations Research* Vol. 31, No. 5, September-October 1983, pp. 835-851.

3. *The equilibrium in 2. consists of two step functions. Generalize your answer to 2. by changing where the "step" occurs from $1/2$ to any given $x \in [0,1]$. Hint: You'll also need to change the values of $S(c)$ and $B(v)$. In this way, you show that there are many different equilibria in the $k = 1/2$ double auction.*

4. *Verify that the following strategies define a Bayesian-Nash equilibrium in the $k = 1/2$ double auction:*

$$S(c) = \begin{cases} \frac{2}{3}c + \frac{1}{4} \text{ if } c \le \frac{3}{4} \\ c \text{ if } c \ge \frac{3}{4} \end{cases},$$

$$B(v) = \begin{cases} v \text{ if } v < \frac{1}{4} \\ \frac{2}{3}v + \frac{1}{12} \text{ if } v \ge \frac{1}{4} \end{cases}.$$

5. *Show in a diagram the values of v, c at which trade occurs in the equilibrium in 4. Note the inefficiency of equilibrium.*

8.F The Possibility of Mistakes: Trembling Hand Perfection

back to games of complete information, for the moment

refinement: a set of principles that allow one to select among equilibria. The principles typically express a sense in which the selected equilibrium is more plausible or more likely to be played by the players. The study of refinements is a rich and rather complicated area within game theory.

We follow here the discussion in Osborne and Rubenstein of an idea first developed by Reinhard Selten (corecipient of the first Nobel Prize awarded in the field of game theory, along with Harsanyi and Nash). The goal here is to develop a principle that selects among multiple Nash equilibria. The intuition is that people may make mistakes motivates the selection.

1\2	A	B	C
A	**0, 0**	0, 0	0, 0
B	0, 0	**1, 1**	2, 0
C	0, 0	0, 2	**2, 2**

Discuss A,A, B,B, and C,C as equilibria. Only BB is *trembling hand perfect*. In the two player case, we are simply ruling out Nash equilibria in which players use weakly dominated strategies. A player can use a weakly dominated strategy in a Nash equilibrium, but doing so may require certainty as to the choices of the opponents. If the other players might fail to play their prescribed strategies, even with a small likelihood, then one might not want to play a weakly dominated strategy. In a sense, this addresses the critical assumption of Nash equilibrium by addressing the correctness of a player's conjectures about his opponents' actions, i.e. is the equilibrium robust to small errors in these conjectures?

The idea of *trembling hand perfection* formalizes this idea. We want to rule out equilibria that depend on specific choices of opponents. The idea will generalize beyond the case of two players, and beyond strategic or normal form games, where it is more interesting than simply eliminating Nash equilibria that involve the use of weakly dominated strategies.

totally mixed strategy: attaches positive probabalistic weight to each pure strategy

A Nash equilibrium σ is *trembling hand perfect* if there exists a sequence of totally mixed strategy profiles $\left(\sigma^k\right)_{k=0}^{\infty}$ such that:

1. $\lim_{k\to\infty} \sigma^k = \sigma$

2. σ_i is a best response to σ_{-i}^k for each i and each k

Notice that a Nash equilibrium in which a player uses a weakly dominated strategy cannot be trembling hand perfect (Proposition 8.F.2 in MWG). This assumes that the weakly dominated strategy is strictly worse than the dominating strategy for some choice of the opponents' actions. The use of weakly dominated strategies in Nash equilibrium is thus ruled out by trembling hand perfection.

We could simply have as our refinement the principle that "no player should use a weakly dominated strategy in Nash equilibrium". This, however, is not motivated by consideration of rationality alone. The idea of mistakes or the "trembling hand" motivates ruling out equilibria in this way, along with other equilibria.

Using Prop. 8.F.1 of MWG as the definition. MWG instead uses a sequence of perturbed games with an associated sequence of Nash equilibria to define trembling hand perfection. The difference in modeling is between players making mistakes vs. errors in knowing the payoffs of the game. The two definitions are formally equivalent.

Apply the criterion to the above game.

Example 39 *8.F.2 Consider the 3 player game below in which each player has 2 choices.*

3: l and r

1/2	L	R		1/2	L	R
T	$\mathbf{1,1,1}$	$1,0,1$		T	$1,1,0$	$0,0,0$
B	$\mathbf{1,1,1}$	$0,0,1$		B	$0,1,0$	$1,0,0$

Notice that l strictly dominates r for player 3 and L strictly dominates R for player 2.

Nash equilibria:

T,L,l, B,L,l. We'll show that T,L,l is a trembling hand perfect Nash equilibrium, but B,L,l is not trembling hand perfect.

Because players 2 and 3 each have a strictly dominant strategy, we don't need to examine the optimality of actions of players 2 and 3 in regard to trembling hand perfection; for each of these players, the equilibrium action is clearly a best response even if the other players make mistakes with a small probability.

Now consider player 1. Letting 2 choose R with probability ε *and 3 choose r with probability* δ*, we have as 1's expected payoff*

$$\begin{aligned} B &: 1\cdot(1-\varepsilon)(1-\delta)+0\cdot(\varepsilon)(1-\delta)+0\cdot(1-\varepsilon)(\delta)+1\cdot(\varepsilon)(\delta) \\ &= (1-\varepsilon)(1-\delta)+\varepsilon\delta \end{aligned}$$

versus

$$\begin{aligned} T &: 1\cdot(1-\varepsilon)(1-\delta)+1\cdot(\varepsilon)(1-\delta)+1\cdot(1-\varepsilon)(\delta)+0\cdot(\varepsilon)(\delta) \\ &= (1-\varepsilon)(1-\delta)+1\cdot(\varepsilon)(1-\delta)+1\cdot(1-\varepsilon)(\delta) \end{aligned}$$

The expected payoff with T is thus greater than the expected payoff with B for small ε, δ*.*

This is related to the independence of trembles, which means that the likelihood of "mistakes" by both 2 and 3 (i.e., R,r) is relatively small in comparison to a mistake by only one of the two players.

This completes the verification that T,L,l is a trembling hand perfect Nash equilibrium, but B,L,l is not trembling hand perfect.

Theorem 40 ***Existence:*** *Selten proved that every finite game has at least one trembling hand perfect Nash equilibrium (possibly in mixed strategies). As a consequence, every finite game has at least one Nash equilibrium in which no player uses a weakly dominated strategy.*

Criticism of trembling hand perfection A trembling hand perfect equilibrium is robust to a particular sequence of "trembles". In other words, the equilibrium is robust to mistakes if the mistakes *are made in a particular way.* This may be better than not being robust to mistakes in any sense, but it seems to fly in the face of the arbitrariness of mistakes (they are *mistakes* after all, not *carefully crafted choices*).

It is a useful principle, however, that will extend to dynamic games.

Problems: 9.B.4, 9.B.5, 9.B.9, 9.B.10, 9.C.2, 9.C.7

9. Dynamic Games

The main issue in this chapter is to refine Nash equilibrium in dynamic games. The refinements focus in particular on *sequential rationality.*

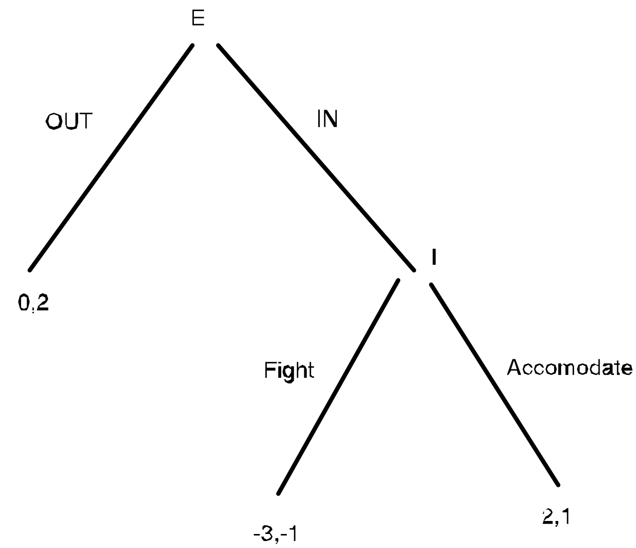
9.B. Sequential Rationality, Backward Induction, and Subgame Perfection

Example 41 *Predation Game*

A component of the chain store paradox, which will be discussed later.

E: entrant

I: incumbent



Draw as normal form:

E/I	*Fight*	*Accomodate*
Out	***0,2***	*0,2*
In	*-3,-1*	***2,1***

There are two Nash equilibria in this game. Notice that the "Out, Fight" Nash equilibrium is not sequentially rational in the sense that if E deviated from his equilibrium strategy and chose "In", then I must choose "Fight" giving it a payoff of -1 *instead of "Accomodate" with a payoff of* 1*.*

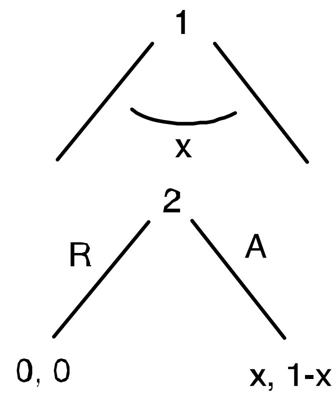
We might rule out "Out, Fight" on the grounds that it isn't trembling hand perfect ("Fight" is a weakly dominated strategy for I). Instead, we'll focus more on the shortcomings of this equilibrium as they appear in the dynamic setting, for the dynamic setting provides its own motivation. Besides, we don't want to consider the strategic or normal form of every dynamic game.

Definition 42 ***history, subgame, equilibrium path*** *(a subgame necessarily starts from a singleton information set)*

Notice that the issue in the above example is behavior **"off the equilibrium path"**: we're concerned about the credibility or reasonableness of a strategy as it specifies actions that are never actually observed as the game is played according to the equilibrium. Recall that a strategy in a dynamic game specifies a player's actions at *all* nodes assigned to him, not simply those that are actually observed when the game is played. Nash equilibrium requires allowing a player to contemplate his payoffs if he chooses to vary his strategy, and doing so requires that he know how his opponents would act if he behaved differently. "Off the equilibrium path" is thus essential in Nash equilibrium. We are now requiring that this "off the equilibrium path" behavior be sensible.

Notion of a "**credible threat**" in the above game. When you contemplate the implications of changing your strategy, you should postulate that your opponents will respond rationally in their own self-interests, each choosing more over less.

Example 43 *"Divide the Dollar" Game. This is sometimes called the Ultimatum Game. Two players have a continuous dollar to divide. Player 1 proposes to divide the dollar at* $x \in [0,1]$*, where he will keep* x *and player 2 will receive* $1-x$*. Player 2 can choose to either accept this division of the dollar, or reject it, in which case each player receives* 0*. The two players thus have a one-shot opportunity to reach agreement through this particular procedure.*



Any division $(p, 1-p)$ of the dollar can be sustained as a Nash equilibrium:

$$\begin{aligned} 1 &: \quad x = p \\ 2 &: \quad \begin{cases} R \text{ if } 1-x < 1-p \\ A \text{ if } 1-x \geq 1-p \end{cases} \end{aligned}$$

Notice that 2's strategy specifies how he responds to any offer, not just the one that 1 actually makes in equilibrium. This is important to understanding why 1 chooses a particular offer in equilibrium. We might interpret this equilibrium as "2 demands at least $1-p$, and 1 offers 2 the minimal amount that 2 will accept." In the case of $p < 1$, however, player 2's strategy involves an "incredible" (i.e., not believable) threat to reject a positive amount $(1-p)$ in favor of 0. This is not rational behavior.

Rationality requires that player 2 accept any positive offer. If we assume that 2 accepts only positive offers, i.e.,

$$2 : \begin{cases} R \text{ if } 1-x = 0. \\ A \text{ if } 1-x > 0 \end{cases},$$

then there is not best response for 1: he wants to choose $x < 1$ as large as possible, which is not well-defined (more on this later – yes, things are different if the dollar isn't infinitely divisible). If 2 accepts any offer, however, then we can define an equilibrium:

$$\begin{aligned} 1 &: \quad x = 1 \\ 2 &: \quad A \end{aligned}$$

2 accepts any offer, and 1's best response is to give 2 nothing.

The interest of this game to behavioral and experimental economics (see Colin Camerer's book). Do people have a sense of fairness in how they play games? If so, should this be modeled as part of economic theory and game theory? The role of anonymity in allowing players to focus on their narrow self-interests.

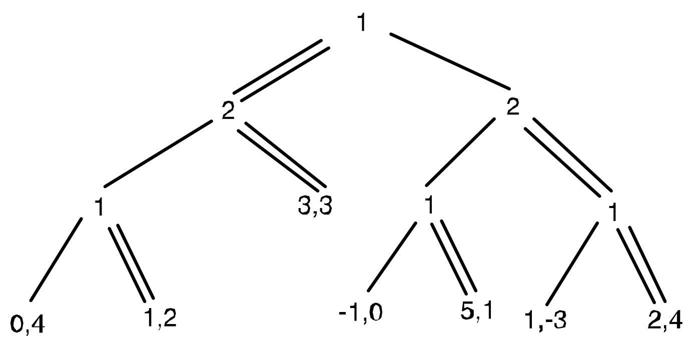
Definition 44 *A Nash equilibrium of a dynamic game is* ***subgame perfect*** *if it defines a Nash equilibrium in every subgame of the game.*

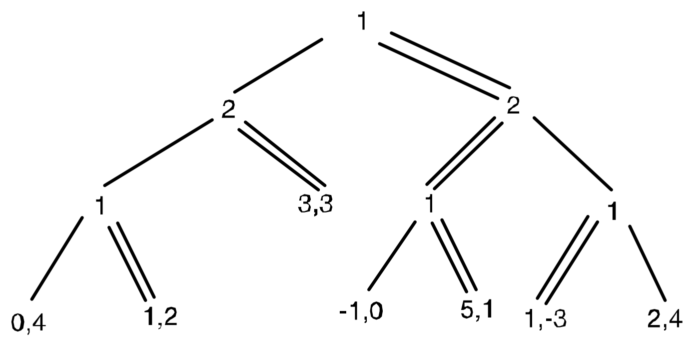
This insures that all behavior in the equilibrium is rational (choosing more over less), even behavior off the equilibrium path. It is another idea due to Reinhard Selten.

Question: As we've noted earlier, rationality alone does not imply Nash equilibrium. Should we simply require that behavior in subgames consist of rationalizable strategies? Subgame perfect Nash equilibria, however, is the most widely applied refinement in extensive form games.

Solving a game of complete and perfect information by **"backwards induction"**

Example 45 *The solution of a game by backwards induction (i.e., the determination of a subgame perfect Nash equilibrium), along with second Nash equilibrium:*





In finite games of complete and perfect information, subgame perfection is exactly the same as solving the game through backwards induction. Why do we bother with it if it is so obvious? It is a useful idea that has significance as a refinement beyond this particular class of games. It is easiest to introduce, however, in the context of this class of games.

Exercise 46 *Determine the subgame perfect Nash equilibria of the "Divide the Dollar" game when the dollar is not infinitely divisible, but is instead divisible into hundredths (e.g., pennies).*

Ans:

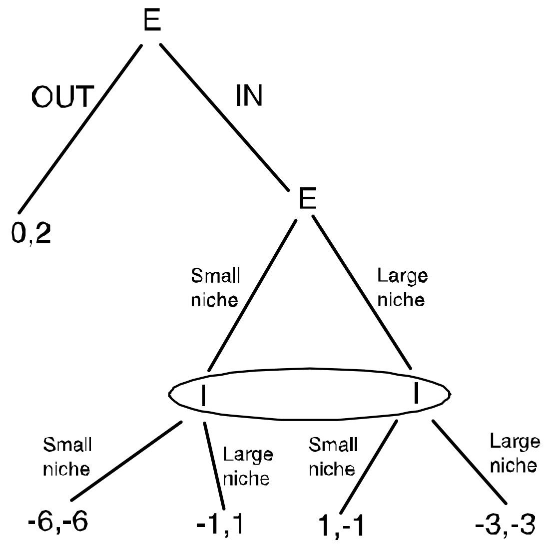
$$\begin{aligned} 1 &: x = 1 \\ 2 &: A \end{aligned}$$

and

$$\begin{aligned} 1 &: x = 0.99 \\ 2 &: \begin{cases} R \text{ } if \text{ } 1 - x = 0 \\ A \text{ } if \text{ } 1 - x \geq 0.01 \end{cases} \end{aligned}$$

Note the representativeness of the solution in the continuum model.

Example 47 *Predation Game (cont.). A simultaneous game is played after entry:*



A "niche" might be interpreted as a type of customer. E and I both choosing "small" means that they compete for a small type of customer, ignoring the large customers.

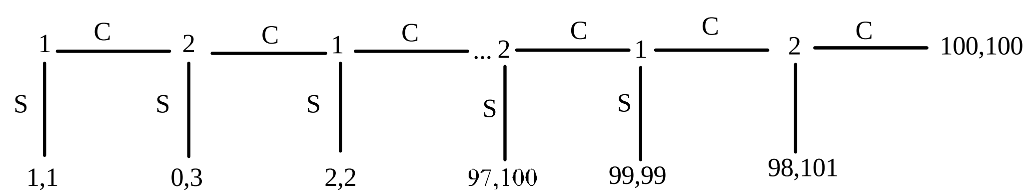
E/I	*small*	*large*
small	$-6,-6$	$\mathbf{-1,1}$
large	$\mathbf{1,-1}$	$-3,-3$

multiple subgame perfect equilibria

Notice that subgame perfection doesn't bind at the bottom information set. This example provides support for insisting that a subgame begin with a singleton information set. In this case (for instance), rational behavior in the subgame clearly allows more than one Nash equilibrium.

Theorem 48 *Existence of a unique pure strategy subgame perfect Nash equilibrium in a finite game of perfect information with "no ties" in payoffs (Zermelo's Theorem – provable by backwards induction). If there are ties, then there can be more than one subgame perfect Nash equilibrium.*

Example 49 *The Centipede Game (Rosenthal)*



a game that helps to clarify the situation:

Each players starts with \$1. When a player chooses C, \$1 is taken from him and \$2 are given to the opponent. If a player chooses S, then play stops and each player leaves with his accumulated money.

Notice the independence of history: a player makes a decision at a node in anticipation of its future consequences and without regard to the sequence of moves that have been made to place him at that node.

Is this what you think would happen if this game were played in an experimental setting? How would you as a player interpret a choice of C by your opponent? If you had seen him choose C repeatedly, would your expectations of the future play of the game be determined solely by looking forward?