Optimality versus Practicality in Market Design: A Comparison of Two Double Auctions*

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Abstract

We consider a market for indivisible items with m buyers, each of whom wishes to buy at most one item, and m sellers, each of whom has one item to sell. The traders privately know their values/costs, which are statistically dependent. Two mechanisms for trading are considered. The *buyer's bid double auction* collects bids and offers from traders and determines the allocation by selecting a market-clearing price. It fails to achieve all possible gains from trade because of strategic bidding by buyers. The *designed mechanism* is a revelation mechanism in which honest reporting of values/costs is incentive compatible and all gains from trade are achieved in equilibrium. This optimality, however, comes at the expense of plausibility: (i) the monetary transfers among the traders are defined in terms of the traders' beliefs about each other's value/cost; (ii) a trader may suffer a loss ex post; (iii) the mechanism may run a surplus/deficit ex post. We compare the virtues of the simple yet mildly inefficient buyer's bid double auction to the flawed yet perfectly efficient designed mechanism.

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Economists evaluate institutions from both normative and positive perspectives. The normative perspective characterizes the optimal mechanism in a given problem. The positive perspective models procedures that are used in practice and evaluates their properties. The approaches are combined when a practical procedure is measured against the optimum. If a practical procedure is not optimal, then the normative approach may provide guidance for improving performance in practice. Alternatively, the normative analysis may be questioned on the grounds that the optimization analysis has failed to address all constraints that matter in practice. A procedure that endures in practice and seems to perform well but not optimally in a theoretical sense compels a reappraisal of the optimization analysis.

We compare in this paper the properties of two mechanisms for organizing trade in a simple model of exchange of homogeneous, indivisible items. There are m buyers, each of whom wishes to buy at most one item, and m sellers, each of whom has one item to sell. In the terminology of auction theory, a correlated, private values model is considered. Each buyer i privately observes the value v_i that he receives if he acquires an item and each seller j privately observes the cost c_j that he bears if he sells his item. Utility for each trader is quasilinear in his value/cost and money. The normative approach we consider generalizes a mechanism devised by McAfee and Reny (1992) in the bilateral case to the case of multiple traders on each side of the market. Using the statistical dependence among values and costs, a revelation mechanism is designed in which (i) honest reporting defines a Bayesian-Nash equilibrium and (ii) all potential gains from trade are achieved in this equilibrium.[1] This is the *designed mechanism* (or *DM*). Alternatively, one can solicit a bid from every buyer, an ask from every seller, and then construct demand and supply curves. The *buyer's bid double auction* (or *BBDA*) selects as the market price the upper boundary of the interval of market-clearing prices with trade occurring among buyers who bid at least this price and sellers whose offers are less than this price. It is a simple model of a call market that is used in practice to organize trade. Traders bid strategically, however, which means that a buyer's value may exceed a seller's cost even though the bid does not exceed the ask. The consequence of this strategic behavior is that the *BBDA* inefficiently fails to achieve all possible gains from trade.

The *DM* is impractical because its monetary transfers among the traders are functions of the probability distribution that models their common beliefs about the value/cost of each other. This is the Wilson Critique of mechanism design (Wilson (1987)), namely, that the field has focused upon mechanisms defined in terms of the agents' beliefs. The assumption of probabilistic beliefs held by the agents are a means to rigorously model the agents' choices under uncertainty; these beliefs are not a datum that is practically available for defining economic institutions.[2] The rules

[1]The use of statistical dependence among the private types of agents to improve mechanism performance originates in Myerson (1981), which presents an example of an auction in which the seller extracts the entire potential revenue by using the dependence among bidder reservation values. This example was formalized into a general theorem for finite auction models by Crémer and McLean (1988). McAfee and Reny (1992) showed that statistical dependence can be used for welfare gains in a variety of mechanism design problems while also extending the analysis to the continuum models that are more commonly used in Bayesian mechanism design.

[2]Econometricians, however, are developing sophisticated methods to identify the beliefs that underlie agents' actions in auctions and bargaining problems. See, for example, Aradillas-López, Gandhi, and Quint (2013), Henderson, List, Millimet, Parmeter, and Price (2012) and Krasnokutskaya (2011). The relevance of this work to mechanism

of the *BBDA*, in contrast, are specified purely in terms of the bids and offers of the traders. It is a mechanism that is robust in the manner that Wilson (1987) advocates.[3]

The focus in this paper is not the failure of the *DM* to satisfy the Wilson Critique;[4] rather, it is to measure the *BBDA* against the *DM* as part of the positive/normative methodology. Though efficient, the *DM* may compel a buyer to bear a loss when he fails to trade and can run either an ex post monetary surplus or deficit because it is only budget balanced in expectation. In contrast, the *BBDA* is ex post individually rational and ex post budget balanced.[5]

Our comparison of the *DM* and *BBDA* is for the case of a particular informational environment and varying sizes m of markets. We focus on the following three questions:

1. How inefficient is the *BBDA*?

2. How significant are the ex post losses that a trader may bear in the *DM*?

3. How large of monetary subsidy may be required ex post to operate the *DM*?

Our results concerning inefficiency in the *BBDA* are drawn from Satterthwaite, Williams, and Zachariadis (2012): (i) a seller reports his cost honestly; (ii) in any symmetric equilibrium strategy, a buyer underbids by an amount that is $O(1/m)$; (iii) the ex ante expected gains from trade that inefficiently fail to be achieved in equilibrium as a fraction of the ex ante expected potential gains from trade is $O(1/m^2)$. Computational evidence that is presented in Section 4 demonstrates that the losses from strategic behavior may be negligible even for market sizes m as small as 8 or 16.[6]

We next consider the *DM*. Let $\underline{U}(m)$ denote for a market with m traders on each side the ex post utility of a buyer who fails to trade in the *DM*. We show that $\underline{U}(m)$ is strictly negative and bounded away from zero for all m and for a robust family of trading problems. A buyer fails to trade in the efficient *DM* if his value is among the m smallest of the $2m$ values/costs of

design remains to be explored.

[3]A second criticism of Wilson (1987) concerns the assumption of common knowledge of beliefs. This criticism is addressed by the growing field of *robust mechanism design* (see Bergemann and Morris (2012, pp. 1-48) and the references therein). The criticism is relevant to both the *DM* and the *BBDA*. Both the *DM* and the *BBDA* have the property that honest reporting is a dominant strategy for sellers and hence beliefs are not needed for seller decision making in either procedure. A buyer, however, must know the distribution of values and costs to verify the optimality of his equilibrium behavior in either the *DM* or the *BBDA*. While honest reporting in the *DM* is arguably simpler than strategic bidding in the *BBDA*, it still requires common knowledge of beliefs to sustain equilibrium. We shall also discuss below a particularly simple form of equilibrium bidding behavior by buyers in the *BBDA*.

[4]It is also well known that the ex post transfers used to sustain incentive compatibility in the designed approach go to infinity in magnitude as the case of independent types is approached, rendering it impractical. See Robert (1991) and Kosmopoulou and Williams (1998).

[5]Kosenok and Severinov (2008) provide necessary and sufficient conditions on the distributions of types for the existence of an ex post efficient, interim individually rational, ex post budget balanced and Bayesian incentive compatible mechanism. Their proof of sufficiency is constructive, and given our focus on the ex post budget imbalance of the *DM*, it would seem that their mechanism would be a more suitable point of comparison than the McAfee-Reny approach that we use here. As in Crémer and McLean (1988), however, Kosenok and Severinov (2008) assume finite type spaces. Generalizing their conditions and construction to our continuous model is far from immediate. Their mechanism holds promise for future evaluation of the designed mechanism approach in the double auction setting.

[6]That these losses are relatively small is also supported by Satterthwaite and Williams (2002), who show in an independent private values model that no incentive compatible, ex ante budget balanced and interim individually rational mechanism achieves a faster rate of convergence to efficiency over all trading environments than the *BBDA*.

all traders. The symmetry of our model implies that a buyer trades with probability 1/2 and so the expected number of buyers who bear an ex post loss is $m/2$. The magnitude of the aggregate expected ex post loss $|\underline{U}(m) \cdot m/2|$ of buyers in the *DM* therefore increases without bound as the market increases in size, which goes against the standard intuition that increasing the size of a market perfects its performance. We then present computational evidence concerning the size of the loss $\underline{U}(m)$ that a buyer may bear in comparison with the increased expected gains from trade that switching from the *BBDA* to the *DM* creates. This evidence suggests that this ex post loss can be a steep price to pay for a relatively modest increase in efficiency, even for relatively small sizes m of the market.

A problem that exists for all sizes of markets is that the *DM* may require a substantial ex post monetary subsidy in order to operate. We show that the sum of transfers from buyers minus the payments to sellers ex post has a worst case value over all samples of values/costs that is strictly negative and decreasing at a linear rate in m to $-\infty$. This is a serious obstacle to using the mechanism.

The paper is organized follows. Section 1 discusses the trading environment that we study along with a summary of relevant results concerning the *BBDA* from other sources. Section 2 first recounts the McAfee-Reny mechanism in the bilateral case of $m = 1$ and then extends it to define the *basic designed mechanism* or *BDM*. This mechanism is incentive compatible, efficient, ex ante individually rational but not ex ante budget balanced. It is an intermediate step to defining the *DM* of interest, which achieves the additional property of ex ante budget balance by adding constant transfers to the *BDM*. Section 3 defines the *DM* and addresses its ex post irrationality/rationality and ex post budget imbalance. Section 4 presents a numerical example for a specific distribution of values/costs and for m ranging from 2 to 16. We then summarize our results in the final section. All proofs are deferred to the Appendix.

1 Model

1.1 The Trading Environment

The values/costs of the $2m$ traders are generated as follows. A state μ is drawn from the uniform diffuse prior on $\mathbb{R}$. We elaborate on our use of this distribution below. For each trader i, a value ε_i is independently drawn from the cumulative distribution F on $\mathbb{R}$, which is absolutely continuous with mean 0. The density f is strictly positive and continuous on $\mathbb{R}$. Trader i privately observes his value/cost

$$\mu + \varepsilon_i.$$

Through the state μ, a trader's beliefs about the distribution of the values/costs of the $2m - 1$ other traders is dependent upon his observation of his own value/cost. The process by which the values/costs of the traders are drawn is assumed to be common knowledge in our analysis of Bayesian-Nash equilibrium. Letting v denote a buyer's value, the buyer's utility if he acquires an item and makes a monetary payment of x is $v - x$; if he fails to trade and makes a payment of x, then his utility is $-x$. Similarly, a seller with cost c who sells his item and receives a payment of x has utility $x - c$; his utility if he does not sell but receives a payment of x equals x.

The uniform diffuse prior can be thought of intuitively as "the uniform distribution across the entire real line." It is an improper prior in the sense that it is not a well-defined probability distribution. Once a trader observes his value, however, his beliefs conditional on his value/cost concerning the distribution of the values and costs of the other traders is well-defined. DeGroot (1980, p. 190) motivates the use of an improper prior as a model of a decision-maker who has little information ex ante concerning future random events but who will receive a valuable signal at the interim stage on which he can base his interim probabilistic beliefs. It may not be worthwhile for the decision-maker to spend time and effort in properly specifying his ex ante beliefs. The diffuse uniform prior is adopted in our model for reasons of mathematical tractability;[7] it implies an invariance of a trader's decision problem to translations of his value/cost that greatly simplifies the problem of studying double auctions in the case of correlated values/costs. This invariance is discussed below. It is also useful in modeling a financial meltdown in which no trader knows anything ex ante about what the ex post price is likely to be. It thus maximally challenges a double auction mechanism to achieve gains from trade in an environment with incomplete information. Further discussion of the use of the uniform diffuse prior in models of double auctions can be found in Satterthwaite, Williams, and Zachariadis (2012).

1.2 A Summary of Results Concerning the Buyer's Bid Double Auction *BBDA*

Trade in the *BBDA* is organized as follows. Buyers and sellers simultaneously submit bids and offers, which are ordered in a list[8]

$$s_{(1)} \leq s_{(2)} \leq ... \leq s_{(2m)}.$$

Assume for the moment that $s_{(m)} \neq s_{(m+1)}$ and let d denote the number of buyers' bids among the top m bids/offers $s_{(m+1)}, ..., s_{(2m)}$. There are $m - d$ buyers' bids among the m lowest bids/offers $s_{(1)}, ..., s_{(m)}$. Because we have assumed that there are exactly m bids/offers among the m lowest bids/offers, there must be d offers of sellers among the m lowest. Selecting a price $p \in [s_{(m)}, s_{(m+1)}]$ therefore equates supply and demand, i.e., the number d of buyers' bids at or above p equals the number d of offers below p.[9] In the case of $s_{(m)} = s_{(m+1)}$, allocate trades on the long side of the market by assigning priority first to the larger bids/smaller offers and then using a fair lottery in the case of ties. The interval $[s_{(m)}, s_{(m+1)}]$ is therefore the interval of market-clearing prices; it could alternatively be derived as the intersection of demand and supply curves constructed from the offers/bids.

[7]The diffuse uniform prior is used for similar purposes in the case of one-sided auctions by Wilson (1998) and Klemperer (1999). It has also proven useful in the theory of global games (see Morris and Shin (2003)).

[8]We use $s_{(t)}$ throughout the paper to denote the t^{th} smallest in a specified sample of either true or reported values and costs.

[9]A minor change in the allocation rule is required in order to clear the market when the price p is selected as $s_{(m)}$: sellers whose offers are at or below p trade with buyers whose bids are strictly more than p.

The *BBDA* is the market procedure that selects $s_{(m+1)}$ as the market price.[10] Because a seller only sells if his ask is below this price, his ask can not influence the price at which he trades. It is straightforward to show that setting his ask equal to his true cost is a weakly dominant strategy for each seller. Tying down the strategic behavior of one side of the market is an attractive feature of the *BBDA* as a theoretical model. We assume the use of this strategy by each seller in the *BBDA* for the rest of the paper.

A buyer, however, sets the price at which he trades if his bid equals $s_{(m+1)}$. He therefore has an incentive to bid less than his value. Items are allocated in the *BBDA* to those traders who submit the m largest bids/offers (i.e., buyers who buy and sellers who do not and therefore retain their items). As a consequence of buyer underbidding, the m largest bids/offers may not have been submitted by the traders with the m largest values/costs. Buyer underbidding may therefore cause inefficiency in the allocation.

The results of Satterthwaite, Williams, and Zachariadis (2012) concern the use of an increasing function $B : \mathbb{R} \to \mathbb{R}$ by each buyer that, together with honest revelation by each seller, defines a Bayesian-Nash equilibrium. The paper shows that in such an equilibrium the inefficiency caused by the strategic behavior of buyers quickly becomes inconsequential as the market size m increases. The results are as follows:[11]

1. *Buyer misrepresentation is* $O(1/m)$: There exists a constant $\kappa(F)$ such that

$$v - B(v) \leq \frac{\kappa(F)}{m} \text{ for all } v \in \mathbb{R}.$$

2. *Relative Inefficiency is* $O(1/m^2)$: Fixing the state μ, let $GFT(m)$ denote the ex ante expected potential gains from trade and $GFT_{BBDA}(B, m, \mu)$ denote the ex ante expected gains from trade earned by the $2m$ traders in the equilibrium determined by the strategy B. *Relative inefficiency* $\mathcal{I}(\mu, m, B)$ is the fraction of $GFT(m)$ that the traders inefficiently fail to achieve in the equilibrium given by B,

$$\mathcal{I}(\mu, m, B) = \frac{GFT(m) - GFT_{BBDA}(B, m, \mu)}{GFT(m)}. \tag{1}$$

There exists a constant $K(F)$ such that

$$\mathcal{I}(\mu, m, B) \leq \frac{K(F)}{m^2} \text{ for all } m, B, \text{ and } \mu.$$

These results are consistent with earlier results proven in a variety of trading environments with a proper prior distribution.[12]

[10] Further discussion of the rules of the *BBDA* can be found in Satterthwaite and Williams (1989).

[11] In addition to the assumptions of section 1.1, these results assume that the density f is symmetric about 0 and that the strategy B and distribution F satisfy the boundary conditions $\lim_{v \to -\infty} B'(v) < \infty$ and $\lim_{v \to -\infty} F(v)/f(v) < \infty$. The assumptions made in section 1.1 are all that are required within this paper.

[12] Relevant references that are not cited elsewhere in this paper include Williams (1991), Rustichini, Satterthwaite, and Williams (1994), Cripps and Swinkels (2006) and Reny and Perry (2006).

A particular aspect of equilibrium in the uniform diffuse prior model is worth mentioning. For whatever value v_i that bidder i observes, he has exactly the same conditional beliefs given v_i about the distribution of the differences of values and costs

$$(v_j - v_i)_{j \neq i}, (c_k - v_i)_{1 \leq k \leq m}$$

of the other $2m - 1$ traders about his value v_i. This is the invariance property of his decision problem that is mentioned above. It is therefore reasonable to conjecture an invariant solution for his choice of a bid, i.e., a bidding strategy B that has the simple form

$$B(v) = v - \lambda$$

for a constant $\lambda(F, m)$. This is an *offset strategy.* Satterthwaite, Williams, and Zachariadis (2012) prove the existence of an offset strategy that solves a buyer's first order condition on B to define a Bayesian-Nash equilibrium. While sufficiency of the first order approach is in general difficult in the double auction setting, numerical calculations demonstrate that such strategies define equilibria for the case of the standard normal distribution that is considered in section 4 and for a variety of other common distributions.[13]

2 The Basic Designed Mechanism *BDM*

2.1 The Bilateral Case

The McAfee-Reny mechanism is defined so that: (i) the seller has a dominant strategy to report honestly; (ii) the buyer's best response is to report honestly; (iii) the allocation is ex post efficient; (iv) the interim expected utility of the buyer equals zero in equilibrium for each of his possible values and all ex ante expected gains from trade therefore go to the seller. The mechanism works as follows. Each trader reports a value/cost and trade occurs if the buyer's reported value is greater than or equal to the seller's reported cost. When trade occurs, the seller receives the buyer's report as his payment. The seller receives no transfer when trade does not occur. Consequently, the seller's dominant strategy is to report honestly.

Let v denote the buyer's value and r his report. The buyer's payment as a function of his report r and the seller's report c is as follows:

$$\begin{array}{rcl} -\dfrac{\Pr(r \geq c\,|r)^2}{\frac{\partial}{\partial v}\Pr(r \geq c\,|v)\big|_{v=r}} \; \textit{if } r & < & c, \\ r + \dfrac{\Pr(r \geq c\,|r)}{\frac{\partial}{\partial v}\Pr(r \geq c\,|v)\big|_{v=r}} - \dfrac{\Pr(r \geq c\,|r)^2}{\frac{\partial}{\partial v}\Pr(r \geq c\,|v)\big|_{v=r}} \; \textit{if } r & \geq & c. \end{array}$$

[13] Offset strategies have also been motivated by their simplicity as a form of cognitive behaviour by bidders in one-sided auctions (Compte and Postlewaite (2012)).

This is derived by starting with the following payment scheme and solving for $\alpha(r)$ and $\beta(r)$:

$$\begin{aligned} -\alpha(r) \ \textit{if} \ r &< c, \\ r + \beta(r) - \alpha(r) \ \textit{if} \ r &\geq c. \end{aligned}$$

The buyer's interim expected utility with value v and report r is

$$u(v,r) = \alpha(r) + (v - r - \beta(r)) \cdot \Pr(r \geq c \,|v\,).$$

We want $u(v,v) = 0$, or

$$\begin{aligned} \alpha(v) - \beta(v) \cdot \Pr(v \geq c \,|v\,) = 0 \Leftrightarrow \\ \alpha(r) = \beta(r) \cdot \Pr(r \geq c \,|r\,). \end{aligned} \tag{2}$$

A second equation is obtained by requiring that

$$\left. \frac{\partial u}{\partial v}(v,r) \right|_{v=r} = 0. \tag{3}$$

This condition is necessary for $u(v,r)$ to have a maximum of 0 at $v = r$, which (together with the sufficient conditions discussed below) insures incentive compatibility.[14] We have

$$\frac{\partial u}{\partial v}(v,r) = \Pr(r \geq c \,|v\,) + (v - r - \beta(r)) \cdot \frac{\partial}{\partial v} \Pr(r \geq c \,|v\,),$$

and so at $v = r$,

$$\begin{aligned} 0 = \Pr(r \geq c \,|r\,) - \beta(r) \cdot \left. \frac{\partial}{\partial v} \Pr(r \geq c \,|v\,) \right|_{v=r} \Leftrightarrow \\ \beta(r) = \frac{\Pr(r \geq c \,|r\,)}{\left. \frac{\partial}{\partial v} \Pr(r \geq c \,|v\,) \right|_{v=r}}. \end{aligned}$$

The denominator is nonzero by assumption, as a sufficient condition for dependence. The formula

$$\alpha(r) = \frac{\Pr(r \geq c \,|r\,)^2}{\left. \frac{\partial}{\partial v} \Pr(r \geq c \,|v\,) \right|_{v=r}}$$

is then determined by (2).

[14] We would normally use

$$\left. \frac{\partial u}{\partial r}(v,r) \right|_{r=v} = 0$$

as a necessary condition for the maximization of $u(v,r)$ at $r = v$ for each v. Because we want $u(v,v) = 0$ for all v, however, it follows that

$$\left. \frac{\partial u}{\partial v}(v,r) \right|_{r=v} + \left. \frac{\partial u}{\partial r}(v,r) \right|_{r=v} = 0,$$

and so the two first order conditions are equivalent. The first order condition with respect to v has the advantage of avoiding derivatives of the "unknowns" $\alpha(r)$ and $\beta(r)$ and the differential equations that result.

The conditions

$$\frac{\partial}{\partial v}\Pr(r \geq c\,|v) < 0$$

and

$$\frac{\partial}{\partial v}\left[v + \frac{\Pr(r \geq c\,|v)}{\frac{\partial}{\partial v}\Pr(r \geq c\,|v)}\right] \geq 0$$

insure that $\partial u/\partial v(v,r)$ changes from positive to negative as v goes from $v < r$ to $v > r$, i.e., $u(v,r)$ achieves its maximum value of 0 at $v = r$. These sufficient conditions are motivated in McAfee and Reny (1992, p. 415-6).

2.2 The Multilateral Case

Ex post efficiency, incentive compatibility, ex ante budget balance and interim individual rationality do not uniquely determine a mechanism in the case of correlated values and costs, and so choices are made below in defining the mechanism. We begin by generalizing from the bilateral case to define a mechanism with the following properties: (i) each seller has a dominant strategy to report honestly; (ii) assuming honest reporting by each seller and every other buyer, each buyer's best response is to report honestly; (iii) the allocation is ex post efficient; (iv) the interim expected utility of the buyer equals zero in equilibrium for each of his possible values. Call this the *basic designed mechanism* (or *BDM*). It is not ex ante budget balanced and should be regarded as an intermediate step to defining the designed mechanism of interest in this paper. We then define a family of mechanisms $\mathcal{DM}$ that have the additional property of ex ante budget balance by adding constants to the transfer functions of every trader in the *BDM.* The designed mechanism *DM* is the member of this family that is of particular interest for investigating the issues of ex post budget imbalance and irrationality.

2.2.1 The Basic Designed Mechanism *BDM*

Consider first the decision problem of a seller who faces a sample of m reported values from buyers and $m-1$ reported costs from sellers. Letting c denote the seller's cost and c^* his report, the following transfer is provided to the seller:

$$\begin{aligned} s_{(m)} \text{ } \textit{if} \text{ } c^* &\leq s_{(m)} \\ 0 \text{ } \textit{if} \text{ } c^* &> s_{(m)}. \end{aligned} \tag{4}$$

If $c^* \leq s_{(m)}$, then the seller's report is among the m smallest of the $2m$ reported values/costs so that he sells his item; if $c^* > s_{(m)}$, then the seller's ask is among the m largest of the $2m$ reported values/costs so that retains his item. The logic of the *BBDA* implies that honest revelation is a dominant strategy for each seller given the transfer formula (4).

We next consider a buyer's decision problem. Assuming honest reporting by every other trader, the sample that he faces is m costs reported by sellers and $m-1$ values reported by buyers. As in the bilateral case, let v denote the value of the selected buyer and r his report. The buyer receives an item if $r \geq s_{(m)}$. We follow the logic of the bilateral case and solve for $\alpha(r)$ and $\beta(r)$ to determine the selected buyer's payment:

$$\begin{aligned} -\alpha(r) \text{ if } r &< s_{(m)}, \\ r + \beta(r) - \alpha(r) \text{ if } r &\geq s_{(m)}. \end{aligned} \tag{5}$$

If $r < s_{(m)}$, then the reported value r is among the m smallest offers/bids and so the selected buyer does not trade; if $r \geq s_{(m)}$, then the reported value r is among the m largest and so he trades.

The buyer's expected utility with value v and report r is

$$u(v,r) = \alpha(r) + (v - r - \beta(r)) \cdot \Pr(r \geq s_{(m)} \,|v\,).$$

$\Pr(r \geq s_{(m)} \,|v\,)$ plays the role that $\Pr(r \geq c\,|v\,)$ has in the bilateral case. The requirement that the buyer's interim expected utility $u(v,v)$ equals zero implies

$$\alpha(r) = \beta(r) \cdot \Pr(r \geq s_{(m)} \,|r\,), \tag{6}$$

which generalizes (2). The first order condition

$$\left.\frac{\partial u}{\partial v}(v,r)\right|_{v=r} = 0$$

is again imposed so that $u(v,r)$ has a maximum of 0 at $v = r$. This implies

$$0 = \Pr(r \geq s_{(m)} \,|r\,) - \beta(r) \cdot \left.\frac{\partial}{\partial v}\Pr(r \geq s_{(m)} \,|v\,)\right|_{v=r} \Leftrightarrow$$

$$\beta(r) = \frac{\Pr(r \geq s_{(m)} \,|r\,)}{\left.\frac{\partial}{\partial v}\Pr(r \geq s_{(m)} \,|v\,)\right|_{v=r}}, \tag{7}$$

and so

$$\alpha(r) = \frac{\left(\Pr(r \geq s_{(m)} \,|r\,)\right)^2}{\left.\frac{\partial}{\partial v}\Pr(r \geq s_{(m)} \,|v\,)\right|_{v=r}}. \tag{8}$$

2.2.2 The Sufficient Conditions

The sufficient conditions of the bilateral case generalize to

$$\frac{\partial}{\partial v}\Pr(r \geq s_{(m)} \,|v\,) < 0, \tag{9}$$

$$\frac{\partial}{\partial v}\left[v + \frac{\Pr(r \geq s_{(m)} \,|v\,)}{\frac{\partial}{\partial v}\Pr(r \geq s_{(m)} \,|v\,)}\right] \geq 0. \tag{10}$$

We now show that condition (9) holds in our model for general F. Invariance implies

$$\Pr(r \geq s_{(m)} \,|v\,) = \Pr(r + \lambda \geq s_{(m)} \,|v + \lambda) \textit{ for all } \lambda \in \mathbb{R}.$$

Consequently,

$$
\begin{aligned}
0 &= \left.\frac{d}{d\lambda}\Pr(r+\lambda \geq s_{(m)}\,|v+\lambda)\right|_{\lambda=0} \qquad (11)\\
&= \frac{\partial}{\partial r}\Pr(r \geq s_{(m)}\,|v\,) + \frac{\partial}{\partial v}\Pr(r \geq s_{(m)}\,|v\,).
\end{aligned}
$$

The desired sufficient condition is therefore equivalent to

$$\frac{\partial}{\partial r}\Pr(r \geq s_{(m)}\,|v\,) > 0.$$

It is clear that this inequality holds because increasing r increases the probability that it is among the m largest reports.

Condition (10) is therefore equivalent to

$$\frac{\partial}{\partial v}\left[v - \frac{\Pr(r \geq s_{(m)}\,|v\,)}{\frac{\partial}{\partial r}\Pr(r \geq s_{(m)}\,|v\,)}\right] \geq 0. \qquad (12)$$

As in the first order approach to the *BBDA*, it is difficult to identify conditions directly upon F that imply (12). Numerical calculations, however, suggest that this condition holds in the case of the standard normal distribution studied in section 4.

2.2.3 The Values of α and β

Given honest reporting, the values of $\alpha(v)$ and $\beta(v)$ are

$$\alpha(v) = \frac{\Pr(v \geq s_{(m)}\,|v\,)^2}{\left.\frac{\partial}{\partial v}\Pr(r \geq s_{(m)}\,|v\,)\right|_{r=v}} \qquad (13)$$

and

$$\beta(v) = \frac{\Pr(v \geq s_{(m)}\,|v\,)}{\left.\frac{\partial}{\partial v}\Pr(r \geq s_{(m)}\,|v\,)\right|_{r=v}}. \qquad (14)$$

The invariance of a buyer's decision problem simplifies the values of $\alpha(v)$ and $\beta(v)$, as summarized in the following theorem.

Theorem 1 *The following properties hold for formulas (13) and (14):*

1. $$\Pr(v \geq s_{(m)}\,|v\,) = \frac{1}{2}. \qquad (15)$$

2. *The derivatives*
$$\left.\frac{\partial}{\partial v}\Pr(r \geq s_{(m)}\,|v\,)\right|_{r=v} = -\left.\frac{\partial}{\partial r}\Pr(r \geq s_{(m)}\,|v\,)\right|_{r=v} \qquad (16)$$
do not depend upon the value of v.

As a consequence of (15) and (16), the values of $\alpha(v)$ and $\beta(v)$ do not depend upon v. We hereafter write $\alpha(m)$ and $\beta(m)$ as we explore their dependence on the market size m. Notice also that (15) and (6) imply

$$\alpha(m) = \frac{\beta(m)}{2}, \tag{17}$$

where

$$\alpha(m) = \frac{\left(\Pr(v \geq s_{(m)}\,|v\,)\right)^2}{\frac{\partial}{\partial v}\Pr(r \geq s_{(m)}\,|v\,)\big|_{v=r}} = -\frac{1}{4}\frac{1}{\frac{\partial}{\partial r}\Pr(r \geq s_{(m)}\,|v\,)\big|_{r=v}} < 0. \tag{18}$$

2.3 The Utility Consequences of the *BDM* for a Buyer

Applying (5), a buyer's ex post utility in the *BDM* when v is his value and all traders report honestly is

$$\begin{aligned} \alpha(m) \ \textit{if}\ v &< s_{(m)}, \\ \alpha(m) - \beta(m) \ \textit{if}\ v &\geq s_{(m)}. \end{aligned}$$

Recall that $s_{(m)}$ here is the m^{th} smallest value/cost among the $2m-1$ values/costs of the other traders. The first line is his ex post utility when he fails to trade and the second is his ex post utility when he trades. Applying (17) and (18), this reduces to

$$\begin{aligned} \alpha(m) \ \textit{if}\ v &< s_{(m)}, \\ -\alpha(m) \ \textit{if}\ v &\geq s_{(m)}, \end{aligned}$$

where

$$\alpha(m) = -\frac{1}{4}\frac{1}{\frac{\partial}{\partial r}\Pr(r \geq s_{(m)}\,|v\,)\big|_{r=v}}$$

is strictly negative and does not depend upon v. Ex post individual rationality is therefore violated in the *BDM* if and only if a buyer fails to trade.

The following theorem bounds this ex post loss away from zero and characterizes its asymptotic value.

Theorem 2 *Assume that the density f is continuous and bounded with $f(\mu_{1/2}) > 0$, where $\mu_{1/2}$ denotes the median of F. The loss $\alpha(m)$ satisfies the inequality*

$$|\alpha(m)| \geq \frac{1}{4\overline{f}} \tag{19}$$

for all $m \in \mathbb{N}$, where

$$\overline{f} = \sup_{x\in\mathbb{R}} f(x).$$

Its limiting value is

$$\lim_{m\to\infty} \alpha(m) = -\frac{1}{4f\left(\mu_{1/2}\right)}. \tag{20}$$

2.4 Ex Ante Budget Balance and the Basic Designed Mechanism

From the ex ante perspective, a seller's expected utility in the *BDM* is the same as it would be in the *BBDA* if (contrary to the incentives of buyers) all traders report their values/costs honestly. A buyer's interim expected utility in the *BDM* is zero by construction, and so his ex ante expected utility equals zero. All of the ex ante gains from trade are thus not distributed among the traders in the *BDM*. As defined in section 1.2, $GFT(m)$ denotes the total ex ante expected gains from trade received by all traders in each state μ in an efficient allocation rule. Let $GFT^*(m)$ denote the ex ante expected gains from trade received by the m sellers in each state μ in the *BBDA* assuming honest revelation by all traders. The difference $GFT(m) - GFT^*(m)$ represents the portion of the ex ante gains from trade allocated to buyers in the *BBDA* when all traders report honestly. The *BDM* must be modified to distribute this quantity among the $2m$ traders in order to achieve ex ante budget balance.

We consider here adding constants

$$\left(\gamma_j^s\right)_{1\leq j\leq m}, \left(\gamma_i^b\right)_{1\leq i\leq m}$$

to the monetary transfers (4) and (5) of sellers and buyers. Seller j receives a payment of

$$\begin{aligned} s_{(m+1)} + \gamma_j^s \textit{ if } c_j &\leq s_{(m)} \\ \gamma_j^s \textit{ if } c_j &> s_{(m)} \end{aligned}$$

with the honest report of his cost c_j, and buyer i pays

$$\begin{aligned} -\alpha(m) - \gamma_i^b \textit{ if } v_i &< s_{(m+1)}, \\ v_i + \beta(m) - \alpha(m) - \gamma_i^b \textit{ if } v_i &\geq s_{(m+1)} \end{aligned}$$

when he honestly reports his value v_i. The order statistics $s_{(m)}$ and $s_{(m+1)}$ above are for the entire sample of $2m$ values/costs. Each trader receives the subsidy of his particular constant for every sample of values and costs. The inclusion of these constants in the transfer functions therefore does not alter the incentives for honest reporting by the traders.[15]

The constraint of ex ante budget balance is satisfied if and only if

$$\sum_{j=1}^{m} \gamma_j^s + \sum_{i=1}^{m} \gamma_i^b = GFT(m) - GFT^*(m). \tag{21}$$

Define $\mathcal{DM}$ as the set of all mechanisms obtained by starting with the *BDM* and adding constant transfers that satisfy (21).

There are two particular mechanisms in the family $\mathcal{DM}$ that are noteworthy for our purposes.

[15]More generally, each trader's subsidy could depend upon the reported values/costs of all other traders.

Let $\gamma(m)$ denote the constant

$$\gamma(m) = \frac{GFT(m) - GFT^*(m)}{m}.$$

Define DM^s as the mechanism *BDM* with the additional constant transfers in the case in which each $\gamma_i^b = 0$ and the sellers share the remaining surplus equally,

$$\gamma_j^s = \gamma(m) \ \textit{for}\ 1 \leq j \leq m.$$

This is a mechanism that most directly generalizes the McAfee-Reny mechanism from the bilateral case in the sense that all ex ante gains from trade are allocated to the sellers. Alternatively, define *DM* as the mechanism *BDM* with the additional constant transfers in the case in which each $\gamma_j^s = 0$ and the buyers share the remaining surplus equally,

$$\gamma_i^b = \gamma(m) \ \textit{for}\ 1 \leq i \leq m.$$

The ex post irrationality of the *BDM* is with respect to buyer payoffs; as discussed below, the mechanism *DM* is most promising for the sake of addressing this flaw of the *BDM*.[16]

We conclude this subsection by characterizing the limiting value of $\gamma(m)$. This will be of interest below in the effort to address the ex post budget imbalance and irrationality of the *BDM*.

Theorem 3 *The constant $\gamma(m)$ satisfies*

$$\lim_{m \to \infty} \gamma(m) = \int_{\mu_{1/2}}^{\infty} v f(v) dv - \mu_{1/2}. \tag{22}$$

3 Ex Post Irrationality and Deficits in the Designed Mechanism *DM*

The *BDM* leaves a buyer with an ex post loss when he fails to trade and may also require an ex post monetary subsidy to operate. We consider in this subsection the possibility of resolving these problems through the constant transfers discussed in subsection 2.4.

Ex Post Irrationality. The ex post utility of a seller in the *BDM* is nonnegative and so any allocation of the ex ante surplus $GFT(m) - GFT^*(m)$ among the sellers wastes a precious resource that could be better applied to resolve the ex post irrationality of the mechanism for buyers. With this in mind, we therefore focus here on *DM* in which all of the surplus is allocated equally among

[16] Given the discussion of DM^s, it makes sense notationally to write DM^b instead of simply *DM* as a way to indicate that it modifies *BDM* by allocating the excess gains from trade equally among buyers. This is the only point of the paper, however, at which DM^s is mentioned. The rest of the paper focuses exclusively on *BDM* and *DM*, and so the superscript b is not needed for our purposes.

the buyers.[17] Let $\underline{U}(m)$ denote a buyer's ex post utility in the *DM* when he fails to trade,

$$\underline{U}(m) = \alpha(m) + \gamma(m). \tag{23}$$

Theorems 2 and 3 together imply

$$\lim_{m\to\infty} \underline{U}(m) = \int_{\mu_{1/2}}^{\infty} vf(v)dv - \mu_{1/2} - \frac{1}{4f\left(\mu_{1/2}\right)}.$$

We show below that this quantity is strictly negative when $F = \Phi$, the cumulative for the standard normal $\mathcal{N}(0,1)$. Consequently, for sufficiently large m, no element of the family $\mathcal{DM}$ is ex post individually rational in this case. The integral on the right-hand side, however, can be arbitrarily large or even infinite depending upon the distribution F.[18] We conclude that there are robust examples of distributions F in which the *DM* is ex post individually rational and robust examples in which it is not.

As a final point in this discussion of individual rationality, we mention the problems posed for the use of a trading mechanism if a trader earns a positive payoff when he fails to trade. This is a feature of any mechanism in the family $\mathcal{DM}$ in which $\gamma_j^s > 0$ for some seller j or $\gamma_i^b + \alpha(m) > 0$ for some buyer i. This does not pose a problem for the rationality of each trader's participation. In a dynamic setting or with repeated use of the mechanism, however, it may provide a trader with an incentive to bid/ask so as to increase the likelihood that he fails to trade so that he can return to the marketplace in a subsequent period to again participate and take his profit. A seller might set his ask far above his cost and a buyer might bid far below his value. It is also the case that the mechanism may attract a con man who participates as a trader, bids/offers so that he almost surely will not trade, and then with high likelihood makes a positive profit. A fraudulent buyer may bid without having the money to truly buy and a fraudulent seller may ask without having an item to sell. Modeling this kind of behavior in a dynamic framework is beyond the scope of this paper. This is, however, a plausible set of problems that a trading mechanism may face in practice if traders profit when they do not trade. It is a twist on the usual individual rationality constraint that to our knowledge has not previously appeared in the market design literature.

Ex Post Budget Imbalance. How large of an external monetary subsidy may be required to operate a member of the family $\mathcal{DM}$? Define $B(\widetilde{v},\widetilde{c})$ as the ex post monetary surplus in *DM*, i.e., the total payments by buyers minus the total payments to sellers in *DM* given the sample of values and costs $\widetilde{v} = (v_i)_{1\leq i\leq m}$ and $\widetilde{c} = (c_j)_{1\leq j\leq m}$. The constraint (21) on the constant transfers $(\gamma_i^b)_{1\leq i\leq m}$, $(\gamma_j^s)_{1\leq j\leq m}$ implies that $B(\widetilde{v},\widetilde{c})$ equals the ex post monetary surplus in any member of the family $\mathcal{DM}$, and so we focus on *DM* below with no loss of generality. Let $\underline{B}(m)$ denote the

[17] There is nothing gained for the sake of reducing the ex post loss of buyers who fail to trade by allowing the constant payments to the buyers to differ, as in (21): diminishing the ex post loss to buyer i by choosing $\gamma_i^b > \gamma(m)$ necessarily worsens the loss of some other buyer because of the ex ante budget constraint (21). This justifies our symmetric treatment of buyers in this discussion.

[18] The Cauchy distribution is one instance in which this integral is infinite. It is relevant here because its "fat-tails" are useful in finance for modeling extreme events that occur with nontrivial probability.

worst case value of $B(\widetilde{v},\widetilde{c})$, i.e.,

$$\underline{B}(m) = \inf_{\widetilde{v},\widetilde{c}} B(\widetilde{v},\widetilde{c}).$$

The following theorem shows that $\underline{B}(m)$ is strictly negative, bounded away from zero for all m, and decreasing at a linear rate in m to $-\infty$.

Theorem 4 *The worst case value of the monetary surplus $B(\widetilde{v},\widetilde{c})$ of any mechanism in $\mathcal{DM}$ is*

$$\underline{B}(m) = m\left(\alpha\left(m\right) - \gamma\left(m\right)\right), \tag{24}$$

which occurs in the event in which all m values of buyers are equal and all m costs of sellers are strictly less than this value. This worst case value satisfies the bound

$$\underline{B}(m) \leq m\left(-\frac{1}{4\overline{f}} - \gamma\left(m\right)\right) \tag{25}$$

for all m. On a per capita basis, $\underline{B}(m)/2m$ has the limiting value

$$\lim_{m\to\infty} \frac{\underline{B}(m)}{2m} = \frac{1}{2}\left(-\frac{1}{4f\left(\mu_{1/2}\right)} - \int_{\mu_{1/2}}^{\infty} vf(v)dv + \mu_{1/2}\right). \tag{26}$$

Because $\alpha\left(m\right) < 0$ and $\gamma\left(m\right) > 0$, (24) implies that $\underline{B}(m)$ is strictly negative for all distributions F, i.e., every mechanism in $\mathcal{DM}$ necessarily requires an external monetary subsidy to operate for some samples of values/costs.

Example. *In the case of $F = \Phi$, the cumulative for the standard normal distribution, we have*

$$\begin{aligned}
\lim_{m\to\infty} \gamma\left(m\right) &= \int_0^{\infty} vf(v)dv \\
&= \int_0^{\infty} v \cdot \frac{1}{\sqrt{2\pi}} e^{-v^2/2} dv \\
&= \int_0^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x} dx \\
&= -\frac{1}{\sqrt{2\pi}} e^{-x}\Big|_0^{\infty} \\
&= \frac{1}{\sqrt{2\pi}}.
\end{aligned}$$

We also have

$$\lim_{m\to\infty} \alpha(m) = -\frac{1}{4f\left(0\right)} = -\frac{\sqrt{2\pi}}{4}.$$

The limiting value of a buyer's ex post utility when he fails to trade is therefore

$$\lim_{m\to\infty} \underline{U}(m) = \lim_{m\to\infty} \gamma\left(m\right) + \alpha(m) = \frac{1}{\sqrt{2\pi}} - \frac{\sqrt{2\pi}}{4} < 0.$$

The limiting value of the worst case budget subsidy per capita is

$$\lim_{m\to\infty}\frac{\underline{B}(m)}{2m}=\frac{1}{2}\left(-\frac{\sqrt{2\pi}}{4}-\frac{1}{\sqrt{2\pi}}\right)=-\frac{1}{2}\left(\frac{\sqrt{2\pi}}{4}+\frac{1}{\sqrt{2\pi}}\right)<0.$$

4 Numerical Example

We numerically investigate the properties of the *DM* relative to the *BBDA* in this section in the case of $F=\Phi$, the cumulative of the standard normal distribution. Table 1 documents the properties of the *DM*.[19] Column 1 lists the number m of traders on each side of the market; the asymptotic limiting values of each column are presented in the bottom row. Column 2 lists the ex ante expected gains from trade in each size of market. This column is included to provide a sense of scale for the other numbers in the table. Column 3 lists the ex post utility $\alpha(m)$ of a buyer who fails to trade in the *BDM*. Column 4 presents the maximal constant transfer $\gamma(m)$ that can be provided to each buyer to lessen this ex post loss while also adapting the *BDM* so that it satisfies ex ante budget balance. This defines the transfer of a buyer in the *DM*. Column 5 then lists the ex post loss $\underline{U}(m)$ of a buyer who fails to trade in the *DM* and the convergence of this value to its asymptotic limit.

Columns 6 and 7 in Table 1 address the ex post surplus/deficit $B(\tilde{v},\tilde{c})$ that the *DM* runs. This surplus/deficit has expectation zero, but its ex post realization can vary significantly. Column 6 shows that the worst case deficit, $|\underline{B}(m)|$, exceeds the *DM*'s expected gains from trade. $|\underline{B}(m)|$ is the monetary reserve that is required if the *DM*'s solvency is to be guaranteed for all possible realizations of values and costs. It may overstate the problem of deficits because the likelihood that $B(\tilde{v},\tilde{c})$ lies within ε of $\underline{B}(m)$ decreases to zero exponentially fast for any sufficiently small ε. Column 7 is therefore relevant. It shows that the standard deviation $SD\left[B(\tilde{v},\tilde{c})\right]$ of $B(\tilde{v},\tilde{c})$ increases at the rate $O(\sqrt{m})$. The problem of deficits and surpluses thus worsens in absolute terms as the market size m increases. Because *GFT* grows linearly in m, however, the ratio $SD\left[B(\tilde{v},\tilde{c})\right]/GFT$ is $O(1/\sqrt{m})$. The budget problem thus diminishes relative to *GFT* when the market size m increases.

The performance of the *BBDA* is presented in Table 2. Columns 2 through 4 demonstrate the properties of the *BBDA* as m increases. *GFT* in column 2 is the ex ante expected gains from trade for an ex post efficient mechanism and GFT_{BBDA} is the expected gains from trade achieved in the market of size m when buyers use an offset equilibrium in the *BBDA*. Column 4 tabulates the relative inefficiency $(GFT-GFT_{BBDA})/GFT$, i.e., the ex ante expected relative loss to the traders from using the inefficient *BBDA*, as defined in (1). Alternatively this number represents the relative gain to the traders that is obtained by switching from the *BBDA* to the *DM*. For m as

[19] The results in this section are computed using a Monte Carlo method. Alternatively, one can follow Serfling (1980, p. 77) and use the fact that the sample median in this case is asymptotically normal with mean zero and variance $\pi/2m$. The values of $\underline{U}(m)$ and $\underline{B}(m)$ can then be estimated using the asymptotic distribution. The estimates calculated in this way closely track the values obtained by Monte Carlo method even for the small values of m considered here.

m	GFT	$\alpha(m)$	$\gamma(m)$	$\underline{U}(m) = \alpha(m) + \gamma(m)$	$\lvert\underline{B}(m)\rvert$	$SD\left[B\left(\widetilde{v}, \widetilde{c}\right)\right]$
2	1.327	-0.7539	0.1833	-0.5707	1.8740	0.8396
4	2.901	-0.6894	0.2874	-0.4019	3.9014	1.0347
8	6.081	-0.6577	0.3407	-0.3170	7.9844	1.4006
16	12.460	-0.6470	0.3696	-0.2724	16.1951	1.9581
∞	∞	-0.6267	0.3987	-0.2278	∞	∞

Table 1: The table documents the properties of the *DM*. Column 1 lists the number m of traders on each side of the market and Column 2 lists the ex ante expected gains from trade for each size of market. Column 3 lists the ex post utility $\alpha(m)$ of a buyer who fails to trade in the *BDM*. Column 4 presents the maximal constant transfer $\gamma(m)$ that can be provided to each buyer to lessen this ex post loss while also adapting the *BDM* so that it satisfies ex ante budget balance. Column 5 then lists the ex post loss $\underline{U}(m)$ of a buyer who fails to trade in the *DM* and the convergence of this value to its asymptotic limit. Column 6 lists $|\underline{B}(m)|$, which is the maximal amount of money that may be needed as an external subsidy to operate the mechanism ex post. Column 7 lists the standard deviation of the ex post budget surplus in each size of market.

m	GFT	GFT_{BBDA}	$\frac{GFT - GFT_{BBDA}}{GFT}$	$\frac{\lvert\underline{U}(m)\rvert}{GFT - GFT_{BBDA}}$
2	1.327	1.222	0.07912	5.4352
4	2.901	2.854	0.01620	8.5511
8	6.081	6.065	0.00263	19.8125
16	12.460	12.452	0.00064	34.0500

Table 2: The table evaluates the performance of the $BBDA$ relative to the DM. Column 1 is the size of the market, column 2 is the expected gains from trade for the efficient DM, column 3 is the expected gains from trade for the imperfectly efficient $BBDA$, column 4 is the relative inefficiency of the $BBDA$. Column 5 measures the relative cost and benefit of switching from the $BBDA$ to DM. It tabulates the magnitude of the loss that a buyer may bear ex post in the DM relative to the increase in the expected gains from trade that switching from the $BBDA$ to the DM generates.

small as 8, the percentage gain in efficiency from switching from the *BBDA* to an ex post efficient mechanism is significantly less than one percent.

Column 5 of Table 2 evaluates the ex post cost of achieving efficiency in the *DM* in comparison with the *BBDA*. A buyer's ex post loss when he fails to trade is $\underline{U}(m)$; bearing this ex post loss allows an increase in the ex ante gains from trade of $GFT - GFT_{BBDA}$ from switching from the *BBDA* to the *DM*. The ratio

$$\frac{|\underline{U}(m)|}{GFT - GFT_{BBDA}}$$

therefore represents the cost that a buyer may bear ex post per dollar increase in the ex ante expected gains from trade. It is worth reemphasizing that these ex post losses fall on buyers who do not trade, an event that has ex ante probability 1/2 for each buyer, regardless of the size m of the market.

5 Conclusion

Statistical dependence among the private signals of agents can be used to design mechanisms that are ex post efficient despite the possibility of strategic behavior. We show, however, in a simple trading problem that a family of designed mechanisms inspired by the methods of McAfee and Reny (1992) can have unattractive properties such as ex post budget imbalance and irrationality. Its flaws do not diminish as the market increases in size, which is counterintuitive to our understanding of a competitive market. We consider these to be undesirable features in comparison to the buyer's bid double auction that, while inefficient, is simply defined, ex post budget balanced, and ex post individually rational. Its inefficiency quickly diminishes as the market increases in size and converges to perfect competition. Using dependence among costs/values to achieve efficiency in a market setting thus appears in this analysis as a complicated exercise that produces an unintuitive mechanism for the sake of only modest gains in efficiency over simple mechanisms such as the *BBDA*.

Appendix

Proof of Theorem 1. The probability (15) can be written as

$$\Pr(v \geq s_{(m)} | v) = \int_{-\infty}^{\infty} \sum_{t=m}^{2m-1} \binom{2m-1}{t} F(v-\mu)^t \, \overline{F}(v-\mu)^{2m-1-t} f(v-\mu) \, d\mu.$$

The change of variable $v - \mu \rightarrow \mu$ and moving the integral sign reduces this to

$$\sum_{t=m}^{2m-1} \binom{2m-1}{t} \int_{-\infty}^{\infty} F(\mu)^t \, \overline{F}(\mu)^{2m-1-t} f(\mu) \, d\mu. \tag{27}$$

The t^{th} integral reduces as follows through integration by parts:

$$\begin{aligned}
& \int_{-\infty}^{\infty} F(\mu)^t \, \overline{F}(\mu)^{2m-1-t} f(\mu) \, d\mu \\
= \; & \left. \frac{F(\mu)^{t+1}}{t+1} \overline{F}(\mu)^{2m-1-t} \right|_{-\infty}^{\infty} \\
& + \int_{-\infty}^{\infty} \frac{2m-1-t}{t+1} F(\mu)^{t+1} \, \overline{F}(\mu)^{2m-2-t} f(\mu) \, d\mu \\
= \; & \int_{-\infty}^{\infty} \frac{2m-1-t}{t+1} F(\mu)^{t+1} \, \overline{F}(\mu)^{2m-2-t} f(\mu) \, d\mu \\
& \vdots
\end{aligned}$$

$$
\begin{aligned}
&= \int_{-\infty}^{\infty} \frac{(2m-1-t)\cdot(2m-2-t)\cdot\ldots\cdot 1}{(t+1)\cdot(t+2)\cdot\ldots\cdot(2m-1)} F(\mu)^{2m-1} f(\mu)\,d\mu \\
&= \binom{2m-1}{t}^{-1} \int_{-\infty}^{\infty} F(\mu)^{2m-1} f(\mu)\,d\mu \\
&= \binom{2m-1}{t}^{-1} \left.\frac{F(\mu)^{2m}}{2m}\right|_{-\infty}^{\infty} \\
&= \binom{2m-1}{t}^{-1} \cdot \frac{1}{2m}.
\end{aligned}
$$

Substitution into (27) implies

$$
\Pr(v \geq s_{(m)} \,|v\,) = \sum_{t=m}^{2m-1} \frac{1}{2m} = \frac{1}{2}.
$$

Statement 2. follows from expanding one of the derivatives and applying a change of variable:

$$
\begin{aligned}
&\frac{\partial}{\partial r} \Pr(r \geq s_{(m)} \,|v\,) \\
&= \frac{\partial}{\partial r} \int_{-\infty}^{\infty} \sum_{t=m}^{2m-1} \binom{2m-1}{t} F(r-\mu)^{t}\, \overline{F}(r-\mu)^{2m-1-t} f(v-\mu)\,d\mu \\
&= \sum_{t=m}^{2m-1} \binom{2m-1}{t} \int_{-\infty}^{\infty} \frac{\partial}{\partial r} \left[F(r-\mu)^{t}\, \overline{F}(r-\mu)^{2m-1-t} \right] f(v-\mu)\,d\mu \\
&= \sum_{t=m}^{2m-1} \binom{2m-1}{t} \int_{-\infty}^{\infty} \left[t-(2m-1)F(r-\mu)\right] \cdot \\
&\qquad F(r-\mu)^{t-1}\, \overline{F}(r-\mu)^{2m-2-t} f(r-\mu) f(v-\mu)\,d\mu.
\end{aligned}
$$

Evaluated at $r = v$, this equals

$$
\begin{aligned}
&\sum_{t=m}^{2m-1} \binom{2m-1}{t} \int_{-\infty}^{\infty} \left[t-(2m-1)F(v-\mu)\right] \cdot \\
&\qquad F(v-\mu)^{t-1}\, \overline{F}(v-\mu)^{2m-2-t} f(v-\mu)^{2}\,d\mu \\
&= \sum_{t=m}^{2m-1} \binom{2m-1}{t} \int_{-\infty}^{\infty} \left[t-(2m-1)F(\mu)\right] F(\mu)^{t-1}\, \overline{F}(\mu)^{2m-2-t} f(\mu)^{2}\,d\mu,
\end{aligned}
$$

where the second line follows from the change of variable $v - \mu \to \mu$. ∎

Proof of Theorem 2. For (19), it is sufficient to bound

$$
\left.\frac{\partial}{\partial r} \Pr(r \geq s_{(m)} \,|v\,)\right|_{r=v}
$$

above by a constant for all m. We have

$$\frac{\partial}{\partial r}\Pr(r \geq s_{(m)}\,|v)\bigg|_{r=v} = \Pr(v = s_{(m)}\,|v)$$

$$= \int_{-\infty}^{\infty}(2m-1)f(v-\mu)\binom{2m-2}{m-1}F(v-\mu)^{m-1}\overline{F}(v-\mu)^{m-1}f(v-\mu)d\mu$$

$$= \int_{-\infty}^{\infty}f(\mu)(2m-1)\binom{2m-2}{m-1}F(\mu)^{m-1}\overline{F}(\mu)^{m-1}f(\mu)d\mu.$$

The expression

$$(2m-1)\binom{2m-2}{m-1}F(\mu)^{m-1}\overline{F}(\mu)^{m-1}f(\mu) \tag{28}$$

is the density of the m^{th} order statistic in a sample of $2m-1$ independent draws according to the distribution F (i.e., the median of the sample). This integral is therefore the expected value of f computed with respect to this density and it is bounded above by $\overline{f}$. The bound (19) follows immediately.

The sample median is asymptotically normal with mean $\mu_{1/2}$ and variance $(4f(\mu_{1/2})^2m)^{-1}$ (Serfling (1980, p. 77)). The density of the m^{th} order statistic therefore converges to a Dirac function at the median $\mu_{1/2}$ as $m \to \infty$. Equation (20) then follows from the assumption that f is continuous and bounded (Serfling (1980, p. 16)). ■

Proof of Theorem 3. The definition of $\gamma(m)$ implies that it equals the ex ante expected gains from trade of a buyer in the *BBDA* assuming that all traders report honestly. The expected gains from trade of a buyer has the same value in every state μ, which allows us to calculate this value despite the improper distribution of μ. We therefore fix $\mu = 0$. A selected buyer's ex ante gains from trade given $\mu = 0$ equals

$$\int_{-\infty}^{\infty}\int_{-\infty}^{v}(v-t)\,(2m-1)\binom{2m-2}{m}F(t)^m\,\overline{F}(t)^{m-2}f(t)\,f(v)\,dtdv. \tag{29}$$

The variable t denotes the $(m+1)^{\text{st}}$ smallest value/cost in a sample of $2m-1$ values/costs given $\mu = 0$, and

$$(2m-1)\binom{2m-2}{m}F(t)^m\,\overline{F}(t)^{m-2}f(t)$$

is the density of this order statistic. This equals

$$\int_{-\infty}^{\infty}\int_{-\infty}^{v}v\,(2m-1)\binom{2m-2}{m}F(t)^m\,\overline{F}(t)^{m-2}f(t)\,f(v)\,dtdv \tag{30}$$

$$-\int_{-\infty}^{\infty}\int_{-\infty}^{v}t\,(2m-1)\binom{2m-2}{m}F(t)^m\,\overline{F}(t)^{m-2}f(t)\,f(v)\,dtdv. \tag{31}$$

We consider these two integrals in separate arguments.

Integral (30) equals

$$\int_{-\infty}^{\infty} v \Pr[v \geq t] f(v) dv.$$

We have by Serfling (1980, Sec. 2.3.3, Cor. A),

$$\lim_{m\to\infty} \Pr[v \geq t] = \begin{cases} 1 \textit{ if } v > \mu_{1/2} \\ 0 \textit{ if } v < \mu_{1/2} \end{cases}.$$

Consequently,

$$\lim_{m\to\infty} \int_{-\infty}^{\infty} v \Pr[v \geq t] f(v) dv = \int_{\mu_{1/2}}^{\infty} v f(v) dv.$$

Reversing the order of integration, the integral in (31) equals

$$\begin{aligned} & \int_{-\infty}^{\infty} \int_{t}^{\infty} t\,(2m-1) \binom{2m-2}{m} F(t)^m \overline{F}(t)^{m-2} f(t) f(v)\, dv dt \\ = & \int_{-\infty}^{\infty} t\,(2m-1) \binom{2m-2}{m} F(t)^m \overline{F}(t)^{m-1} f(t)\, dt. \end{aligned}$$

This is the expected value of the $(m+1)^{\text{st}}$ order statistic in a sample of $2m-1$ values from the distribution F. Its limit is $\mu_{1/2}$ (Serfling (1980, p. 77)), which completes the proof. ■

Proof of Theorem 4. We pair (i) each buyer who trades with a seller who trades and (ii) each buyer who does not trade with a seller who does not trade. We then consider the increment that each pair adds to $B(\widetilde{v}, \widetilde{c})$. If a buyer and a seller trade in the *DM*, the buyer pays

$$v + \alpha(m) - \gamma(m),$$

and the seller receives $s_{(m+1)}$. Each consummated trade therefore adds

$$\begin{aligned} & v + \alpha(m) - \gamma(m) - s_{(m+1)} \\ = & \left(v - s_{(m+1)}\right) + \alpha(m) - \gamma(m) \end{aligned}$$

to $B(\widetilde{v}, \widetilde{c})$. A buyer trades only if $v \geq s_{(m+1)}$; this term therefore attains its minimum value of $\alpha(m) - \gamma(m)$ at $v = s_{(m+1)}$. A seller who does not trade receives nothing while a buyer who does not trade pays $-\alpha(m) - \gamma(m)$. Each buyer/seller pair that do not trade therefore adds $-\alpha(m) - \gamma(m)$ to $B(\widetilde{v}, \widetilde{c})$. Because $\alpha(m) < 0$, we have

$$\alpha(m) - \gamma(m) < -\alpha(m) - \gamma(m).$$

A buyer/seller pair therefore contribute more to $B(\widetilde{v}, \widetilde{c})$ when they do not trade than when they trade and the buyer's value equals $s_{(m+1)}$. The monetary surplus therefore has the minimum value of (24).

The bound (25) then follows from (19) and the limit (26) follows from (20) and (22). ■

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