Lecture, January 20, 2015

Chapter 7 Basic Elements of Noncooperative Games

A game requires the specification of the players, the actions available to each player, and the order in which actions are taken. We might consider this to be the *rules* of the game. Games are interesting when some outcome affects the well-being of the different agents (i.e., what any player receives depends upon the actions of all players). This is the *interdependence* referred to in MWG.

perspective of MWG: dwells on the process of *modeling* a situation with a game rather than simply taking the standard economic models as reality. It's deliberate for a reason. Be patient if you already know some game theory.

Extensive Form Representation of a Game

decision nodes vs. **terminal nodes**

Example 1 *Matching Pennies and Tic-Tac-Toe*

verbal description of the two games, vs. their extensive form representations

games of **perfect information** (all decisions of all players are publicly observable) vs. game of **imperfect information** (some decisions are only observed by a proper subset of players)

information set in Matching Pennies to denote that Player 2 does not observe the choice of Player 1 at the time he makes his choice (2 versions of matching pennies)

Strategies and the Normal Form Representation of the Game

strategy sets, outcome sets, outcome mapping

noncooperative: assumes that agents act independently of each other

Strategy: complete contingent plan that specifies an agent's actions at each information set assigned to him

Example 2 *Normal form representation of the two forms of Matching Pennies (perfect and imperfect information)*

diagram of a game

Normal (or strategic) form representation of an extensive form game

Randomized choices (behavioral strategies)

Example 3 *battle of the sexes*

F/M	B	F
B	$2,1$	$0,0$
F	$0,0$	$1,2$

Example 4 *Meeting in NY:*

$1/2$	ESB	GCS
ESB	$100,100$	$0,0$
GCS	$0,0$	$100,100$

Chapter 8: Simultaneous-Move Games

solution concept: A story or theory of what happens when the game is played – a notion of equilibrium. Solution concepts typically depend upon what the players know and the different senses in which an action can be interpreted as in the best interest of a player (e.g., does the optimality of an agent's action depend upon some specification of actions for the other agents?).

game theory: precise language of incentives.

Dominant and Dominated Strategies

Example 5 *Example: prisoner's dilemma (p. 236)*

1/2	DC	C
DC	$-2,-2$	$-10,-1$
C	$-1,-10$	$-5,-5$

definition of a dominant strategy

strictly dominate, (weakly) dominate, (weakly) dominated, strictly dominant, strictly dominated

I'd like to change the definition of a (weakly) dominant strategy from the way in which it is defined in MWG.

MWG: A strategy $s_i^{'}$ is *weakly dominant* for player i if it weakly dominates every other strategy $s_i \neq s_i^{'}$, i.e.,

$$u_i(s_i^{'}, s_{-i}) \geq u_i(s_i, s_{-i})$$

for all s_{-i}, *with strict inequality for at least one* s_{-i}.

This definition is unusual because of the clause "*with strict inequality for at least one* s_{-i}". One consequence of this definition is that a player can have at most one weakly dominant strategy. If he instead had two dominant strategies $s_i^{'}$ and $s_i^{''}$, then

$$u_i(s_i^{'}, s_{-i}) > u_i(s_i^{''}, s_{-i})$$

for some at least one s_{-i} because $s_i^{'}$ weakly dominates $s_i^{''}$, but

$$u_i(s_i^{'}, s_{-i}) \leq u_i(s_i^{''}, s_{-i})$$

because $s_i^{''}$ weakly dominates $s_i^{'}$.

The more commonly used definition of a dominant strategy is as follows: $s_i^{'}$ is a dominant strategy iff for s_i and s_{-i},

$$u_i(s_i^{'}, s_{-i}) \geq u_i(s_i, s_{-i})$$

The clause "*with strict inequality for at least one* s_{-i}" has thus been dropped. A player can have more than one dominant strategy with this definition, though any two of his dominant strategies must be *payoff-equivalent* in the sense of providing him with exactly the same utility for all choices of his opponents' strategies: if $s_i^{'}$ and $s_i^{''}$ are both dominant strategies, then

$$u_i(s_i^{'}, s_{-i}) = u_i(s_i^{''}, s_{-i})$$

for all s_{-i}.

Consideration of Exercise 8.B.2. (p. 262) suggests that MWG in fact use this more conventional definition of a dominant strategy.

Example 6 *A simple model of an auction of an indivisible item. n bidders, n real valuations, quasilinear utility. Discuss randomized allocation in the case of ties.*

$u_i(v_i, p) = v_i - p$

Example 7 *Dominance and the First-Price Auction*

Consider bidder i with value v_i who considers the bid b_i. Let $\overline{b}$ denote the maximal bid of the other $n-1$ bidders.

No bid is strictly dominated. In comparing bids b_i and b_i', suppose $b_i, b_i' < \overline{b}$, i.e., one of the other bidders bids so much that bidder i loses with either bid b_i or b_i'. In such a case, both bids produce a utility of 0 and so neither bid can strictly dominate the other.

Bidding one's value or more than one's value is dominated by any bid less than one's value. We compare the consequences of bidding $b_i < v_i$ with bidding $b_i' \geq v_i$ in the table below:

bid\value of $\overline{b}$	$\overline{b} < b_i$	$\overline{b} = b_i$	$b_i < \overline{b} < b_i'$	$\overline{b} = b_i'$	$b_i' < \overline{b}$
$b_i < v_i$	$v_i - b_i > 0$	$\delta(v_i - b_i) > 0$	0	0	0
comparison	$>$	$>$	$\geq$	$\geq$	$=$
$b_i' \geq v_i$	$v_i - b_i^{'} \leq 0$	$v_i - b_i^{'} \leq 0$	$v_i - b_i^{'} \leq 0$	$\delta\left(v_i - b_i^{'}\right) \leq 0$	0

Here, $\delta \leq 1/2$ reflects the randomization in the case of ties and depends upon the number of bidders who submit the bid $\overline{b}$.

Example 8 *Dominance and the Second Price Auction*

Bidding one's valuation weakly dominates any other strategy. It is the unique dominant strategy for each bidder.

Consider bidder i with value v_i who considers the bid b_i. Let $\overline{b}$ denote the maximal bid of the other $n-1$ bidders. Bidder i's utility as a function of v_i, b_i, and $\overline{b}$ is

$$u_i(v_i, b_i, \overline{b}) = \begin{cases} v_i - \overline{b} \text{ if } b_i > \overline{b} \\ \delta\left(v_i - \overline{b}\right) \text{ if } b_i = \overline{b} \\ 0 \text{ if } b_i < \overline{b} \end{cases}$$

We see here the utility consequences of bidder i being the high bidder, tying as the high bidder, or losing the auction. Here, $\delta \leq 1/2$ reflects the randomization in the case of ties and depends upon the number of bidders who submit the bid $\overline{b}$. For any given bids of the other bidders (which bidder i doesn't know when he chooses his bid), bidder i's choice of bid b_i doesn't change the 3 possible utilities that he might receive: it only affects which of the 3 values that he receives. We can order the possible utilities as follows:

- *if* $v_i > \overline{b}$, *then* $v_i - \overline{b} > \delta\left(v_i - \overline{b}\right) > 0$.
- *if* $v_i = \overline{b}$, *then* $v_i - \overline{b} = \delta\left(v_i - \overline{b}\right) = 0$.
- *if* $v_i < \overline{b}$, *then* $0 > \delta\left(v_i - \overline{b}\right) > v_i - \overline{b}$.

The bids of the other bidders present bidder i with a choice among 3 possible utilities. The key point to notice is that choosing $b_i = v_i$ insures that bidder i receives the largest of these 3 payoffs for all possible profiles of bids by the other bidders: if he happens to be the high bidder with $v_i = b_i > \overline{b}$, then $v_i - \overline{b}$ is the maximal possible utility available to him; if he ties as high bidder with $v_i = b_i = \overline{b}$, then he is indifferent between winning and losing (0 is the best that he can do); if he loses with $v_i = b_i < \overline{b}$, then he's glad to lose because the bids of the other bidders simply do not present him with an opportunity to make a profit. For all possible profiles of bids by the other bidders, bidding his value maximizes i's utility. It is therefore a dominant strategy.

It is interesting that a bidder would not benefit from learning the bids of the other bidders in advance, for bidding his value is optimal regardless of the bids of the other bidders.

How does one argue that it is bidder i's unique dominant strategy? Any other bid is not optimal for some choice of $\overline{b}$. Consider for instance, $v_i < b_i$. If $v_i < \overline{b} < b_i$, then bidder i wins the auction and suffers a loss of $v_i - \overline{b}$. Consider $b_i < v_i$. If $b_i < \overline{b} < v_i$, then bidder i loses the auction and forgoes the positive profit he could have won by bidding between $\overline{b}$ and v_i. Bidding one's value is the only bid with the property that it maximizes one's payoff against all possible specifications of the bids of the others.

Exercise 9 *7.c.1, 7.d.1, 7.d.2, 8.b.1, 8.b.2, 8.b.3, 8.b.4,8.b.5, 8.d.1, 8.d.2*

Example 10 *(This example was not covered in class.) There are two divisions of a firm (1 and 2) that would benefit from a research project conducted by the headquarters (HQ) of the firm. The value of the project to division i is v_i and the division receives 0 if the project is not provided. Providing the project would cost HQ an amount c. HQ is concerned about the welfare of the entire firm. It therefore would like to provide the project if and only if*

$$v_1 + v_2 \geq c.$$

Each division knows the value that it would receive from the project if it is provided. HQ, however, does not know these values and must ask the divisions to report them. If a division anticipates how its report will influence the outcome, then it may choose to misrepresent the benefit that it would receive from the project. In the discussion below, let v_i^* denote a reported benefit by division i.

Suppose HQ decides to charge division i the amount

$$p_i = c - v_{-i}^*$$

when $v_1^* + v_2^* \geq c$. Here, v_{-i}^* again denotes the report of the other division $-i$. No money is exchanged when the project is not provided. This may not seem particularly intuitive as a rule for the price, but it has several virtues that will be made apparent through this example.

What is division i's payoff as a function of its report and the report of the other division?

$$u_i(v_i^*, v_{-i}^*) = \begin{cases} v_i + v_{-i}^* - c \text{ if } v_i^* + v_{-i}^* \geq c \\ 0 \text{ otherwise} \end{cases}$$

Show that reporting honestly (i.e., $v_i^* = v_i$) dominates any report that is less than v_i. Show next that honest reporting dominates any report that is greater than v_i. Conclude that the honest report is a dominant strategy for division i.

The division's report determines whether or not it receives $v_i + v_{-i}^* - c$ or 0. Ideally, division i would like to receive $v_i + v_{-i}^* - c$ if and only if it is nonnegative. Honest reporting insures this outcome and therefore is a dominant strategy.

Following the question, compare the report of v_i to $v_i^* < v_i$. The following table presents the payoff to division i given its report and the report of division $-i$:

report	$v_{-i}^* \leq c - v_i$	$c - v_i < v_{-i}^* < c - v_i^*$	$c - v_i^* \leq v_{-i}^*$
v_i	0	$v_i + v_{-i}^* - c > 0$	$v_i + v_{-i}^* - c$
$v_i^* < v_i$	0	0	$v_i + v_{-i}^* - c$

Notice that the payoff from honest reporting is either the same or strictly more than from reporting $v_i^* < v_i$. Honest reporting thus weakly dominates under-reporting.

Now compare the report of v_i to $v_i^* > v_i$. The following table presents the payoff to division i given its report and the report of division $-i$:

report	$v_{-i}^* \leq c - v_i^*$	$c - v_i^* < v_{-i}^* < c - v_i$	$c - v_i \leq v_{-i}^*$
v_i	0	0	$v_i + v_{-i}^* - c$
$v_i^* > v_i$	0	$v_i + v_{-i}^* - c < 0$	$v_i + v_{-i}^* - c$

Notice that the payoff from honest reporting is either the same or strictly more than from reporting $v_i^* > v_i$. Honest reporting thus weakly dominates over-reporting.

Does division i have any other dominant strategies? Explain (your answer to 2. may be helpful).No – this is shown by the above tables.

Assuming that each division uses its unique dominant strategy, show that the HQ provides the project exactly when it should be provided.

With honest reporting, the project is provided if and only if $v_1 + v_2 \geq c$, which is exactly when it should be provided.

How much of a deficit does HQ incur in following this procedure?

HQ collects $2c - (v_1 + v_2)$ from the divisions and and funds the project at a cost of c. Its deficit is therefore equal to the negative of the benefit to the firm as a whole from the project, $-(v_1 + v_2 - c)$.

Now assume that in addition to paying $c - v_{-i}^*$ when the project is provided, each division is required to pay a constant tax $t_i \in \mathbb{R}$ regardless of whether or not the project is provided.

Verify that honest reporting remains the unique dominant strategy of each division.

This changes division i's payoff to

$$u_i(v_i^*, v_{-i}^*) = \begin{cases} v_i + v_{-i}^* - c - t_i \text{ if } v_i^* + v_{-i}^* \geq c \\ -t_i \text{ otherwise} \end{cases}$$

The logic of the above analysis remains the same: honest reporting insures that the division receives the larger of its two possible payoffs for all v_{-i}.

What is the deficit incurred by HQ in equilibrium?

$$t_1 + t_2 - (v_1 + v_2 - c)$$

Suppose that v_1 and v_2 are jointly distributed according to the distribution F. Show that t_1 and t_2 can be chosen so that the expected deficit of HQ is zero.

This is easy: choose any t_1, t_2 such that

$$t_1 + t_2 = E_F [v_1 + v_2 - c] .$$

Example 11 (Pigou (1920)) *A suburb is connected to a train station by two highways, one old and one modern. A large number n of people in the suburb commute daily from the suburb to the station on one of these two highways. Travel time on the old highway from the suburb to the station is one hour. Travel time* *on the modern highway equals x hours, where $x \in [0,1]$ is the fraction of the n commuters who choose this modern highway. We make two observations. First, choosing the modern highway is a weakly dominant strategy for each commuter (it is strictly better except in the case in which every commuter chooses the modern highway). Second, there are Pareto superior outcomes. Suppose half the commuters choose the old highway and half choose the new highway. Those on the new highway have travel time equal to $1/2$ hour, and so the average commuting time is $3/4$ hour. It is straightfoward to show that this is the minimum possible average commuting time and $x = 1/2$ is the unique value of x that achieves this minimum.*

The average commuting time equals

$$\frac{n \cdot (1-x) \cdot 1 + n \cdot x \cdot x}{n} = (1-x) + x^2,$$

which is minimized at $x = 1/2$.

Pigou's Example is an early example in which self-interested behavior by individuals does not lead to an efficient outcome. The Prisoner's Dilemma is another example of this phenomenon. Contrast it with the Adam Smith's idea that self-interested behavior by firms and consumers leads to an outcome that is good for everyone. You saw this idea formalized as the First Fundamental Theorem of Welfare Economics in the fall semester.

Iterative Deletion of Strictly Dominated Strategies

define *rationality* as maximization of a "reasonable" utility function

MWG: The approach here is, "How far can we get in the analysis of the game from the assumption that a player acts rationally? Or instead, with the assumption that all players act rationally and know that all other players act rationally? Etc.. Dominance takes us as far as we can under the assumption that each individual player is rational. We now work with the assumption that it is common knowledge that all players are rational.

This subsection begins the discussion of one player analyzing his opponents' payoffs to understand what they may do in the game.

Example 12 *The DA's Brother vs. The Prisoner's Dilemma*

1/2	DC	C
DC	$0, -2$	$-10, -1$
C	$-1, -10$	$-5, -5$

1/2	DC	C
DC	$-2, -2$	$-10, -1$
C	$-1, -10$	$-5, -5$

In comparison with the Prisoner's Dilemma, here player 1 receives preferential treatment ($0 > -2$) if neither player confesses (he's the "DA's brother"). Notice that C no longer dominates DC for player 1, for he would prefer to choose DC in the event that 2 chooses DC. If player 1 knows 2's payoffs, however, and if he knows that 2 is rational, then he can deduce that 2 would never choose DC (it is a strictly dominated strategy for him). Therefore, 1 chooses C because he deduces that 2 will also choose C.

Example 13

1/2	L	C	R
U	$0, 3$	$3, 2$	$4, 1$
M	$-1, -2$	$1, 0$	$2, -1$
D	$1, 1$	$4, 3$	$3, 2$

$\Rightarrow$

1/2	L	C
U	$0, 3$	$3, 2$
M	$-1, -2$	$1, 0$
D	$1, 1$	$4, 3$

$\Rightarrow$

1/2	L	C
D	$1, 1$	$4, 3$

$\Rightarrow$

1/2	C
D	$4, 3$

or

1/2	L	C	R
U	$0, 3$	$3, 2$	$4, 1$
M	$-1, -2$	$1, 0$	$2, -1$
D	$1, 1$	$4, 3$	$3, 2$

$\Rightarrow$

1/2	L	C	R
U	$0, 3$	$3, 2$	$4, 1$
D	$1, 1$	$4, 3$	$3, 2$

$\Rightarrow$

1/2	L	C
U	$0, 3$	$3, 2$
D	$1, 1$	$4, 3$

$\Rightarrow$

1/2	L	C
D	$1, 1$	$4, 3$

$\Rightarrow$

1/2	C
D	$4, 3$

Issues concerning iterative deletion:

1. Which player starts?
2. Is it essential that one eliminate all possible strategies at each step?
3. Can both players eliminate strategies in each iteration?
4. It doesn't solve all games (e.g., *Battle of the Sexes* or *Meeting in New York*)

Successive elimination of strictly dominated strategies does not depend upon the order or the thoroughness of elimination in a finite game. (Exercise 8.B.4: Consider the strategies available at stage n; for a strictly dominated strategy s_i, there must exists a strategy s_i^* that strictly dominates it that itself is not strictly dominated. The strategy s_i^* will carry over to strictly dominate s_i at a future stage.) Emphasize how the answer depends upon strict dominance.

Successive elimination of weakly dominated strategies can depend on the order.

Example 14

$$\begin{array}{ccc} 1/2 & L & R \\ U & 5,1 & 4,0 \\ M & 6,0 & 3,1 \\ D & 6,4 & 4,4 \end{array} \Rightarrow \begin{array}{ccc} 1/2 & L & R \\ U & 5,1 & 4,0 \\ D & 6,4 & 4,4 \end{array} \Rightarrow \begin{array}{cc} 1/2 & L \\ U & 5,1 \\ D & 6,4 \end{array} \Rightarrow \begin{array}{cc} 1/2 & L \\ D & 6,4 \end{array}$$

or

$$\begin{array}{ccc} 1/2 & L & R \\ U & 5,1 & 4,0 \\ M & 6,0 & 3,1 \\ D & 6,4 & 4,4 \end{array} \Rightarrow \begin{array}{ccc} 1/2 & L & R \\ M & 6,0 & 3,1 \\ D & 6,4 & 4,4 \end{array} \Rightarrow \begin{array}{cc} 1/2 & R \\ M & 3,1 \\ D & 4,4 \end{array} \Rightarrow \begin{array}{cc} 1/2 & R \\ D & 4,4 \end{array}$$

The rationale behind eliminating a player's weakly dominated strategy is that the opponent may use *any* of his strategies (i.e., one can focus on strategies by the opponents in which a given strategy is strictly dominated as grounds for eliminating it). The logic of successive elimination, however, is that the opponent may not necessarily use all of his strategies. Successively eliminating weakly dominated strategies therefore is not as sensible or coherent as a method for solving a game as successively eliminating strictly dominated strategies. The complements the fact that it may be ambiguous in the solutions that it provides.

Criticism of this solution concept:

1. Elimination of either weakly or strictly dominated strategies doesn't get us very far with the first price auction (we can conclude that every bidder should bid less than his value). This is a problem with these solution concepts: there are many games that they fail to solve.

2. A second criticism is that the sequence of logical deductions may exceed the abilities of most humans.

3. A third criticism is that using the solution concept requires that a player believe that his opponents are also applying it correctly.