Price Discovery Using a Double Auction

by

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Why study double auctions?

- effect of strategic behavior on efficiency
- comparative analysis of market mechanisms
- relevance to experimental testing
- equilibrium price verification vs. equilibrium price discovery

Accomplished here:

- both correlated private (CPV) and interdependent values/costs (CIV)

- computable model of trading

- generality of the informational enviroment traded for deeper insight

- meaningfulness of rates of convergence

- *numerical result* vs. *theorem*

The Buyer's Bid Double Auction

m buyers each of whom wishes to buy one item

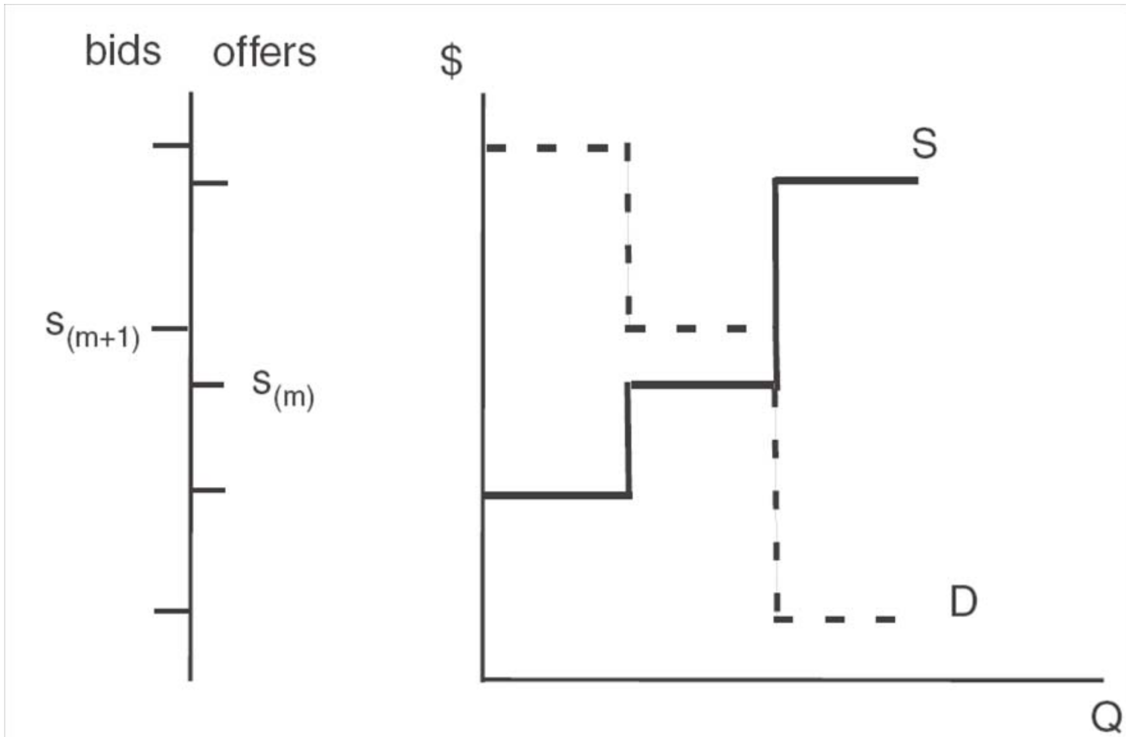
n sellers, each of whom wishes to sell one item

- buyers and sellers simultaneously submit bids/offers

- bids/offers are ordered in a list:

$$s_{(1)} \leq s_{(2)} \leq \ldots \leq s_{(m)} \leq s_{(m+1)} \leq \ldots \leq s_{(m+n)}$$

- buyers whose bids are at or above $s_{(m+1)}$ trade with sellers whose offers are below $s_{(m)}$ at the market price of $p = s_{(m+1)}$



bids
offers
$
S
$s_{(m+1)}$
$s_{(m)}$
D
Q

Novel Feature: Values/Costs and Signals

- a state μ is drawn from the *uniform improper prior* on $\mathbb{R}$
- buyer i's value is $v_i = \mu + \varepsilon_i$ and seller j's cost is $c_j = \mu + \varepsilon_j$, where ε_i, $\varepsilon_j \backsim G_\varepsilon$
- quasilinear utility
- a *correlated, private value model (CPV)*
- *correlated interdependent value model* (CIV): each trader observes a noisy signal $\sigma_i = z_i + \delta_i$ of his value/cost z_i, where $\delta_i \backsim G_\delta$

The Uniform Improper Prior

- models complete ignorance about the distribution of values/costs and the likely price ex ante
- DeGroot:
  - forming a prior is costly
  - good information is on the way at the interim stage
  - beliefs conditioned on an observed signal are well-defined
- Maximal test of the BBDA institution

- methodological: *invariance*

- Cripps and Swinkels (2006) in CPV case, Reny and Perry (2006) in CIV case:

  - large numbers of traders

  - no examples

- robustness check: $\mu \backsim \mathcal{N}(0, var)$

Invariance of a Trader's Decision Problem

For all $j, k \neq i$, the distributions of

$$c_j - \sigma_i, v_k - \sigma_i$$

and

$$\sigma_j - \sigma_i$$

are the same for all $\sigma_i \in \mathbb{R}$

- conjectured form of symmetric equilibrium: each buyer i uses

  $B(\sigma_i) = \sigma_i + \lambda_B$ and each seller j uses $S(\sigma_j) = \sigma_j + \lambda_S$ for $\lambda_B, \lambda_S \in \mathbb{R}$

- *offset strategies and offset equilibrium*

First Order Conditions for Equilibrium

Buyer:

$$\pi_b(b|\sigma) = (\mathbb{E}[v|\sigma, x = b] - b)\, f^B_{x|\sigma}(b|\sigma) - \mathsf{Pr}\,[x < b < y|\sigma] = 0 \Leftrightarrow$$

$$\begin{aligned} b &= \mathbb{E}[v|\sigma, x = b] - \frac{\mathsf{Pr}\,[x < b < y|\sigma]}{f^B_{x|\sigma}(b|\sigma)} \\ &= \text{price-taking term - strategic term} \end{aligned}$$

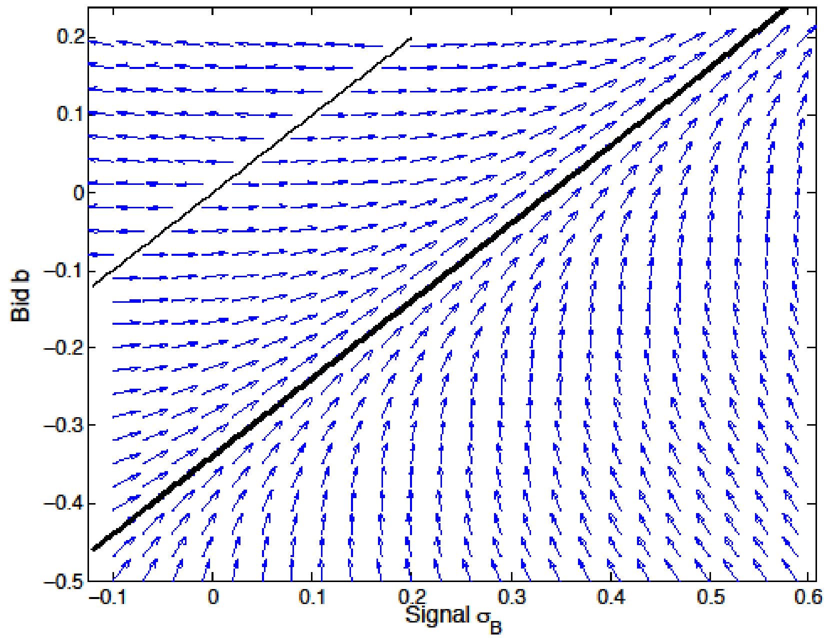
Seller:

$$\pi_a^S(a|\sigma_S) = a - \mathbb{E}[c|\sigma_S, x = a] = 0 \Leftrightarrow$$

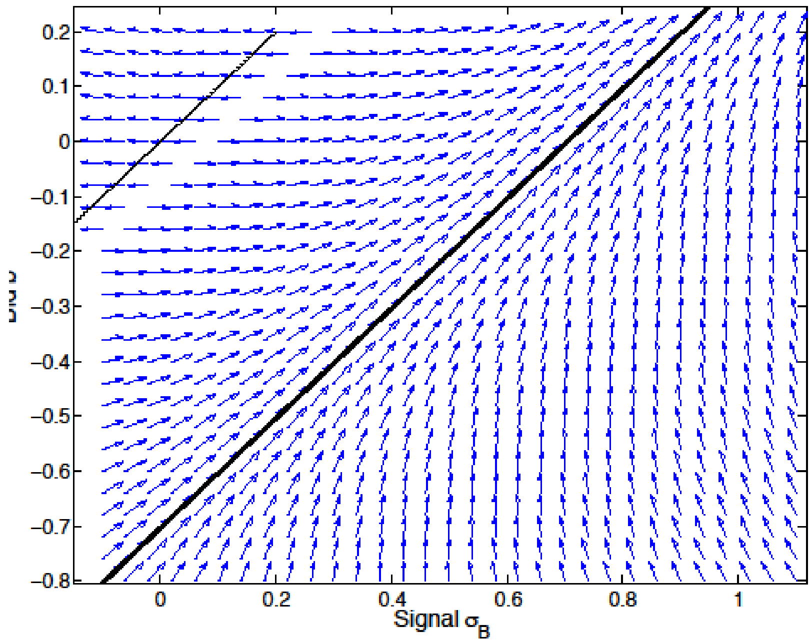
$$a = \mathbb{E}[c|\sigma_S, x = a]$$

$$= \text{price-taking term}$$

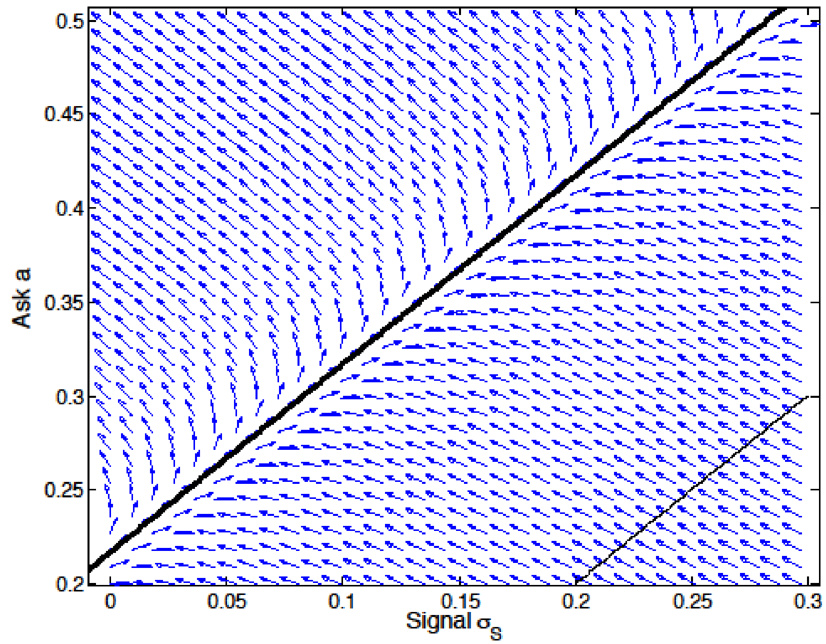
FOCs define a vector field $\overrightarrow{\mathcal{V}} = \left(\dot{b}, \dot{\sigma}_B, \dot{\sigma}_S\right)$



The normalized vector field $\overrightarrow{\mathcal{V}}$ for buyers in the CPV case ($m = n = 4$, G_ε standard normal).

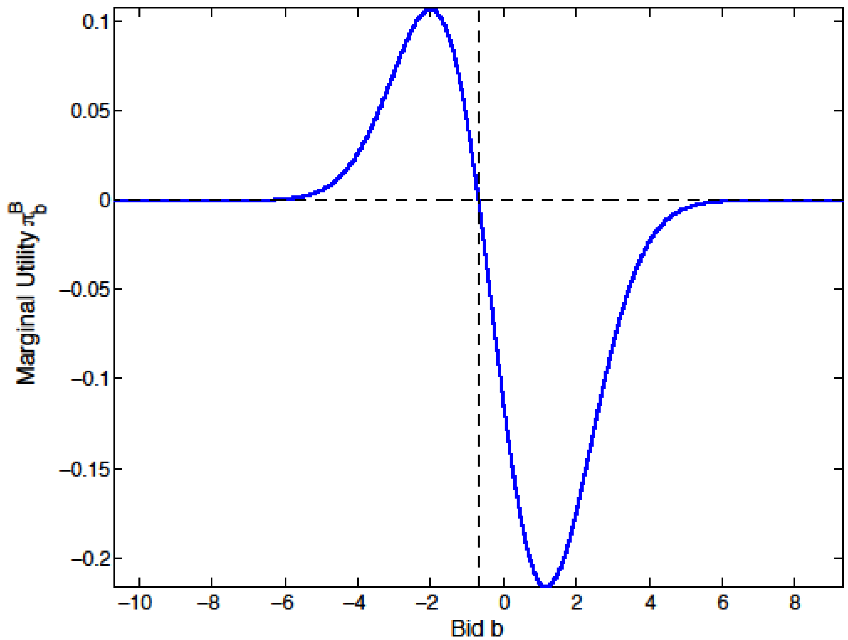


The normalized vector field $\overrightarrow{\mathcal{V}}$ for buyers in the CIV case ($m = n = 4$, G_ε, G_δ standard normal). $S(\sigma_S) = \sigma_S + 0.2172$

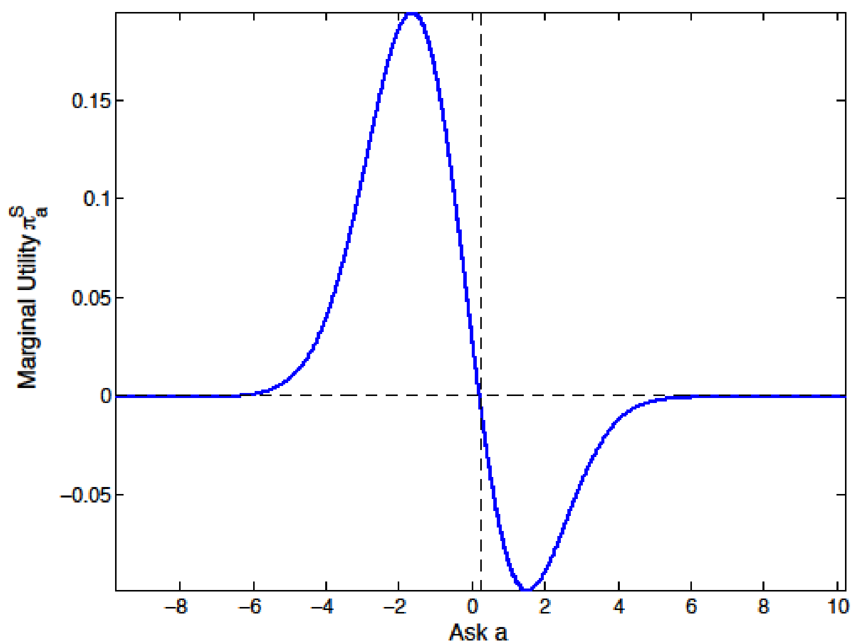


The normalized vector field $\overrightarrow{\mathcal{V}}$ for sellers in the CIV case ($m = n = 4$, G_ε, G_δ standard normal). $B(\sigma_B) = \beta = \sigma_B - 0.7036$

Sufficiency of FOC Verified Numerically:



Marginal expected utility for focal buyer ($m = n = 4$, G_ε, G_δ standard normal)). The vertical dashed line ($b = -0.7036$) indicates the offset solution to the focal trader's FOC.



Marginal expected utility for focal seller ($m = n = 4$, G_ε, G_δ standard normal)).

Results

- Numerical Result I: Existence and uniqueness of symmetric equilibrium in CIV and CPV cases
  - Theorem: Existence of offset solution to buyer's FOC in CPV case

- Numerical Results II-III: Fixed m, n, ηm buyers and ηn sellers
  - Equilibrium strategic term of buyers is $O(1/\eta)$:

$$\frac{\Pr\left[x < b < y|\sigma\right]}{f^{B}_{x|\sigma}\left(b|\sigma\right)} \leq \frac{K_1\left(m, n, G_\varepsilon, G_\delta\right)}{\eta}$$

– Relative inefficiency is $O(1/\eta^2)$:

$$\frac{\overline{GFT}^{\mathrm{pt}} - \overline{GFT}^{\mathrm{e}}}{\overline{GFT}^{\mathrm{pt}}} \leq \frac{K_2\left(m, n, G_\varepsilon, G_\delta\right)}{\eta^2}$$

- Numerical Result IV: convergence to REE

  – theorems in the CPV case

2	4	8	16
$-1.3404, 0.4124$	$-0.8372, 0.4912$	$-0.3642, 0.6508$	$0.0361, 0.8546$
$-1.2189, 0.1332$	$-0.7036, 0.2172$	$-0.2657, 0.3948$	$0.1128, 0.6192$
$-1.2084, -0.1712$	$-0.7431, -0.0787$	$-0.3417, 0.1091$	$0.0212, 0.3494$
$-1.3011, -0.4677$	$-0.8853, -0.3756$	$-0.5175, -0.1886$	$-0.1754, 0.0614$

Equilibrium offsets λ_B, λ_S for different values of m and n in the case of G_ε, G_δ standard normal.

η	λ_B	$\overline{GFT}^{\text{pt}}$	$\overline{GFT}^{\text{eq}}$	$(\overline{GFT}^{\text{pt}} - \overline{GFT}^{\text{eq}})/\overline{GFT}^{\text{pt}}$
2	-0.6896	1.3265	1.2221	0.0795
4	-0.3398	2.9008	2.8535	0.0163
8	-0.1639	6.0812	6.0653	0.0026
16	-0.0805	12.4604	12.4516	0.0007

CPV case ($m = n = 1$, G_ε standard normal)

η	$\frac{\Pr[x<\lambda_B<y \mid \sigma_B]}{f_x^B(\lambda_B \mid \sigma_B)}$	$\overline{GFT}^{\text{pt}}$	$\overline{GFT}^{\text{eq}}$	$(\overline{GFT}^{\text{pt}} - \overline{GFT}^{\text{eq}})/\overline{GFT}^{\text{pt}}$
2	0.9279	0.9395	0.7151	0.2389
4	0.4864	2.075	1.9354	0.0594
8	0.2326	4.3011	4.2434	0.0134
16	0.1139	8.8093	8.776	0.0037

CIV case ($m = n = 1$, G_ε, G_δ standard normal)

Limit Market

- *limit market* in each state μ: m times a unit mass of buyers and n times a unit mass of sellers with values/costs and signals generated using the distributions G_ε, G_δ

- $V(\sigma) \equiv \mathbb{E}[z|0, \sigma]$ assumed increasing

- *REE function* $P^{\text{REE}} : \mathbb{R} \rightarrow \mathbb{R}$

- invertible. Let Λ denote the function that recovers the state μ from the REE price, $\Lambda\left(p^{\text{REE}}\right) = \mu$.

- importance of revealing μ

- $P^{\text{REE}}(\mu) = p^{\text{REE}}$ clears the limit market in the state μ. Each trader learns his private signal σ, observes p^{REE}, and calculates his expected value/cost $\mathbb{E}\left[z|\Lambda\left(p^{\text{REE}}\right), \sigma\right]$.

$$q \equiv \frac{m}{m+n},\ \xi_q^{\varepsilon+\delta} \equiv G_{\varepsilon+\delta}^{-1}(q)$$

Consider the CIV case. For fixed m and n, consider the limit market. Then:

- The unique REE price in state μ is
$$p^{\mathsf{REE}} \equiv \mu + V\left(\xi_q^{\varepsilon+\delta}\right).$$
The one-to-one mapping from the REE price to the state is $\Lambda\left(p^{\mathsf{REE}}\right) = p^{\mathsf{REE}} - V\left(\xi_q^{\varepsilon+\delta}\right)$.

- In the BBDA, all traders play the equilibrium offset $\lambda_B = \lambda_S = V\left(\xi_q^{\varepsilon+\delta}\right) - \xi_q^{\varepsilon+\delta}$. This results in the equilibrium price $\mu + V\left(\xi_q^{\varepsilon+\delta}\right)$.

Strategic Error vs. Sampling Error

Absolute Error in the strategic market price p^e as an estimate of $p^{\mathsf{REE}} \equiv \mu + V\left(\xi_q^{\varepsilon+\delta}\right)$:

$$AE = \left|p^e - p^{\mathsf{REE}}\right|$$

$$\leq \left|p^e - p^{pt}\right| + \left|p^{pt} - p^{\mathsf{REE}}\right|$$

$$= \textit{Strategic Error} + \textit{Sampling Error}$$

Numerical Result IV

- For every sample of values/costs, strategic error is $O\left(1/\eta\right)$

- Sampling error is a random variable that can achieve any value in $\mathbb{R}^+$

- $\mathbb{E}\left[\textit{sampling error}\,|\mu\right]$ is $\Theta(1/\sqrt{\eta})$, i.e.,

$$0 < \frac{k_1}{\sqrt{\eta}} \leq \mathbb{E}\left[\textit{Sampling Error}\,|\mu\right] \leq \frac{k_2}{\sqrt{\eta}}.$$

- Expected total error is $\Theta(1/\sqrt{\eta})$

- The effect of strategic behavior is swamped by the error inherent in the finiteness of the market and the noisiness of the signals

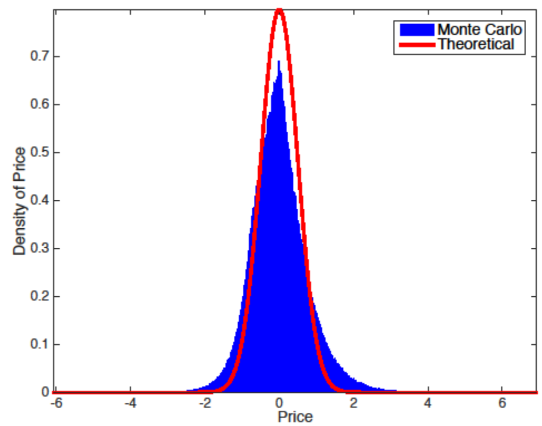
- This holds as a theorem in the CPV case if G_ε satisfies two regularity conditions on its downward tail

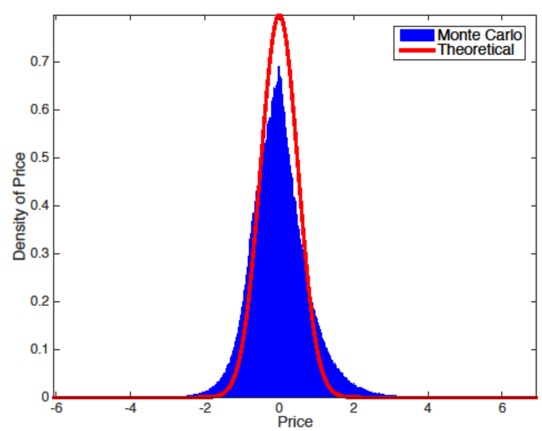
Asymptotics

CPV case: For each μ, p^{pt} and p^{eq} share the same asymptotic distribution,

$$p^{\text{pt}},\, p^{\text{eq}} \sim \mathcal{AN}\left(\mu + \xi_q^{\varepsilon}, \frac{mn}{\eta\,(m+n)^3\, g_{\varepsilon}^2\left(\xi_q^{\varepsilon}\right)}\right)$$

- each is an asymptotically unbiased and consistent estimate of $\mu + \xi_q^{\varepsilon}$
- holds despite the fact that $\mathbb{E}\left[p^{\text{pt}} - p^{\text{eq}}\,|\mu\right] > 0$ for all η
- result concerning p^{pt} is standard; result concerning p^{eq} is new


Monte Carlo
Theoretical
Density of Price
Price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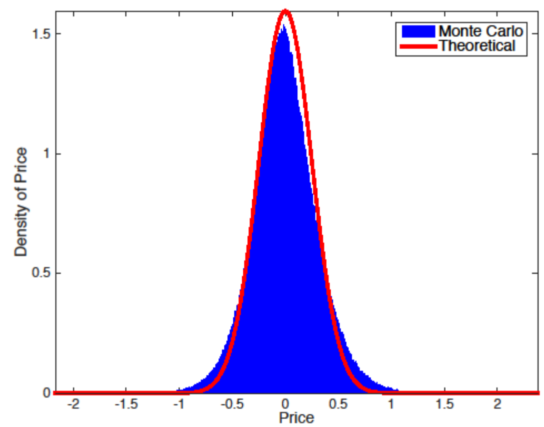

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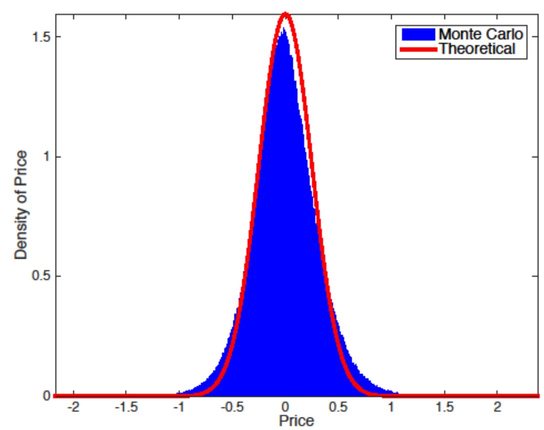


Monte Carlo
Theoretical
Density of Price
Price

η	$\text{VAR}\left(p^{\text{pt}} - p^{\text{REE}} \mid \mu\right)$	$\text{VAR}\left(p^{\text{eq}} - p^{\text{REE}} \mid \mu\right)$	$\frac{1}{8\eta\phi^2(0)}$
2	0.3646	0.3834	0.3927
4	0.1887	0.1901	0.1963
8	0.0954	0.0958	0.0981
16	0.0482	0.0483	0.0491

CPV case ($m = n = 1$, G_ε standard normal)

η	Exp. Sampling Error $\mathbb{E}\left[\left\|p^{\text{pt}} - p^{\text{REE}}\right\| \|\mu\right]$	Exp. Total Error $\mathbb{E}\left[\left\|p^{\text{eq}} - p^{\text{REE}}\right\| \|\mu\right]$	Exp. Strategic Error $\mathbb{E}\left[\left\|p^{\text{eq}} - p^{\text{pt}}\right\| \|\mu\right]$
2	0.7546	0.7327	0.5895
4	0.5174	0.4968	0.3354
8	0.3597	0.3509	0.1682
16	0.2526	0.2491	0.0871

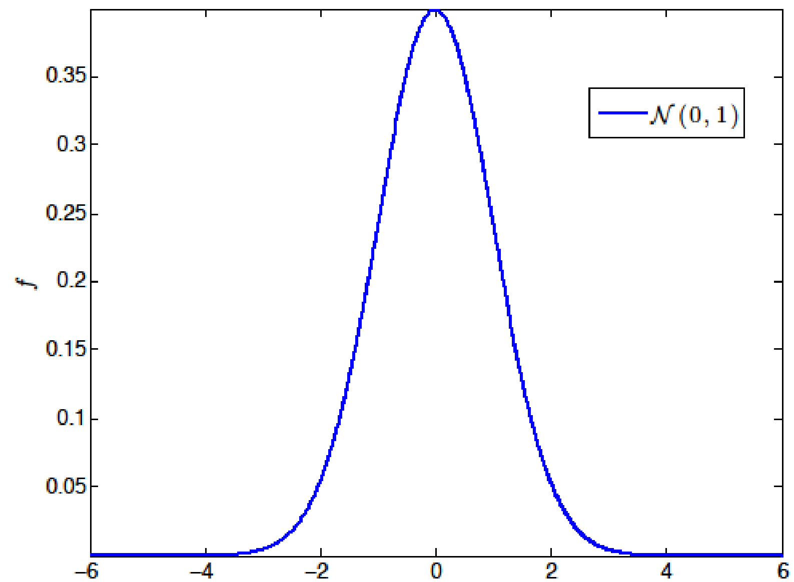
CPV case ($m = n = 1$, G_ε standard normal)

Conclusion

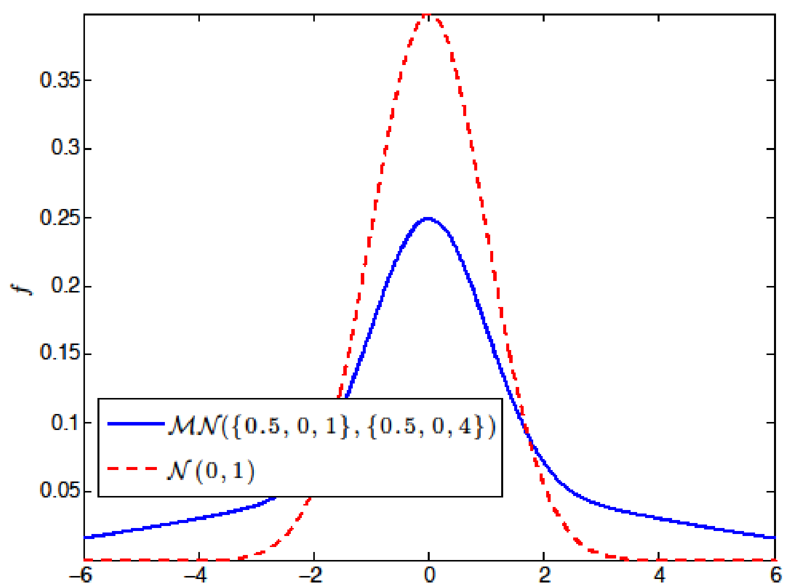
- informational environment:
  - simple enough: formal analysis, computational work, and the display of equilibrium
  - rich enough to include the CPV and CIV cases
- Previous work: the asymptotic properties of large markets.
- Private information marginally affects the market's performance relative to price formation, allocative efficiency, and the estimation of the REE price.

Asymptotic FOC in CPV and its Solution

$$\lambda_{\mathsf{approx}}(\eta) = \frac{1}{(m+n)\eta - 1} \frac{1}{g_{\varepsilon}(\xi_q)}$$



0.35
0.3
0.25
0.2
0.15
0.1
0.05
f
−6
−4
−2
0
2
4
6
$\mathcal{N}(0,1)$



0.35
0.3
0.25
0.2
0.15
0.1
0.05
f
−6
−4
−2
0
2
4
6
$\mathcal{MN}(\{0.5, 0, 1\}, \{0.5, 0, 4\})$
$\mathcal{N}(0, 1)$

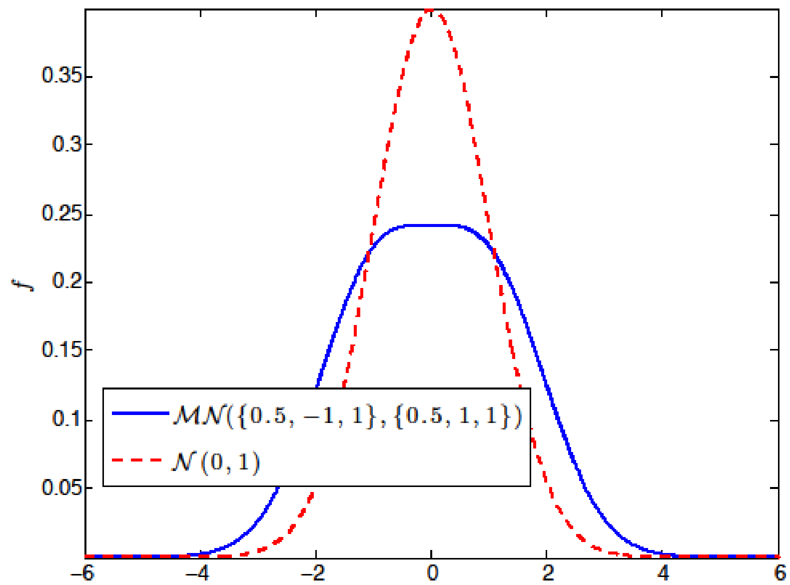
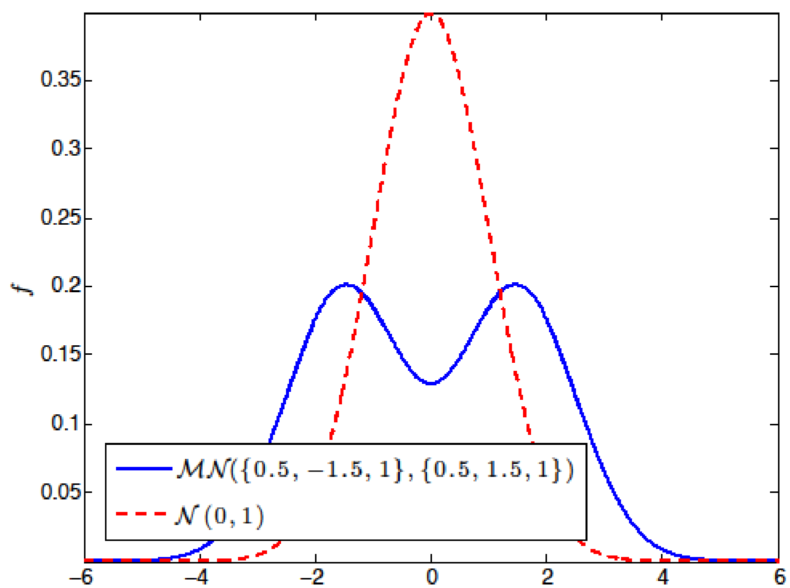


Figure 1:

Panel A: $\mathcal{N}(0, 1)$, $\xi_q = 0$, $f(\xi_q) = 0.3989$.



0.35
0.3
0.25
0.2
0.15
0.1
0.05
f
–6
–4
–2
0
2
4
6
$\mathcal{MN}(\{0.5, -1.5, 1\}, \{0.5, 1.5, 1\})$
$\mathcal{N}(0, 1)$

Figure 2:

η	λ	λ_{approx}	$\lvert\lambda_{\text{approx}} - \lambda\rvert$	$\frac{\lvert\lambda_{\text{approx}} - \lambda\rvert}{\lambda}$
2	0.6896	0.8355	0.1459	0.2116
4	0.3398	0.3581	0.0183	0.0539
8	0.1639	0.1671	0.0031	0.0195
16	0.0805	0.0809	0.0004	0.0050

Panel B: $\mathcal{MN}\left(\{0.5, 0, 1\}, \{0.5, 0, 4\}\right)$, $\xi_q = 0$, $f(\xi_q) = 0.2992$.

η	λ	λ_{approx}	$\lvert\lambda_{\text{approx}} - \lambda\rvert$	$\frac{\lvert\lambda_{\text{approx}} - \lambda\rvert}{\lambda}$
2	0.9304	1.1141	0.1837	0.1974
4	0.4617	0.4775	0.0158	0.0342
8	0.2215	0.2228	0.0065	0.0293
16	0.1077	0.1078	0.0001	0.0009

Panel C: $\mathcal{MN}\left(\{0.5, -1, 1\}, \{0.5, 1, 1\}\right)$, $\xi_q = 0$, $f(\xi_q) = 0.2420$.

η	λ	λ_{approx}	$\lvert\lambda_{\text{approx}} - \lambda\rvert$	$\frac{\lvert\lambda_{\text{approx}} - \lambda\rvert}{\lambda}$
2	1.0468	1.3776	0.3308	0.3160
4	0.5305	0.5904	0.0599	0.1129
8	0.2610	0.2755	0.0145	0.0556
16	0.1296	0.1333	0.0037	0.0285

Panel D: $\mathcal{MN}\left(\{0.5, -1.5, 1\}, \{0.5, 1.5, 1\}\right)$, $\xi_q = 0$, $f(\xi_q) = 0.1295$.

η	λ	λ_{approx}	$\lvert\lambda_{\text{approx}} - \lambda\rvert$	$\frac{\lvert\lambda_{\text{approx}} - \lambda\rvert}{\lambda}$
2	1.4650	2.5737	1.1087	0.7568
4	0.7626	1.1030	0.3404	0.4464
8	0.3948	0.5147	0.1199	0.3037
16	0.2084	0.2491	0.0407	0.1953

η	$\lambda_{1,2}$	$\lambda_{2,1}$	$\lvert\lambda_{1,2}-\lambda_{2,1}\rvert$	$\frac{\lvert\lambda_{1,2}-\lambda_{2,1}\rvert}{\lambda_{1,2}}$
2	0.5027	0.5085	0.0058	0.0115
4	0.2433	0.2441	0.0008	0.0033
8	0.1184	0.1185	0.0001	0.0008
16	0.0583	0.0583	0	0

For different market sizes η and F standard normal, the equilibrium offset $\lambda_{1,2}$ for the case of $m = 1$ buyer, $n = 2$ sellers is compared to the equilibrium offset $\lambda_{2,1}$ for the case of $m = 2$ buyers, $n = 1$ seller.

η	$\lambda_{1,2}$	$\lambda_{2,1}$	$\lvert\lambda_{1,2} - \lambda_{2,1}\rvert$	$\frac{\lvert\lambda_{1,2} - \lambda_{2,1}\rvert}{\lambda_{1,2}}$
2	0.5027	0.5085	0.0058	0.0115
4	0.2433	0.2441	0.0008	0.0033
8	0.1184	0.1185	0.0001	0.0008
16	0.0583	0.0583	0	0