- First class:
  - introduction
  - elementary model of the k-DA (IPV)
  - bilateral case
  - multilateral case
  - convergence: price-taking and efficiency

- Second class:
  - discussion of the convergence results
  - numerical work
  - discussion of related work
- Third class:
  - asymptotic distributions
  - asymptotic FOC and solution

Issues and Motivations

- experimental evidence dating from the 1960s
  - "clearinghouse" (one-shot) versus continuous time
- price discovery vs. price verification: where do prices come from?
- M. A. Satterthwaite

- market design
  - design of algorithms for computerized trading
  - Budish, Cramton and Shim (2014)
  - Loertscher, Marx and Wilkening (2015)
  - Loertscher and Mazzetti (2014)

Elementary Trading Environment

- m buyers and n sellers
- unitary demand/supply
- indivisible units
- redemption value/cost: $v_i \in [\underline{v}, \overline{v}]$, $v_i \backsim G$, $c_i \in [\underline{c}, \overline{c}]$, $c_j \backsim F$.
  - $[\underline{v}, \overline{v}] = [\underline{c}, \overline{c}] = [0, 1]$
- quasilinear utility

For $k \in [0,1]$, k-Double Auction in Bilateral Case

- bid b, ask a

- trade iff $b \geq a$ at price $kb + (1-k)a$

- $k = 1$: buyer's bid double auction

  - dominant strategy of seller to submit his cost as his ask

Buyer FOC

$$\pi^B(v,b) = \int_0^{S^{-1}(b)} (v - (kb + (1-k)S(c)))f(c)dc$$

marginal utility:

$$(v - \left(kb + (1-k)S(S^{-1}(b))\right) f(S^{-1}(b)) \cdot \frac{1}{S'\left(S^{-1}(b)\right)} - kF\left(S^{-1}(b)\right)$$

$$= (v-b) \cdot \frac{f(S^{-1}(b))}{S'\left(S^{-1}(b)\right)} - kF\left(S^{-1}(b)\right)$$

Seller FOC

$$\pi^S(c,a) = \int_{B^{-1}(a)}^{1} ((kB(v) + (1-k)a) - c)g(v)dv$$

marginal utility:

$$-(\left(kB(B^{-1}(a)) + (1-k)a\right) - c)g(B^{-1}(a)) \cdot \frac{1}{B'\left(B^{-1}(a)\right)}$$

$$+ (1-k)\left(1 - G\left(B^{-1}(a)\right)\right)$$

$$= -(a-c) \cdot \frac{g(B^{-1}(a))}{B'\left(B^{-1}(a)\right)} + (1-k)\left(1 - G\left(B^{-1}(a)\right)\right)$$

Equilibrium

$$0 = (v - B(v)) \cdot \frac{f(S^{-1}(B(v))}{S'\left(S^{-1}(B(v)\right)} - kF\left(S^{-1}(B(v))\right)$$

$$= -(S(c) - c) \cdot \frac{g(B^{-1}(S(c))}{B'\left(B^{-1}(S(c)\right)} + (1-k)\left(1 - G(B^{-1}(S(c))\right)$$

"linked" differential equations

Chatterjee-Samuelson Solution in Uniform Case

$$B(v) = \begin{cases} \frac{2}{3}v + \frac{1}{12} \text{ if } v \geq \frac{1}{4} \\ v \text{ if } v \geq \frac{1}{4} \end{cases}$$

$$S(c) = \begin{cases} \frac{2}{3}c + \frac{1}{4} \text{ if } c \leq \frac{3}{4} \\ c \text{ if } c \geq \frac{3}{4} \end{cases}$$

Sufficiency of FOC

evaluating the buyer's FOC at bid b, value $B^{-1}(b)$:

$$0 = \left(B^{-1}(b) - b\right) \cdot \frac{f(S^{-1}(b))}{S'\left(S^{-1}(b)\right)} - kF\left(S^{-1}(b)\right) \Leftrightarrow$$

$$\left(B^{-1}(b) - b\right) = \frac{kF\left(S^{-1}(b)\right)}{S'\left(S^{-1}(b)\right)} S'\left(S^{-1}(b)\right)$$

marginal utility with value v and bid b:

$$\begin{aligned}
\pi_b^B(v,b) &= (v-b)\cdot\frac{f(S^{-1}(b))}{S'\left(S^{-1}(b)\right)} - kF\left(S^{-1}(b)\right) \\
&= \frac{f(S^{-1}(b))}{S'\left(S^{-1}(b)\right)}\left[(v-b) - \frac{kF\left(S^{-1}(b)\right)}{f(S^{-1}(b))}S'\left(S^{-1}(b)\right)\right] \\
&= \frac{f(S^{-1}(b))}{S'\left(S^{-1}(b)\right)}\left[(v-b) - \left(B^{-1}(b)-b\right)\right] \\
&= \frac{f(S^{-1}(b))}{S'\left(S^{-1}(b)\right)}\left[(v-B^{-1}(b))\right]
\end{aligned}$$

Geometric Representation

$$0 \leq c \leq \lambda \leq v \leq 1,$$

$$0 = (v - \lambda) \cdot f(c)\dot{c} - kF(c)$$

$$0 = -(\lambda - c) \cdot g(v)\dot{v} + (1 - k)(1 - G(v))$$

$$\dot{v} = \frac{(1-k)\left(1-G\left(v\right)\right)}{(\lambda - c)\cdot g(v)}$$

$$\dot{c} = \frac{kF\left(c\right)}{(v-\lambda)\cdot f(c)}$$

$$\dot{v} = 0 : v = 1 \qquad \dot{v} = \infty : c = \lambda$$

$$\dot{c} = 0 : c = 0 \qquad \dot{v} = \infty : v = \lambda$$

Existence of "Double Continuum" of Equilibria

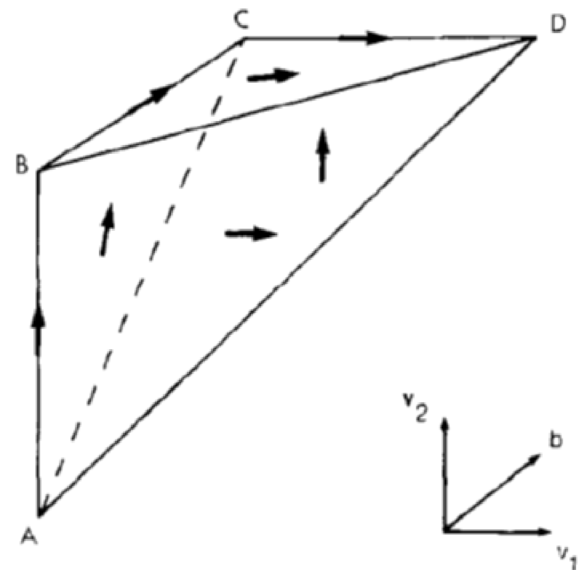


FIG. 3.1. Tetrahedron $0 \leqslant v_1 \leqslant b \leqslant v_2 \leqslant 1$ that contains solutions. The arrows indicate the imit of the normalized vector field on the tetrahedron's faces and edges.

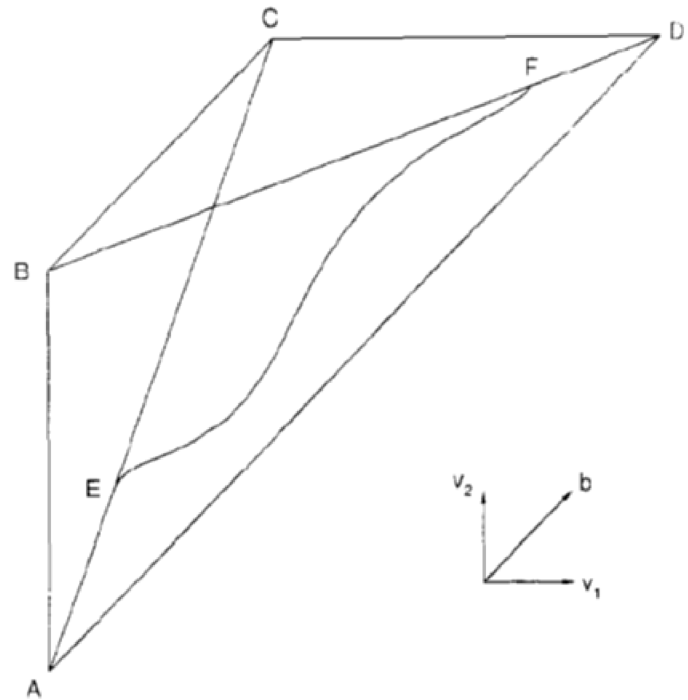


FIG. 4.1. Solution through $(v_1, v_2, b) = (0.375, 0.625, 0.45)$ shown within tetrahedron. The solution enters the tetrahedron at point E and exits through point F.

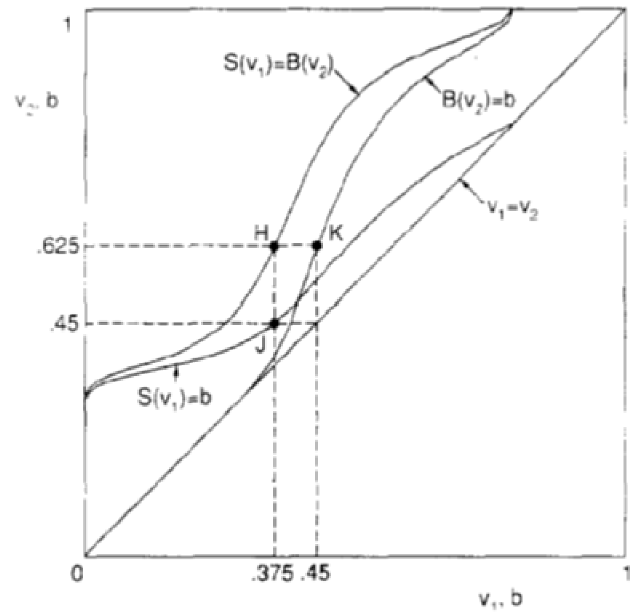
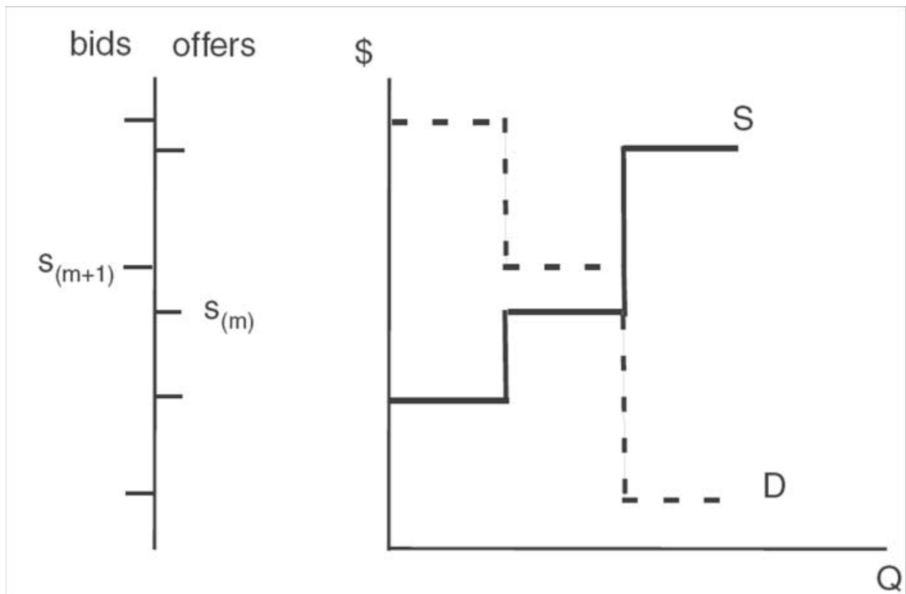


FIG. 4.2. Solution through $(v_1, v_2, b) = (0.375,\ 0.625,\ 0.45)$. Point $H = (v_1, v_2) = (0.375, 0.625)$ is on the trading boundary where $S(0.375) = B(0.625) = 0.45$. Point $J = (v_1, S(v_1)) = (0.375,\ 0.45)$ is on the graph of the seller's strategy. Point $K = (B(v_2), v_2) = (0.45,\ 0.625)$ is on the graph of the buyer's strategy. The ex ante expected utility is 0.0654 for the seller and 0.0725 for the buyer.

Multilateral Case

$$s_{(1)} \leq s_{(2)} \leq \ldots \leq s_{(m+n)}$$



$$s_{(m)} < s_{(m+1)}$$

	bids	asks
$\geq s_{(m+1)}$	t	
$\leq s_{(m)}$	$m - t$	t

$$s_{(m)} = s_{(m+1)},\ s + x > 1$$

	bids	asks
$> s_{(m)} = s_{(m+1)}$	t	$n - x - y$
$= s_{(m)} = s_{(m+1)}$	s	x
$< s_{(m)} = s_{(m+1)}$	$m - s - t$	y

$$t + (n - x - y) < n \Rightarrow t < x + y$$

$$y + (m - s - t) < m \Rightarrow y < s + t$$

Increasing Strategies

- probability of trading must be nondecreasing in a buyer's value and nonincreasing in a seller's cost

- "no flat spots" in the multilateral case

- Leininger, Linhart and Radner (1989): step function equilibria

- no trade equilibrium

Buyer's Bid Double Auction (BBDA)

- $p = s_{(m+1)}$ with sellers trading only if their asks are strictly less than the price

FOC:

$$\pi_b^B(v, b) = (v - b)\,\mathsf{Pr}(b = s_{(m)}) - \mathsf{Pr}(s_{(m)} < b < s_{(m+1)})$$

$$\mathsf{Pr}(s_{(m)} < b < s_{(m+1)}) =$$

$$\sum_{i=0}^{m-1} \binom{m-1}{i} \binom{n}{m-i} G\left(B^{-1}(b)\right)^{i} \left(1 - G\left(B^{-1}(b)\right)\right)^{m-1-i} \cdot$$

$$F\,(c)^{m-i}\,(1 - F(c))^{n-m+i}$$

$$\Pr(b = s_{(m)}) =$$

$$nf(b) \cdot \sum_{i=0}^{m-1} \left(\begin{array}{c} m-1 \\ i \end{array} \right) \left(\begin{array}{c} n-1 \\ m-1-i \end{array} \right) G\left(B^{-1}(b)\right)^i \cdot$$
$$\left(1 - G\left(B^{-1}(b)\right)\right)^{m-1-i} F(c)^{m-1-i} (1 - F(c))^{n-m+i}$$

$$+(m-1)\frac{g(B^{-1}(b))}{B'(B^{-1}(b))} \cdot \sum_{i=0}^{m-1} \left(\begin{array}{c} m-2 \\ i \end{array} \right) \left(\begin{array}{c} n-2 \\ m-1-i \end{array} \right) G\left(B^{-1}(b)\right)^i \cdot$$
$$\left(1 - G\left(B^{-1}(b)\right)\right)^{m-2-i} F(c)^{m-1-i} (1 - F(c))^{n-m+i+1}$$

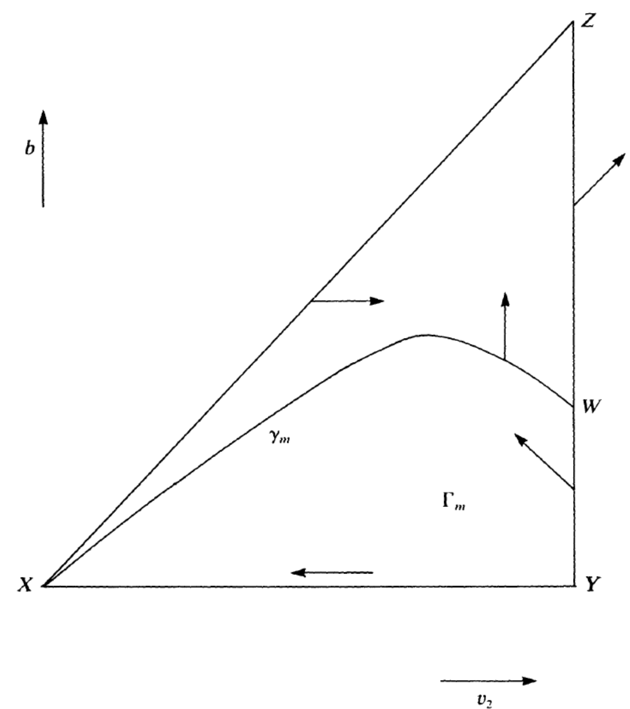


FIGURE 4.1

If $\langle \tilde{S}, B \rangle$ is an equilibrium then the graph of B lies in the triangle XYZ defined by the inequalities $0 \leqq b \leqq v_2 \leqq 1$. The arrows show the direction of the vector field $(\dot{v}_2, \dot{b})$ on the edges and at a point on γ_m

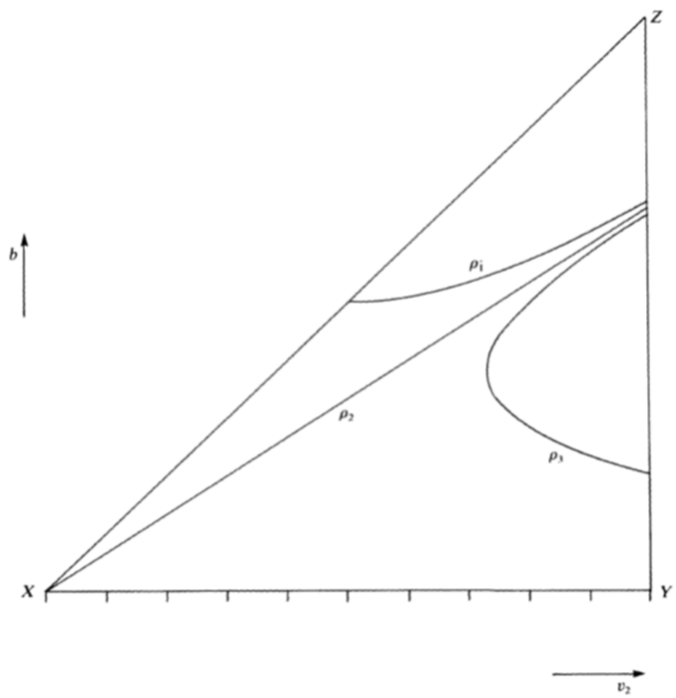


FIGURE 4.2

The curves ρ_1, ρ_2, and ρ_3 are solutions to the differential equation (3.6)–(3.7) when $m = 2$ and reservation values are distributed uniformly. Only ρ_2 defines an equilibrium

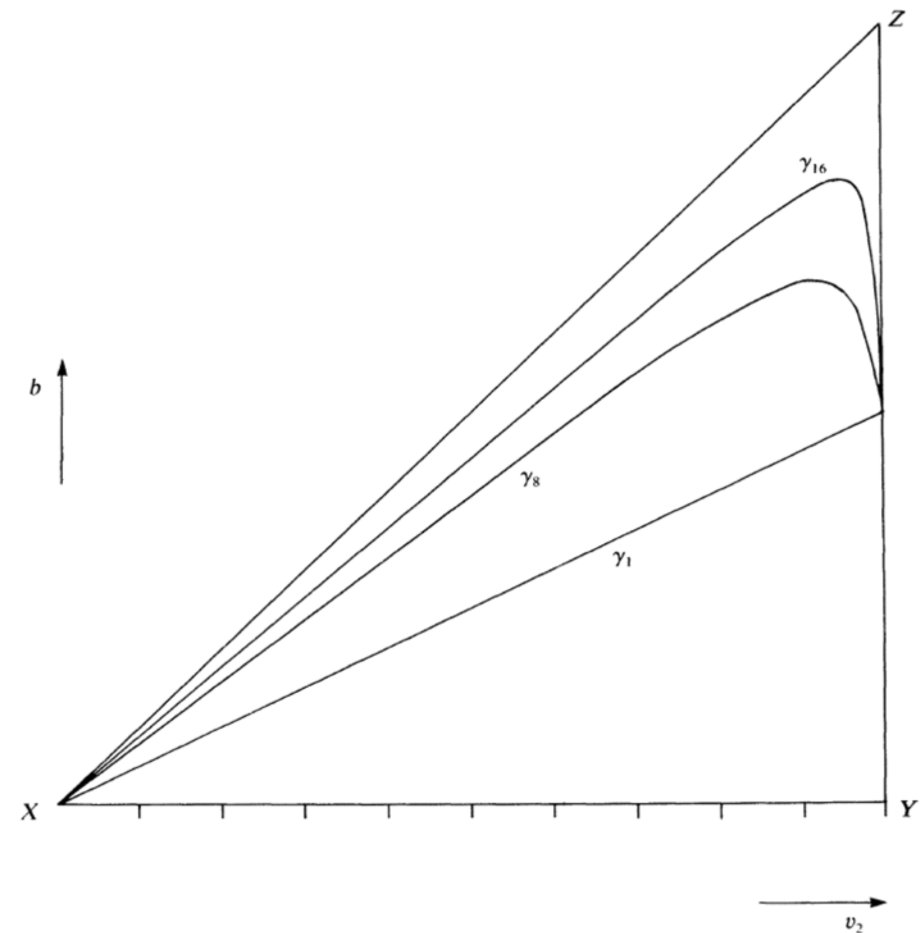


FIGURE 4.3

The boundaries γ_1, γ_8, and γ_{16} are shown for the uniform case. The graph of any equilibrium strategy B in a market with $2m$ traders must lie above γ_m almost everywhere. The edge XZ corresponds to the strategy of truthful revelation

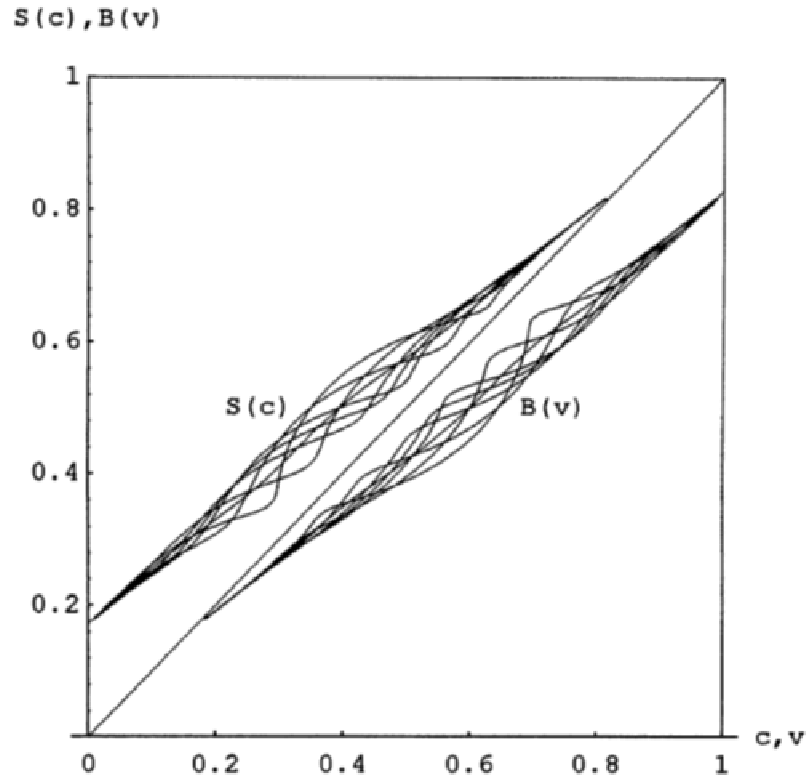


FIGURE 3.—A bundle of equilibrium strategies in the 0.5-DA for uniform F and G and $m = n = 2$. Buyers' strategies lie below the diagonal, sellers' strategies lie above it, and each buyer's strategy is paired with a particular seller's strategy to form an equilibrium.

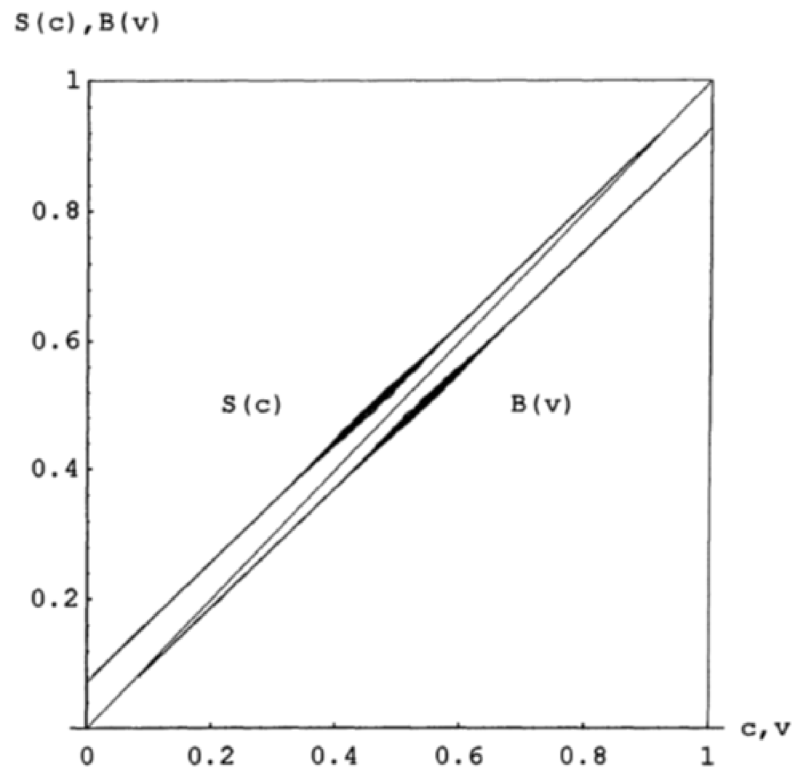


FIGURE 4.—A bundle of equilibrium strategies in the 0.5-DA for uniform F and G and $m = n = 6$.

Convergence Results

- m, n satisfy the bounds $\frac{1}{K} \le \frac{m}{n}, \frac{n}{m} \le K$

- Convergence to price-taking behavior at the rate $O(1/m)$:

$$v - B(v), S(c) - c \le \frac{\kappa_1(K, F, G)}{m}$$

Convergence to Efficiency

- relative inefficiency:

$$\frac{GFT^{pt} - GFT^{e}}{GFT^{pt}}$$

- relative inefficiency is $O(1/m^2)$:

$$\frac{GFT^{pt} - GFT^{e}}{GFT^{pt}} \leq \frac{\kappa_2\,(K, F, G)}{m^2}$$

- meaning of rates

Numerical Results

EXPECTED INEFFICIENCIES OF THE OPTIMAL MECHANISM, THE LEAST AND MOST INEFFICIENT EQUILIBRIA OF THE 0.5-DOUBLE AUCTION, THE DUAL PRICE MECHANISM, AND THE FIXED PRICE MECHANISM FOR DIFFERENT MARKET SIZES IN THE CASE OF UNIFORM F AND G.
(For the dual price mechanism the first number listed includes in the gains from trade only the profits of the traders while the second number also includes the specialist's profits.)

$m = n$	Optimal Mechanism	0.5-DA Least	0.5-DA Most	Dual Price Mechanism	Fixed Price Mechanism
1	0.16	0.16	1.00	0.25 (0.25)	0.25
2	0.056	0.056	0.063	0.21 (0.18)	0.22
4	0.015	0.015	0.016	0.16 (0.075)	0.18
6	0.0069	0.0069	0.0070	0.12 (0.040)	0.16
8	0.0039	0.0039	0.0039	0.099 (0.024)	0.15

Notes: The values of the optimal and fixed price mechanisms are taken from Tables I and II respectively of Gresik and Satterthwaite (1989). We calculated the values for the dual price mechanism by numerical integration; our values for $m = 2$ and 4 agree with the values from a simulation that McAfee (1992) reported in his Table I. Finally, the values for the 0.5-DA were obtained by numerical integration using the equilibria that we computed employing the procedure described in footnote 12. Calculation of the values for the 0.5-DA posed numerical difficulties; consequently values are reported to only two significant digits.

Is This Fast?

the k-DA is *worst case asymptotic optimal* among all BIC, IIR and EABB mechanisms

- asymptotic: mechanisms are compared using the rates at which relative inefficiency converges to zero

- worst-case: each mechanism is evaluated in its least favorable environment

Result

- relative inefficiency of the *constrained efficient mechanism* $\geq \gamma/m^2$ in the uniform case for some $\gamma > 0$

- We believe that this holds for all F, G that are reasonably well-behaved

- relative inefficiency of any mechanism in its worst case $\geq$

  relative inefficiency of constrained efficient mechanism in its worst case $\geq \gamma/m^2$

T. A. Gresik and M. A. Satterthwaite (1989)

- multilateral case
- markets with ηm buyers and ηn sellers for $\eta \in \mathbb{N}$
- relative inefficiency in the constrained efficient mechanism is $O\left(\frac{\ln \eta}{\eta^2}\right)$
- notable for the question that it pursues

R. Wilson (1985)

- If **min** $\{m, n\}$ is sufficiently large, then an equilibrium of a k-DA is interim incentive efficient

- imputes welfare weights from the allocation rule in the k-DA in equilibrium

- the issue is whether or not these imputed weights are positive

- these imputed weights converge uniformly for all trader types to 1 as

  min $\{m, n\} \to \infty$

Wilson Critique

- procedures that are not defined in terms of the probabilistic beliefs of the agents (traders)

- relaxing the assumption of common knowledge of beliefs

- "The practical advantage of a double auction is that its rules for trades and payments do not invoke the data that are common knowledge among the agents – namely, the numbers of buyers and sellers, the joint probability distribution of their types, and the functional dependence of their reservation prices on the type parameter. Instead, the burden of coping with the complexity of the common knowledge features is assumed by the traders in the construction of their strategies."

P. McAfee (1992)

$b_{(1)}$	$s_{(n)}$
$b_{(2)}$	.
.	.
.	
$b_{(k)}$	$s_{(k+1)}$
$b_{(k+1)}$	$s_{(k)}$
.	.
.	.
.	.
$b_{(m)}$	$s_{(1)}$

$b_{(1)}$	$s_{(n)}$
$b_{(2)}$	.
.	$s_{(k+1)}$
.	
$b_{(k)}$	
$b_{(k+1)}$	$s_{(k)}$
.	.
.	.
.	.
$b_{(m)}$	$s_{(1)}$

$+$: dominant strategies

$+$: if the monetary surplus of the "specialist" is counted among the gains from trade, then expected inefficiency is $O\left(1/\left(m+n\right)\right)$

-: does not converge to efficiency if the monetary surplus is treated as a cost of trading to the traders

- Loertscher, Marx and Wilkening (2015), Anbarci and Roy (2015)

Large Double Auctions

- 1. S-W
  - focus on the first order conditions
  - analyzed using combinatorics
- large double auctions: assumes a sufficiently large number of traders
  - results of probability and statistics become applicable
  - remains a model of strategic price discovery

– rarely the production of an equilibrium or any connection to smaller markets

– motivation: strategic foundation for competitive equilibrium and for REE

M. W. Cripps and J. M. Swinkels (2006)

- As the number of traders grows, every nontrivial equilibrium of the double auction converges to the Walrasian outcome. Relative inefficiency disappears at the rate $1/n^{2-\alpha}$ for any $\alpha > 0$

- correlated, private values in $[0, 1]$

  - asymmetry of the distribution and across the strategies used by each side of the market is allowed

  - *no asymptotic gaps*, *no asymptotic atoms*

  - for $z \in (0, 1]$, *z-independence*

- symmetry and "purification" of equilibrium strategies as the number of traders grows

- n "quite large" is necessary

P. J. Reny and M. Perry (2006)

- a strategic foundation for rational expectations equilibrium
- affiliated, interdependent values/costs
- limit market: BNE equilibrium in increasing strategies that implements REE price
- continuity as the number of traders and the number of possible bids/asks goes to infinity

- all traders are fully rational and strategic: no noise traders and sellers are active (unlike auction models)

- no indication of how large a market is required, no examples in finite markets

R. C. Shafer (2015)

- P. B. Linhart and R. Radner (1989)

- Does the emergence of price-taking behavior as the market increases in size fundamentally require that traders be Bayesians?

- minimax regret and maxmin: behavior invariant to the size of the market
  - culprit: this is true of any decision rule that satisfies the axiom of symmetry
  - Γ-minimax regret, and Γ-maxmin; minimizing maximum expected regret

J. H. Kagel and W. Vogt (1993)

- experimental design
  - $m = 2$ and $m = 8$ traders on each side
- few sellers played their dominant strategies, causing inefficiency
- buyers underbid by less than the equilibrium prediction
- change from $m = 2$ to $m = 8$ notable but not as much as predicted by theory
- opportunities for learning in BBDA

Continuous Bid/Ask Market

- R. Wilson, "Equilibria of Bid-Ask Markets," 1987.

- D. Easley and J. Ledyard, "Theories of Price Formation and Exchange in Double Oral Auctions," 1993.

- D. Friedman, "A Simple Testable Model of Double Auction Markets," 1991.

- D. Friedman, "The Double Auction Market Institution: A Survey, "1993.

- *The Double Auction Market: Institutions, Theories and Evidence*, D. Friedman and J. Rust, eds.

Asymptotics in the CPV Model

- identify the asymptotic distribution of the BBDA's price
- identify the asymptotic limits of the probabilities in a trader's FOC
- formulate the asymptotic FOCs (AFOCs) and solve
- compare the solutions to the AFOCs to computed equilibrium
- AFOCs identify what is "first order" in a trader's decision problem

Review

- a state μ is drawn from the *uniform improper prior* on $\mathbb{R}$

- buyer i's value is $v_i = \mu + \varepsilon_i$ and seller j's cost is $c_j = \mu + \varepsilon_j$, where

  ε_i, $\varepsilon_j \backsim G_\varepsilon$

- a *correlated, private value model (CPV)*

- convergence results

Limit Market

- $q \equiv \frac{m}{m+n}$, $\xi_q \equiv G_{\varepsilon}^{-1}(q)$

- measure q of buyers and measure $1 - q$ of sellers

- values/costs z, which conditional on μ, are i.i.d. according to $G_{\varepsilon}(z - \mu)$.

REEE

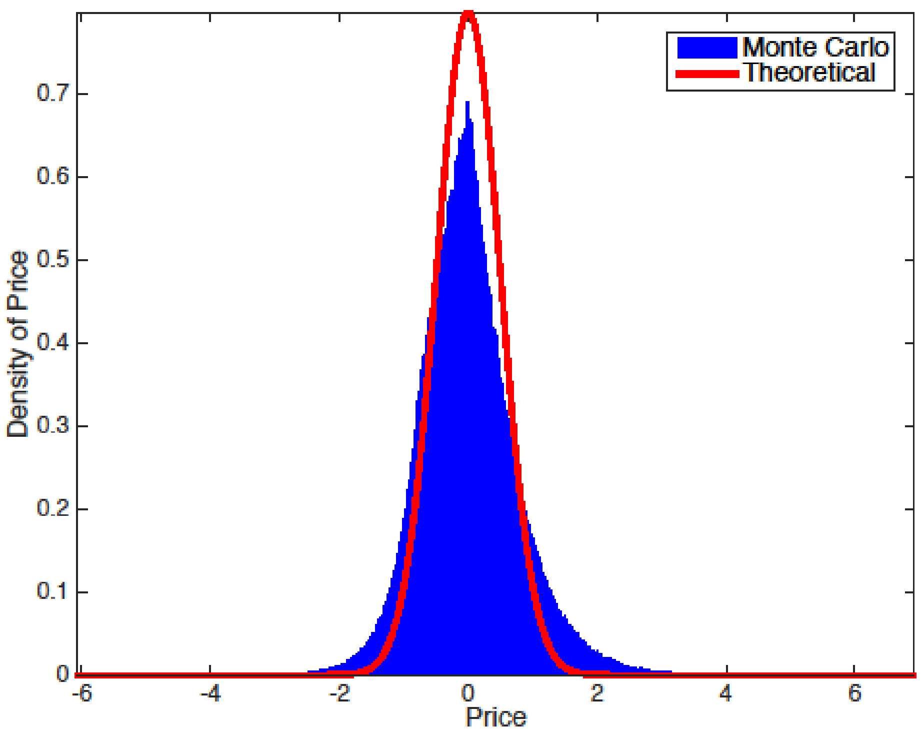
- REE: The unique REE price in state μ is $p^{\mathsf{REE}} \equiv \mu + \xi_q$
- *REE function* $P^{\mathsf{REE}} : \mathbb{R} \to \mathbb{R}$
  - invertible
  - $P^{\mathsf{REE}}(\mu) = p^{\mathsf{REE}}$ clears the limit market in the state μ

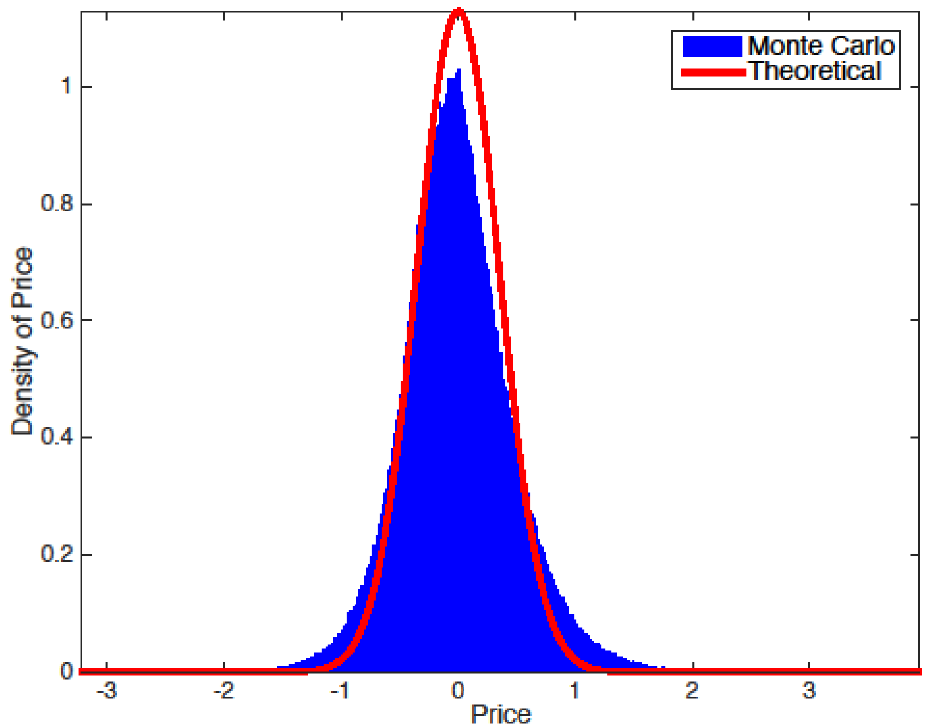
Asymptotics

CPV case: For each μ, p^{pt} and p^{eq} share the same asymptotic distribution,

$$p^{\mathsf{pt}},\, p^{\mathsf{eq}} \sim \mathcal{AN}\left(\mu + \xi_q^{\varepsilon}, \frac{mn}{\eta\,(m+n)^3\, g_{\varepsilon}^2\left(\xi_q^{\varepsilon}\right)}\right)$$

- each is an asymptotically unbiased and consistent estimate of $\mu + \xi_q^{\varepsilon}$
- holds despite the fact that $\mathbb{E}\left[p^{\mathsf{pt}} - p^{\mathsf{eq}}\,|\mu\right] > 0$ for all η
- result concerning p^{pt} is standard; result concerning p^{eq} is new


Monte Carlo
Theoretical
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
Density of Price
-6
-4
-2
0
2
4
6
Price


Monte Carlo
Theoretical
Density of Price
Price
1
0.8
0.6
0.4
0.2
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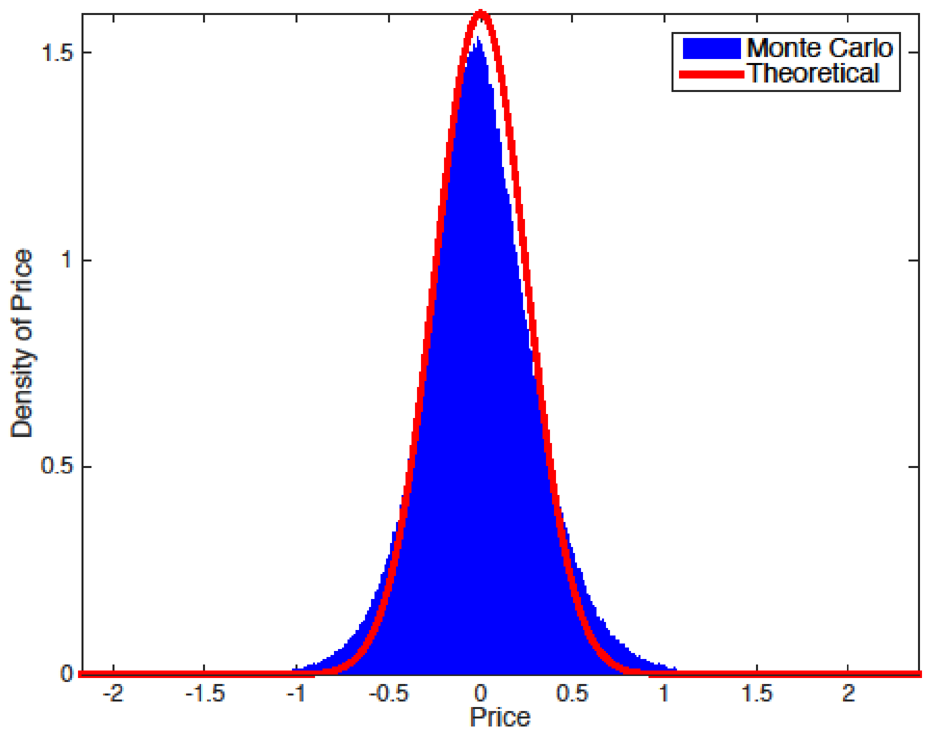
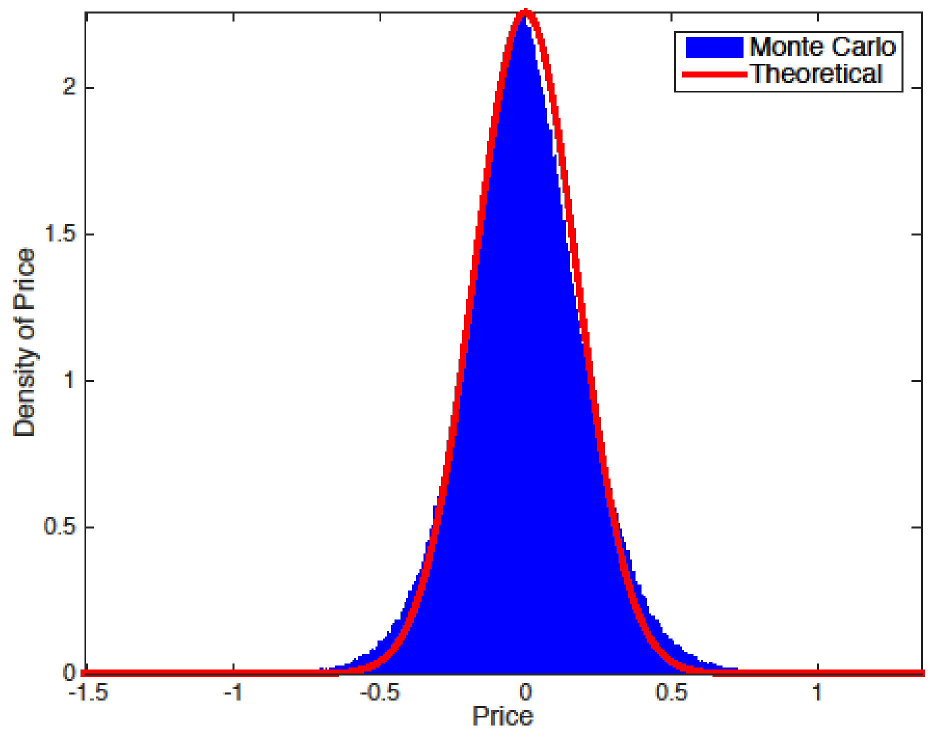
-3
-2
-1
0
1
2
3


Monte Carlo
Theoretical
Density of Price
Price
1.5
1
0.5
0
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2



Monte Carlo
Theoretical
Density of Price
Price
2
1.5
1
0.5
0
-1.5
-1
-0.5
0
0.5
1

CPV case ($m = n = 1$, G_ε standard normal)

η	$\mathsf{VAR}\left(p^{\mathsf{pt}} - p^{\mathsf{REE}} \mid \mu\right)$	$\mathsf{VAR}\left(p^{\mathsf{eq}} - p^{\mathsf{REE}} \mid \mu\right)$	$\frac{1}{8\eta\phi^2(0)}$
2	0.3646	0.3834	0.3927
4	0.1887	0.1901	0.1963
8	0.0954	0.0958	0.0981
16	0.0482	0.0483	0.0491

Asymptotic FOC in CPV and its Solution

fix n, m; markets with ηm buyers and ηn sellers

$x(\eta) = s_{\eta m:\eta(m+n)-1}$, $y(\eta) = s_{\eta m+1:\eta(m+n)-1}$,

$x(\eta)$, $y(\eta)$ are asymptotically consistent, unbiased and normal estimators of the REE price in state μ:

$$x(\eta), y(\eta) \sim \mathcal{AN}\left(p^{\mathsf{REE}}, \frac{m\,n/(m+n)^2}{[\eta(m+n)-1]\,g_{\varepsilon}^2(\xi_q)}\right)$$

FOC

$$(v-b)f_{x(\eta)|v}(b|v) - \mathsf{Pr}[x(\eta) \le b \le y(\eta)|v] = 0$$

$$\lambda_{\mathsf{approx}}(\eta) = \frac{1}{(m+n)\eta - 1}\frac{1}{g_{\varepsilon}(\xi_q)}$$

- no distinction between m and n except in determining q
- dependence on $g_{\varepsilon}(\xi_q)$Buyer's Bid Double Auction (BBDA)

- $p = s_{(m+1)}$ with sellers trading only if their asks are strictly less than the price

FOC:

$$\pi_b^B(v, b) = (v - b)\,\mathsf{Pr}(b = s_{(m)}) - \mathsf{Pr}(s_{(m)} < b < s_{(m+1)})$$

$$\Pr(s_{(m)} < b < s_{(m+1)}) =$$

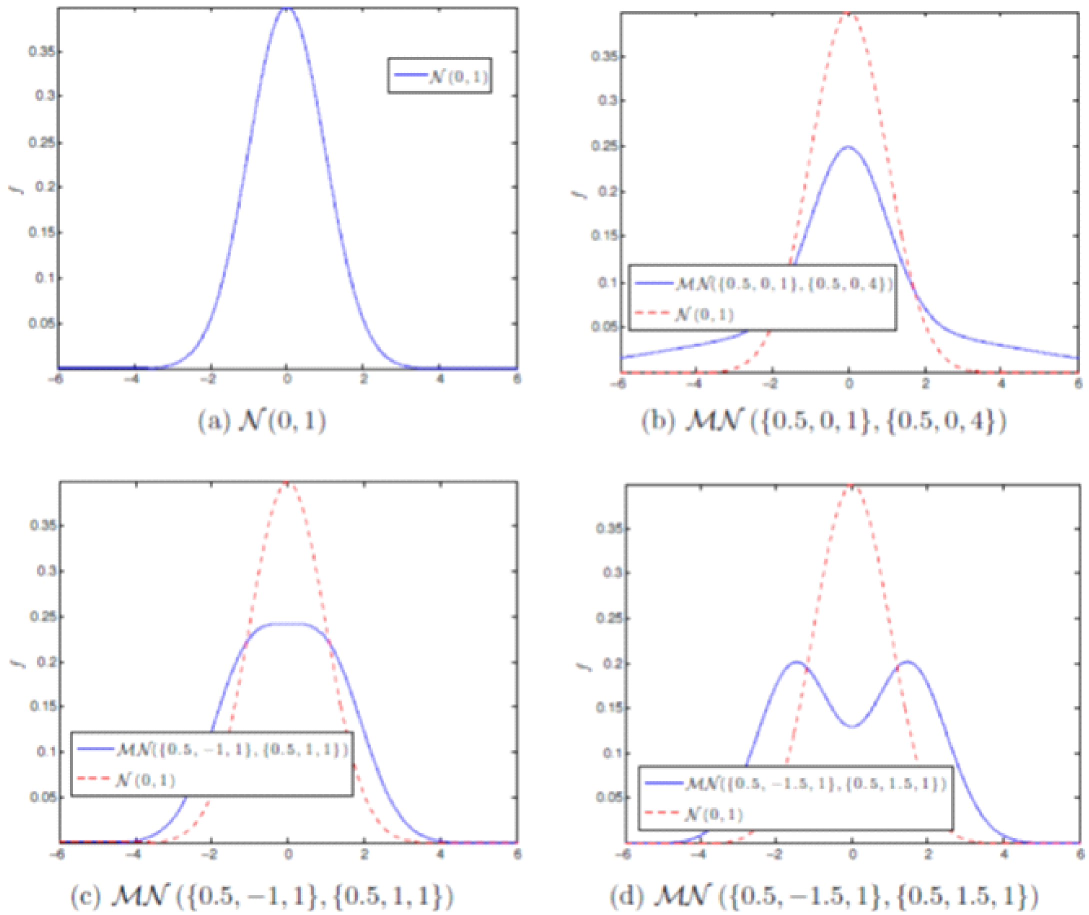
$$\sum_{i=0}^{m-1} \binom{m-1}{i} \binom{n}{m-i} G\left(B^{-1}(b)\right)^{i} \left(1 - G\left(B^{-1}(b)\right)\right)^{m-1-i} \cdot$$

$$F\left(c\right)^{m-i} \left(1 - F(c)\right)^{n-m+i}$$

$$\mathsf{Pr}(b = s_{(m)}) =$$

$$nf(b) \cdot \sum_{i=0}^{m-1} \left(\begin{array}{c} m-1 \\ i \end{array} \right) \left(\begin{array}{c} n-1 \\ m-1-i \end{array} \right) G\left(B^{-1}(b)\right)^{i} \cdot$$
$$\left(1 - G\left(B^{-1}(b)\right)\right)^{m-1-i} F(c)^{m-1-i} (1 - F(c))^{n-m+i}$$

$$+(m-1)\frac{g(B^{-1}(b))}{B'(B^{-1}(b))} \cdot \sum_{i=0}^{m-1} \left(\begin{array}{c} m-2 \\ i \end{array} \right) \left(\begin{array}{c} n-2 \\ m-1-i \end{array} \right) G\left(B^{-1}(b)\right)^{i} \cdot$$
$$\left(1 - G\left(B^{-1}(b)\right)\right)^{m-2-i} F(c)^{m-1-i} (1 - F(c))^{n-m+i+1}$$



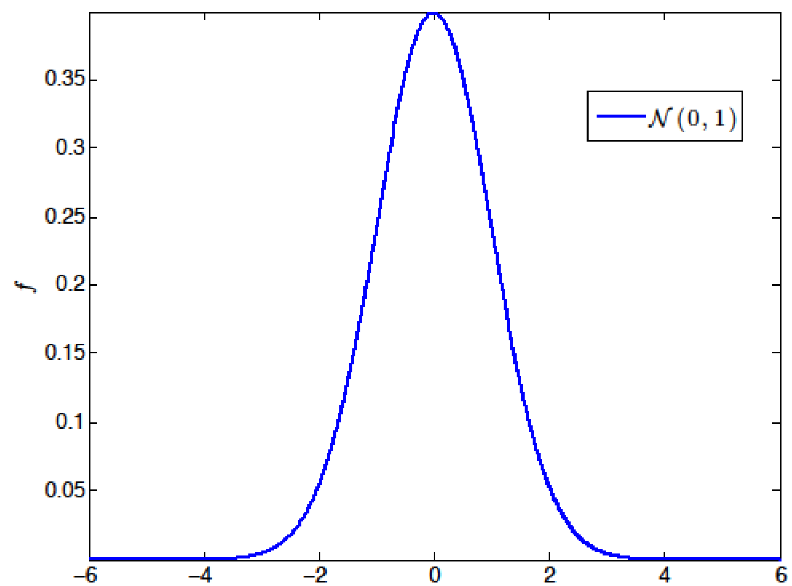
(a) $\mathcal{N}(0,1)$

(b) $\mathcal{MN}\left(\{0.5,0,1\},\{0.5,0,4\}\right)$

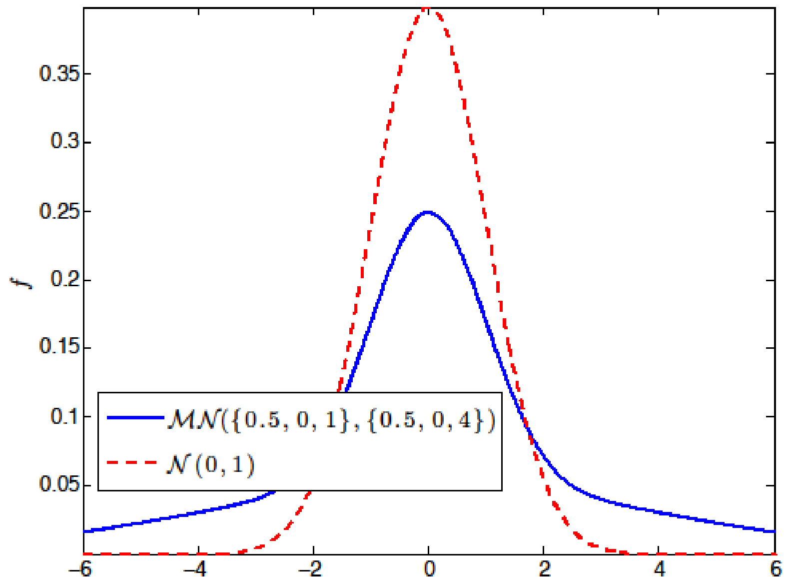
(c) $\mathcal{MN}\left(\{0.5,-1,1\},\{0.5,1,1\}\right)$

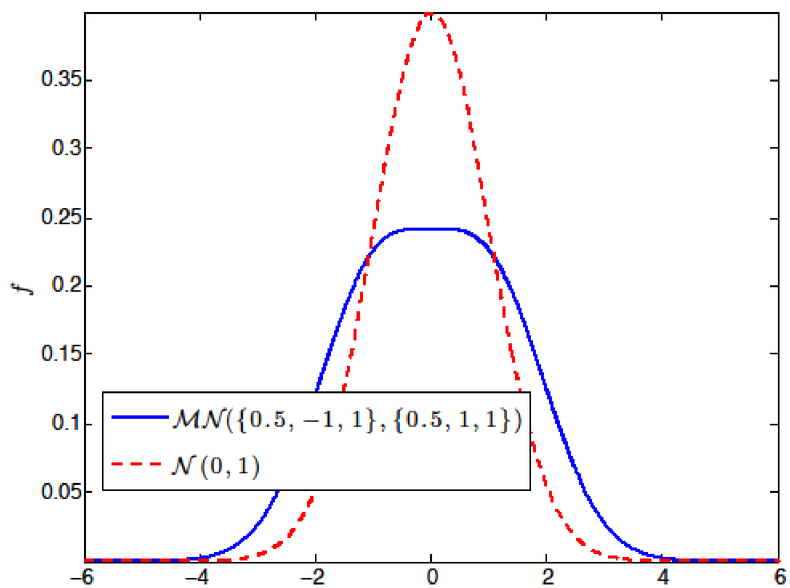
(d) $\mathcal{MN}\left(\{0.5,-1.5,1\},\{0.5,1.5,1\}\right)$

Figure 1: The densities f of the mixture of normals that we use for our numerical illustration.

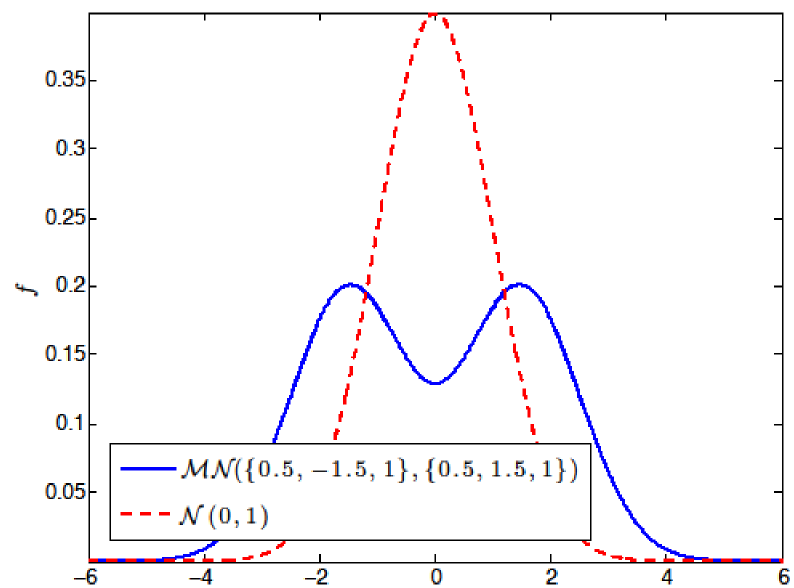


0.35
0.3
0.25
0.2
0.15
0.1
0.05
f
–6
–4
–2
0
2
4
6
$\mathcal{N}(0,1)$


0.35
0.3
0.25
0.2
0.15
0.1
0.05
f
–6
–4
–2
0
2
4
6
$\mathcal{MN}(\{0.5, 0, 1\}, \{0.5, 0, 4\})$
$\mathcal{N}(0, 1)$



0.35
0.3
0.25
0.2
0.15
0.1
0.05
f
–6
–4
–2
0
2
4
6
$\mathcal{MN}(\{0.5, -1, 1\}, \{0.5, 1, 1\})$
$\mathcal{N}(0, 1)$



0.35
0.3
0.25
0.2
0.15
0.1
0.05
f
–6
–4
–2
0
2
4
6
$\mathcal{MN}(\{0.5, -1.5, 1\}, \{0.5, 1.5, 1\})$
$\mathcal{N}(0, 1)$

Panel A: $\mathcal{N}(0,1)$, $\xi_q = 0$, $f(\xi_q) = 0.3989$.

η	λ	λ_{approx}	$\lvert\lambda_{\text{approx}} - \lambda\rvert$	$\frac{\lvert\lambda_{\text{approx}} - \lambda\rvert}{\lambda}$
2	0.6896	0.8355	0.1459	0.2116
4	0.3398	0.3581	0.0183	0.0539
8	0.1639	0.1671	0.0031	0.0195
16	0.0805	0.0809	0.0004	0.0050

Panel B: $\mathcal{MN}\left(\{0.5, 0, 1\}, \{0.5, 0, 4\}\right)$, $\xi_q = 0$, $f(\xi_q) = 0.2992$.

η	λ	λ_{approx}	$\lvert\lambda_{\text{approx}} - \lambda\rvert$	$\frac{\lvert\lambda_{\text{approx}} - \lambda\rvert}{\lambda}$
2	0.9304	1.1141	0.1837	0.1974
4	0.4617	0.4775	0.0158	0.0342
8	0.2215	0.2228	0.0065	0.0293
16	0.1077	0.1078	0.0001	0.0009

Panel C: $\mathcal{MN}\left(\{0.5, -1, 1\}, \{0.5, 1, 1\}\right)$, $\xi_q = 0$, $f(\xi_q) = 0.2420$.

η	λ	λ_{approx}	$\lvert\lambda_{\text{approx}} - \lambda\rvert$	$\frac{\lvert\lambda_{\text{approx}} - \lambda\rvert}{\lambda}$
2	1.0468	1.3776	0.3308	0.3160
4	0.5305	0.5904	0.0599	0.1129
8	0.2610	0.2755	0.0145	0.0556
16	0.1296	0.1333	0.0037	0.0285

Panel D: $\mathcal{MN}\left(\{0.5, -1.5, 1\}, \{0.5, 1.5, 1\}\right)$, $\xi_q = 0$, $f(\xi_q) = 0.1295$.

η	λ	λ_{approx}	$\lvert\lambda_{\text{approx}} - \lambda\rvert$	$\frac{\lvert\lambda_{\text{approx}} - \lambda\rvert}{\lambda}$
2	1.4650	2.5737	1.1087	0.7568
4	0.7626	1.1030	0.3404	0.4464
8	0.3948	0.5147	0.1199	0.3037
16	0.2084	0.2491	0.0407	0.1953

η	$\lambda_{1,2}$	$\lambda_{2,1}$	$\lvert\lambda_{1,2} - \lambda_{2,1}\rvert$	$\frac{\lvert\lambda_{1,2}-\lambda_{2,1}\rvert}{\lambda_{1,2}}$
2	0.5027	0.5085	0.0058	0.0115
4	0.2433	0.2441	0.0008	0.0033
8	0.1184	0.1185	0.0001	0.0008
16	0.0583	0.0583	0	0