Finite Repetition of a Game

stage game: the game that is played repeatedly

n stage game: repetition of the game n times, with payoffs accumulating. For now, we won't address discounting.

Example 50 *Example from Friedman's book (p. 78-79), "Game Theory with Applications to Economics". Notice the large number of strategies and the increase in the number of Nash equilibria with repetition.*

Theorem 51 *If a game has a unique Nash equilibrium, then its finite repetition has a unique SPNE*

Proof by induction on the number n of repetitions. Consider the n stage game.

1. The case of $n = 1$ is trivial.
2. In the nth repetition, consider each $n - 1$ stage game that follows as a subgame determined by the outcome of the first play of the game. Subgame perfection requires a subgame perfect NE in each subgame, and the induction hypothesis therefore uniquely characterizes the play of the game in stages 2 through n (regardless of the outcome of stage 1).
3. The play of the game in stage 1 does not alter the play of the game in all subsequent stages. Consequently, the assumption of Nash equilibrium for the n-stage game requires the play of the unique Nash equilibrium in the first stage.

Notice that the result concerns not only the equilibrium path, but also the specification of the strategies off the equilibrium path. As we'll see in the example from Friedman's book that follows below, there can be other Nash equilibria that are not subgame perfect and that share the same equilibrium path as the unique subgame perfect equilibrium.

Note also the independence of history that is described by the theorem. Long-term relationships ought to be different, but modeling an enduring relationship as a finite repetition of a game may fail produce interesting or different results. This is one of the reasons that so much of dynamic game theory focuses on infinitely repeated games.

Example 52 *prisoner's dilemma (p. 236)*

1/2	DC	C
DC	$-2,-2$	$-10,-1$
C	$-1,-10$	$-5,-5$

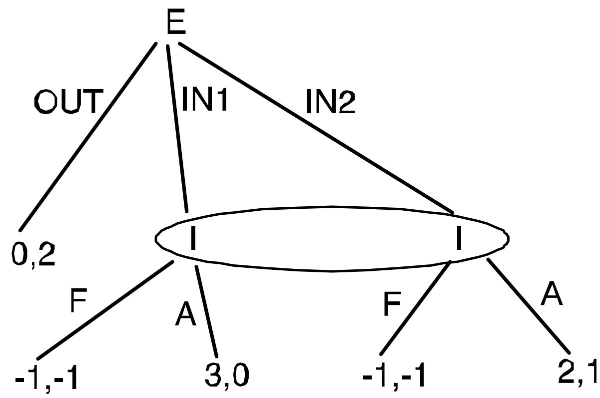
Example 53 *Battle of the Sexes*

1/2	B	F
B	$2,1$	$0,0$
F	$0,0$	$1,2$

Consider playing this 5 *times. Note the multiplicity of subgame perfect Nash equilibria.*

9.C. Beliefs and Sequential Rationality

Example 54 *Entrant has two strategies in addition to entry and the incumbent does not observe which one has been selected*



Two pure strategy Nash equilibria:

Out, Fight if entry occurs

In1, Accomodate if entry occurs

There is only one subgame of the game (the entire game itself), and so subgame perfection is ineffective as a refinement.

Bayesian Hypothesis: *A person has probabilistic beliefs about anything and everything that he does not know.*

sequential rationality given system of beliefs

Definition 55 ***weak perfect Bayesian equilibrium****: actions consistent with beliefs, and beliefs consistent with actions (whereever the actions restrict the beliefs)*

weak: any beliefs can be assigned at an information set that is reached with zero probability

It is important to understand the following perspective of game theory: we include all possible outcomes and behavior as "equilibrium" behavior unless we have some principle for ruling it out. Here, for instance, we allow a player to "rationalize" his choices with any beliefs that he may possibly concoct, so long as the beliefs are consistent with the equilibrium behavior in the game. Unless we have a principled argument for ruling out certain behaviors, we'll keep them around as possibilities.

Example 56 *Returning to the above example, let μ denote the probability that the Incumbent assigns to being at the left hand node. WPBE means that, conditional on choosing In1 or In2, μ is the probability that E chooses In1, and $1-\mu$ is the probability that E chooses In2.*

Expected payoff of I from choosing F: -1

Expected payoff of I from choosing A: $0+(1-\mu)=1-\mu$

It is not possible to assign beliefs for I that would induce him to choose F. We thus have a principle that rules out "Out, Fight if entry occurs" as an equilibrium.

A weak perfect Bayesian Nash equilibrium:

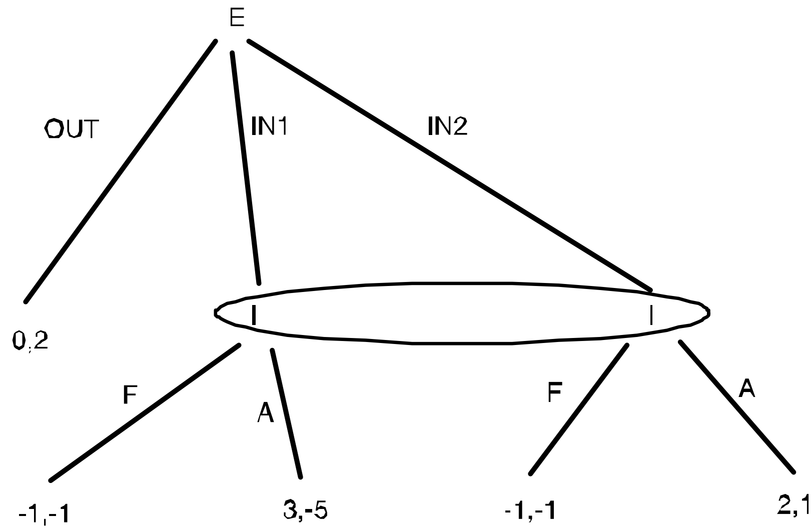
E: In1

I: A and $\mu=1$.

Notice how the strategies pin down the beliefs.

Note: It is typically assumed in game theory that a player knows his opponents' strategies in a Nash equilibrium. With this knowledge, he can verify that deviations are not beneficial. How he gains this knowledge of his opponents' strategies is not well understood (recall our discussion of rationalizability, and the assumption of correct beliefs about one's opponents' actions in Nash equilibrium). *With weak perfect Bayesian Nash equilibrium, we require that the player's beliefs at any information set be consistent with his knowledge of his opponents' strategies and his own strategy.*

Example 57 *Let's change the above example to make it more interesting. The game above has the property that A dominates F for I. The payoff of* 0 *for I has been changed to* -5 *so that I might plausibly choose either A or F:*



Expected payoff of I from choosing F: -1

Expected payoff of I from choosing A: $-5\mu + (1-\mu) = 1 - 6\mu$

When could I choose F? If μ *satisfies*

$$-1 \geq 1 - 6\mu$$
$$6\mu \geq 2$$
$$\mu \geq \frac{1}{3}$$

The weak perfect Bayesian Nash equilibrium from before is no longer an equilibrium:

E: In1

I: A and $\mu = 1$.

With these beliefs, I would choose F over A.

Let's be more careful. If $\mu > \frac{1}{3}$*, then I chooses F. If I chooses F, the E should choose Out. This defines a weak perfect Bayesian Nash equilibrium. It will also be an equilibrium if* $\mu = 1/3$.

If $\mu < \frac{1}{3}$*, then I chooses A. If I chooses A, the E should choose In1. Consistency of beliefs with actions would then require that* $\mu = 1$*, which is a contradiction.*

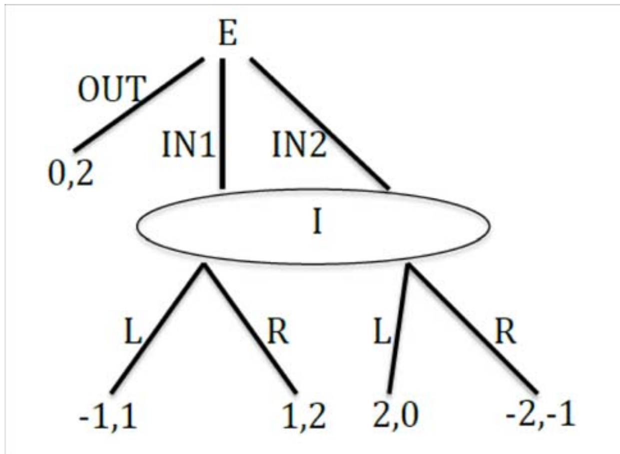
Thus, there is only one weak perfect Bayesian-Nash equilibrium:

E: Out

I: Fight, with $\mu \geq 1/3$.

As with subgame perfect equilibrium, our refinement is very much dependent on behavior off the equilibrium path. In this example, it is beliefs off the equilibrium path that sustain the behavior on the equilibrium path as rational.

Example 58 *Here's a problem that was on the 2014 midterm: Determine all weak perfect Bayesian-Nash equilibria of the following game.*



Let μ denote the probability that I assigns to being at the left hand node in his information set. We first determine the values of μ that support I's choice of L over R:

$$L : 1 \cdot \mu + 0 \cdot (1-\mu) = \mu$$
$$R : 2\mu + -1 \cdot (1-\mu) = 3\mu - 1$$
$$L \text{ is preferred to } R \text{ iff } \mu > 3\mu - 1 \Leftrightarrow 1 > 2\mu \Leftrightarrow \mu < \frac{1}{2}$$

We first try to construct an equilibrium in which I chooses L. This leads E to choose IN2, in which case $\mu = 0$. We have our first WPBNE: E chooses IN2, I chooses L, $\mu = 0$.

We next try to construct an equilibrium in which I chooses R. In this case, E chooses IN1, resulting in $\mu = 1$. We therefore have a second WPBNE: E chooses IN1, I chooses R, $\mu = 1$.

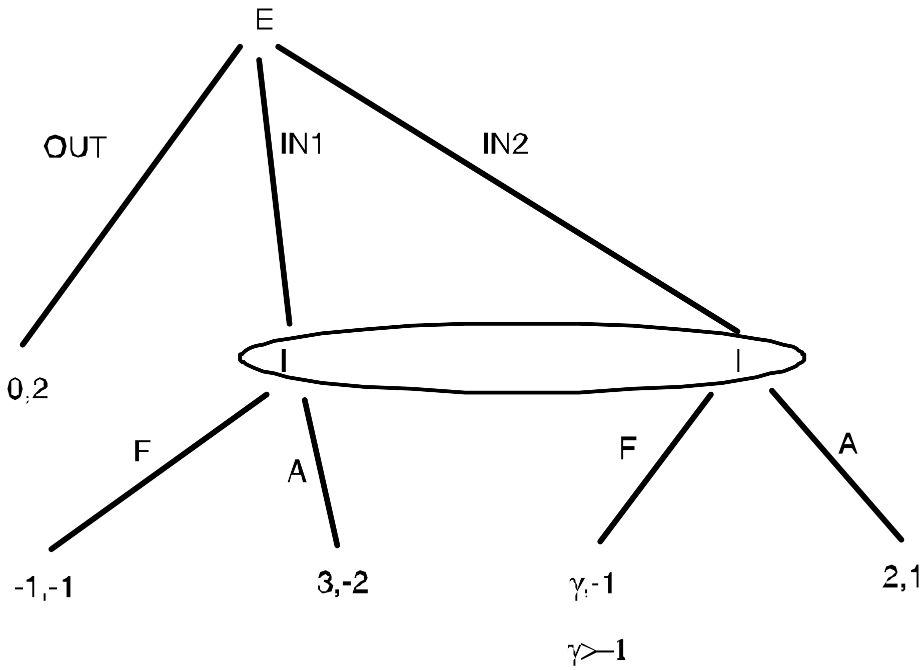
We consider $\mu = \frac{1}{2}$ so that I is indifferent between L and R. Let σ denote the probability that I chooses L. If E is to choose IN1 and IN2 with positive probability, we must have

$$\begin{aligned} IN1 &: -\sigma + (1-\sigma) = 2 \cdot \sigma + -2 \cdot (1-\sigma) : IN2 \\ IN1 &: -2\sigma + 1 = 4\sigma - 2 : IN2 \Leftrightarrow \\ \sigma &= \frac{1}{2} \end{aligned}$$

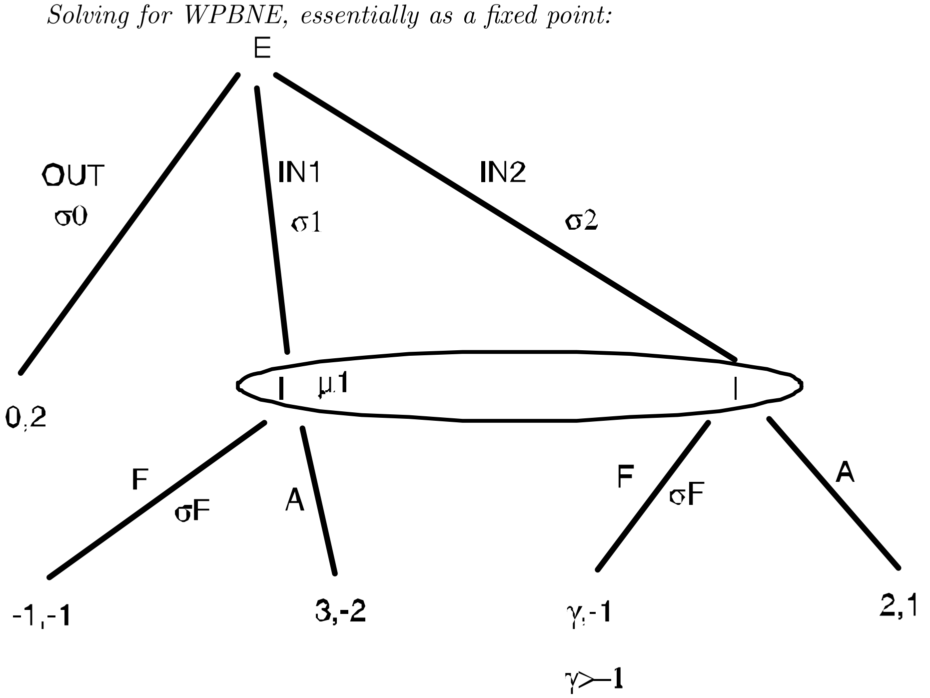
With $\sigma = \frac{1}{2}$, E is indifferent among OUT, IN1 and IN2, as each gives him an expected payoff of zero. We thus have a family of WPBNE: For any $\varepsilon \in [0,1]$, E plays OUT with probability $1 - 2\varepsilon$ and IN1 and IN2 each with probability ε. We have $\mu = \frac{1}{2}$, and I plays each of L, R with probability $\frac{1}{2}$.

Are there any more WPBNE? We're still working in the case of $\mu = \frac{1}{2}$. For $\sigma \neq \frac{1}{2}$, one of IN1, IN2 produces a produces a positive expected payoff for E and the other produces a negative expected payoff. I's best response is therefore exactly one of IN1, IN2, which contradicts $\mu = \frac{1}{2}$. We are therefore done.

Example 59 *9.C.3. This example is not simplified by the identification of a dominant strategy for some player. The good news is that the analysis is mostly arithmetic. If $\gamma < -1$ in the game below, then IN1 strictly dominates IN2 for player E, which makes the game less interesting.*



Solving for WPBNE, essentially as a fixed point:



- *I chooses F if*

$$\mu_1 \geq \frac{2}{3}$$

  expected payoff from choosing F$= -1$

  expected payoff from choosing A: $-2\mu_1 + (1-\mu_1) = 1 - 3\mu_1$

  F$\geq$*A iff* $-1 \geq 1 - 3\mu_1 \Leftrightarrow \mu_1 \geq \frac{2}{3}$

- *Suppose* $\mu_1 < \frac{2}{3}$. *Then firm I chooses A with probability* 1, *and so* $\sigma_1 = 1$. *This contradicts* $\mu_1 < \frac{2}{3}$. *We conclude that* $\mu_1 \geq \frac{2}{3}$ *in a WPBNE*

- *Suppose* $\mu_1 > \frac{2}{3}$. *Firm I must therefore choose F with probability* 1.
- *If* $\gamma > 0$, *then E chooses In2, and so* $\sigma_2 = 1$. *This contradicts* $\mu_1 > \frac{2}{3}$.
- *If* $\gamma < 0$, *then E chooses Out, and we have our first WPBNE in this example.*
- *If* $\gamma = 0$, *then E is indifferent between Out and In2. E would choose In1 with probability* 0, *however, and so he must choose In2 with probability 0 also in order to support I's beliefs at its information set. And so we are back at the WPBNE that we've just described.*
- ***Summary to this point:*** *We have completed the analysis of the case of* $\gamma > 0$ *and we hereafter restrict attention to the case of* $\gamma \in (-1, 0]$. *For* $\gamma \in (-1, 0]$, *we've derived one WPBNE: E chooses Out, and I chooses F because* $\mu_1 > \frac{2}{3}$. *We assume that* $\mu_1 = \frac{2}{3}$ *and* $\sigma_1 = 2\sigma_2$ *throughout the remainder of the example. The condition is required for I's beliefs to be consistent with E's strategy. Note that* $\sigma_1 = \sigma_2 = 0$ *remains as a possibility.*
- *Assuming that E chooses In1 or In2 with positive probability, Firm I must randomize between F and A to make E indifferent between In1 and In2 (this is a property of mixed strategy equilibrium):*

E's expected payoff from playing In1: $\sigma_F(-1) + (1 - \sigma_F)(3) = -4\sigma_F + 3$
E's expected payoff from playing In2: $\sigma_F(\gamma) + (1 - \sigma_F)(2) = (\gamma - 2)\sigma_F + 2$

$$-4\sigma_F + 3 = (\gamma - 2)\sigma_F + 2$$

$$1 = (\gamma + 2)\sigma_F$$

$$\sigma_F = \frac{1}{\gamma + 2}$$

With I's randomization, E's payoff from playing either In1 or In2 is

$$\begin{aligned} -4\sigma_F + 3 &= \frac{-4}{\gamma + 2} + 3 \\ &= \frac{-4 + 3\gamma + 6}{\gamma + 2} \\ &= \frac{3\gamma + 2}{\gamma + 2} \end{aligned}$$

We now have 3 cases:

1. $\gamma > \frac{-2}{3}$: *This expression is positive. E would therefore never choose Out (i.e.,* $\sigma_0 = 0$*). The WPBNE equilibrium is completed by setting* $\sigma_1 = \frac{2}{3}$ *and* $\sigma_2 = \frac{1}{3}$, *with* σ_F *as above and* $\mu_1 = 2/3$.
2. $\gamma < \frac{-2}{3}$: *This expression is negative, and so E chooses Out (i.e.,* $\sigma_0 = 1$*). A WPBNE is completed by setting* σ_F *as above and* $\mu_1 = 2/3$.
3. $\gamma = \frac{-2}{3}$: *E is indifferent between choosing Out, In1 and In2. Any* $\sigma_0, \sigma_1, \sigma_2$ *that satisfy*

$$\sigma_0 + \sigma_1 + \sigma_2 = 1$$

and

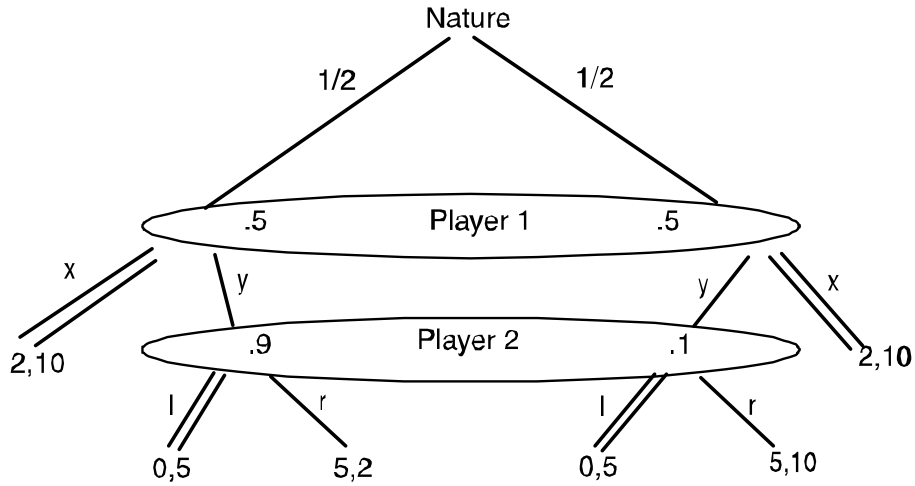
$$\frac{\sigma_1}{\sigma_1 + \sigma_2} = \frac{2}{3} \text{ and } \frac{\sigma_2}{\sigma_1 + \sigma_2} = \frac{1}{3}$$

therefore defines a WPBNE (with σ_F *as above and* $\mu_1 = 2/3$*). Notice that there are many solutions of these equations.*

Problems with the Weak Perfect Bayesian Equilibrium Concept

Off the equilibrium path beliefs – should they be "sensible" in some way?

Example 60 *9.C.4*



The move of Nature determines the payoff to Player 2 along the equilibrium path y,r. This is a game of incomplete information in which neither player knows with certainty the payoffs of 2.

Let's verify that we have a WPBNE:

l vs. r for player 2:

$$5 = (.9)(5) + (.1)(5) > (.9)(2) + (.1)(10) = 2.8$$

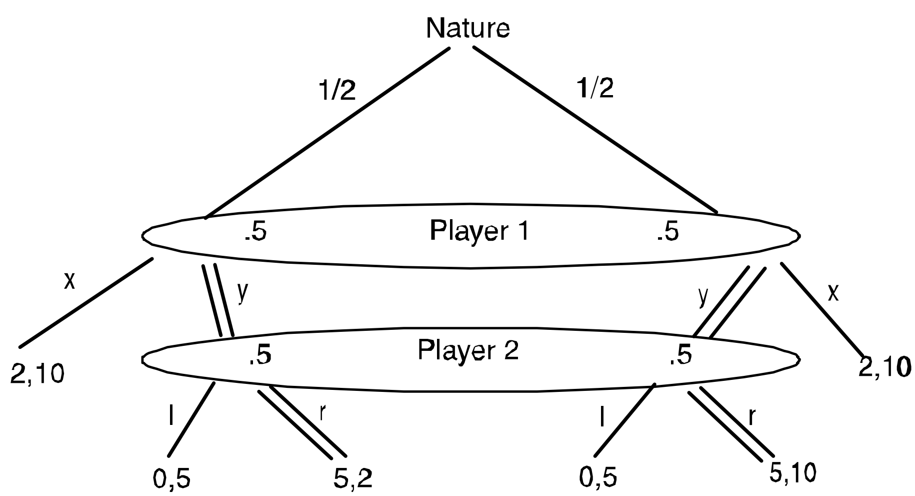
Therefore, l is best for 2 given his beliefs. Player 1 has a choice between 2 *from choosing x and* 0 *from choosing y; x is clearly his best response.*

The problem is that WPBNE does not restrict beliefs off the equilibrium path, even though those beliefs are typically important in sustaining equilibrium (i.e.,insuring that players' choices are best responses).

Clearly, we could perturb the beliefs of player 2 without breaking this equilibrium ($5 > 2.8$)*.*

One might expect in this example that, knowing the move of Nature, 2 should assign equal probability to each of the nodes in his information set (notice that 1 must make the same choice at each of his nodes). This doesn't follow from Bayes Rule, however; 2's information set is reached with probability zero in this equilibrium.

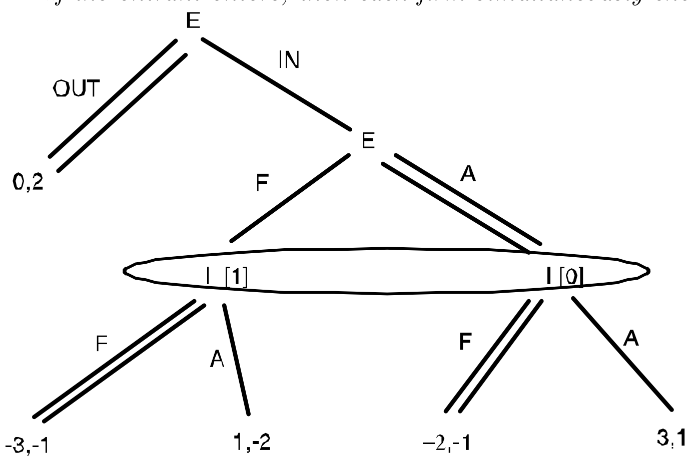
Let's concoct another equilibrium:



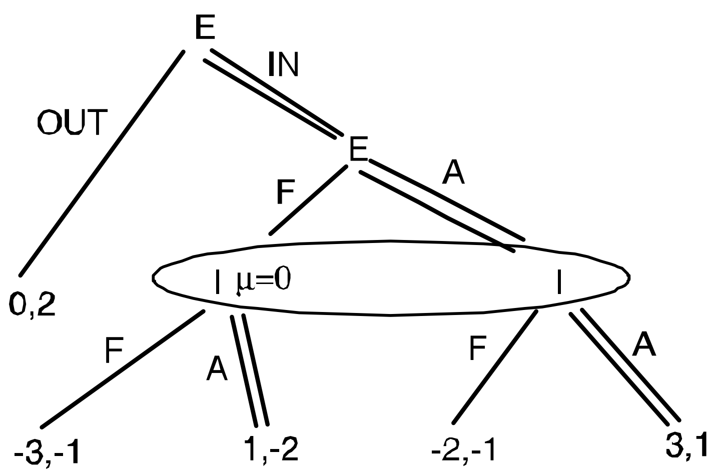
What principle supports this equilibrium over the first one discussed for this game? *We don't just want to be correct in particular examples, we want principles that explain the right answers (that's what game theory is about). Remember that the "weak" in "weak perfect Bayesian" refers to the lack of* *restrictions on off-the-equilibrium path beliefs. We're headed toward restricting these beliefs in a suitable way.*

Example 61 *9.C.5 A WPBNE need not be subgame perfect.*

If the entrant enters, then each firm simultaneously chooses F or A.



This is a WPBNE that is not SPNE (not a NE in the subgame following entry: I's choice at his information set is not a best response to E's choice of A). The problem is that Firm I's beliefs at its information set are unrestricted because this is off the equilibrium path.



Another WPBNE: I chooses F over A if

$$
\begin{aligned}
-1(\mu) - 1(1-\mu) &\geq -2(\mu) + 1(1-\mu) \\
-2 &\geq -3(\mu) \\
\frac{2}{3} &\leq \mu
\end{aligned}
$$

So we'll assume $\mu \leq \frac{2}{3}$. E will therefore choose In and A, and the equilibrium is completed by setting $\mu = 0$.