The Asymptotics of Price and Strategy in the Buyer's Bid Double Auction

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Abstract

The analysis of equilibrium in double auctions is complicated, with almost no closed-form examples and with formal results limited to bounds on bidding behavior. We consider the buyer's bid double auction in a correlated, private values model. Sellers have the incentive to submit their costs as their asks while buyers strategically underbid. We determine (i) the asymptotic distribution of the equilibrium market price and (ii) the asymptotic limits of terms in the first order condition for a buyer's selection of his bid. Part (i) reveals the properties of the strategically determined market price as an estimator of the rational expectations equilibrium price. Part (ii) provides a simple formula for a buyer's bid that is shown numerically to closely approximate his equilibrium bid even in relatively small markets. This formula reveals how equilibrium varies with the numbers of buyers and sellers and the distribution of their values/costs.
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1 Introduction

We consider the *buyer's bid double auction* (BBDA), which is a simple model of a call market in which traders interact strategically to determine a market-clearing price and allocation. This paper considers a particular *correlated private values* (CPV) environment, in which case the rules of the BBDA ensure that submitting one's true cost as his ask is a weakly dominant strategy for each seller. The focus is therefore upon the strategic effort by each buyer to influence price in his favor and the solution concept is Bayesian-Nash equilibrium. We carry out an asymptotic analysis of the probabilities that are relevant to a buyer's decision problem in choosing his bid and obtain two results. First, we determine the asymptotic distribution of the BBDA's equilibrium

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price about the *rational expectations equilibrium* (REE) price, thereby determining the properties of the strategically determined price as an estimator of the REE price. Second, we obtain a simple approximation for a buyer's equilibrium bid as a function of his probabilistic beliefs and the numbers of traders on each side of the market. Calculating equilibrium exactly is a formidable task in all but the smallest of markets; the simple approximating formula is easily applied in all sizes of markets and also provides insight into the dependence of equilibrium upon the fundamentals of the model. Finally, numerical work demonstrates the accuracy of the approximation even in relatively small markets.

The market consists of m buyers, each of whom wishes to buy at most one item, and n sellers, each of whom has one item to sell. Each buyer i privately observes the value v_i that he receives if he acquires an item and each seller j privately observes the cost c_j that he bears if he sells his item. Utility for each trader is quasilinear in his value/cost and money. Our trade mechanism is a follows: a bid is collected from every buyer, an ask from every seller, which are then used to construct demand and supply curves. The BBDA selects as the market price the upper boundary of the interval of market-clearing prices with trade occurring among buyers who bid at least this price and sellers whose asks are less than this price.

The values/costs of the $m + n$ traders are generated as follows. A *state* μ is drawn from the uniform improper prior on $\mathbb{R}$, which can be thought of intuitively as "the uniform distribution on the entire real line". The use of the uniform improper prior is discussed in Satterthwaite, Williams, and Zachariadis (2014, Introduction);[1] its main value in our analysis is that it implies an invariance property for a trader's decision problem that greatly simplifies the analysis of equilibrium. For each trader i, a value ε_i is then independently drawn from the cumulative distribution F on $\mathbb{R}$, which is absolutely continuous with finite first moment. The density f is strictly positive on $\mathbb{R}$ and symmetric about zero. Trader i privately observes his value/cost $\mu + \varepsilon_i$. Through its dependence on the state μ, a trader's value/cost is correlated with the values/costs of others. We consider equilibria in which all buyers employ the same increasing, differentiable, non-dominated strategy B and all sellers ask honestly. A strategy B is assumed to have these three properties throughout the paper.

We use here several results from SWZ. The *first order condition* (FOC) for a buyer's decision problem is derived in its section 3.1. An *offset strategy* for buyers is one in which the amount $v - B(v)$ by which a buyer underbids is constant for all v. Existence of an offset solution to a buyer's FOC is proven in its Theorem 1; numerical work then establishes that this offset solution defines an equilibrium and in fact appears to be the unique equilibrium strategy. For fixed m, n and for a sequence of markets with ηm buyers and ηn sellers with *market size* $\eta \in \mathbb{N}$, SWZ (Thm. 3) establishes that the amount by which a buyer underbids in a symmetric equilibrium offset strategy is $O(1/\eta)$.[2] Finally, the paper uses standard methodology to determine the REE price as a function of the state μ, m, n and the distribution F.

[1]Hereafter we refer to Satterthwaite, Williams, and Zachariadis (2014) as SWZ.

[2]For functions $u_1(\eta)$, $u_2(\eta) : \mathbb{N} \to \mathbb{R}_+$, $u_1(\eta) = O(u_2(\eta))$ means that there exist constants $k \in \mathbb{R}_+$ and $\eta_0 \in \mathbb{N}$ such that $u_1(\eta) < ku_2(\eta)$ for all $\eta > \eta_0$.

We have two main results. First, we determine the asymptotic distribution of the BBDA's equilibrium price. It is shown to be a consistent, asymptotically unbiased and normal estimator of the REE price in each state μ. We thus show in our elementary trading environment that a simple procedure can approximate the REE price despite the strategic behavior of traders. In this way, we advance the research agenda initiated by Reny and Perry (2006) that aims to build a strategic foundation for rational expectations equilibrium based upon the double auction as the market mechanism.

The FOC for the buyer's decision problem is difficult to use because of the complexity of the probabilities of the relevant order statistics among bids/asks. Our second result addresses this by identifying the asymptotic values of these probabilities. This asymptotic analysis models how a sophisticated but boundedly rational buyer might approach his decision problem in the BBDA as opposed to solving exactly for equilibrium. Identifying the asymptotic probabilities in the first order condition produces the *asymptotic first order condition* (AFOC), which we are able to solve for a rich family of distributions (i.e., mixtures of normals). Its solution for this family is a unique constant offset that we call the *asymptotic offset*. This formula is notable because there are so few closed-form examples of equilibria in the double auctions literature.[3] Numerical calculations suggest that the asymptotic offset approximates the equilibrium offset quite well even in small markets (e.g., with $m = n$ as small as 8 or 16). More importantly, the asymptotic offset identifies the parameters of the model that are of first order in determining a buyer's equilibrium bid. As an explicit formula in m, n and the distribution F, it permits a comparative statics analysis in these variables. This is significant for both empirical application and experimental testing of the model.[4]

The paper is organized as follows. Section 2 presents the model. Section 3 discusses the buyer's FOC, the asymptotics of its probabilities, and the asymptotic distribution of the BBDA's equilibrium price in relation to the REE price. Using these results in Section 4 we obtain the AFOC and the asymptotic offset. Section 5 concludes. All proofs are in the Appendices.

2 The Model

Trade in the BBDA is organized as follows. Buyers and sellers simultaneously submit bids/asks, which are ordered in a list[5]

$$s_{(1)} \leq s_{(2)} \leq ... \leq s_{(m+n)}.$$

Assume for the moment that $s_{(m)} \neq s_{(m+1)}$ and let d denote the number of buyers' bids among the top m bids/asks $s_{(m+1)}, ..., s_{(m+n)}$. There are $m - d$ buyers' bids among the m lowest bids/asks $s_{(1)}, ..., s_{(m)}$ and so there are d sellers' asks among the m lowest. Selecting a price $p \in [s_{(m)}, s_{(m+1)}]$

[3] The two examples that we are aware of are limited to the case of independent, uniformly distributed private values/costs: the linear equilibrium of the $k = 1/2$ bilateral double auction of Chatterjee and Samuelson (1983); the linear equilibrium of the multilateral BBDA of Satterthwaite and Williams (1989) and Williams (1991).

[4] Experimental tests of the multilateral double auction include Kagel and Vogt (1993) and Cason and Friedman (1997).

[5] We use $s_{(t)}$ throughout the paper to denote the t^{th} smallest in a specified sample of bids/asks. We use $s_{t:r}$ for the t^{th} smallest element when we need to specify the number r of bids/asks in the sample.

therefore equates supply and demand, i.e., the number d of buyers' bids at or above p equals the number d of asks below p.[6] In the case of $s_{(m)} = s_{(m+1)}$, allocate trades on the long side of the market by assigning priority first to the larger bids/smaller asks and then using a fair lottery in the case of ties.

The BBDA selects $s_{(m+1)}$ as the market price. Because a seller only sells if his ask is below this price, his ask cannot influence the price at which he trades. It is straightforward to show that setting his ask equal to his true cost is a weakly dominant strategy for each seller. Tying down the strategic behavior of one side of the market is an attractive feature of the BBDA as a theoretical model. We assume the use of this strategy by each seller in the BBDA for the rest of the paper. A buyer, however, sets the price at which he trades if his bid equals $s_{(m+1)}$. He therefore has an incentive to bid less than his value.

3 First Order Condition

We discuss in this section a buyer's FOC for the optimal selection of his bid, as derived in SWZ (Sec. 3.1). A goal is to highlight the complexity of dealing with the probabilities involved in a buyer's decision problem and thus to motivate the asymptotic analysis.

Pick a focal buyer. Assuming that the n sellers submit their costs as their asks and the $m-1$ nonfocal buyers use strategy B, order their bids/asks into a vector: $s_{(1)} \leq s_{(2)} \leq \cdots \leq s_{(m)} \leq s_{(m+1)} \leq \cdots \leq s_{(m+n-1)}$. This is the random vector against which the focal buyer, as a function of his value v, chooses his bid b to maximize his expected utility let $\pi(v,b)$. Let x denote $s_{(m)}$ and let y denote $s_{(m+1)}$ in the bid/ask vector. Also, let $f_{x|v}(\cdot|v)$ denote the density of x conditional on v. The focal buyer's marginal expected utility is

$$\frac{\partial \pi}{\partial b}(v,b) = (v-b) f_{x|v}(b|v) - \Pr[x \leq b \leq y|v]. \tag{1}$$

The first term on the right times Δb is the focal buyer's expected gain from increasing his bid by Δb: he passes x with probability $f_{x|v}(b|v)\Delta b$, goes from not trading to trading and earns $v - b - \Delta b$, which in the limit as $\Delta b \to 0$ gives him a marginal expected gain of $(v-b)f_{x|v}(b|v)$. The second term is the expected cost of increasing his bid by Δb: with probability $\Pr[x \leq b \leq y|v]$ his bid b sets the price and increasing b by Δb forces the market clearing price he pays up by Δb. Setting the marginal expected utility equal to zero yields the FOC for b given v:

$$(v-b) f_{x|v}(b|v) - \Pr[x \leq b \leq y|v] \quad = \quad 0. \tag{2}$$

Taking advantage of the conditional independence in the state μ of values/costs, this can be written

[6] A minor change in the allocation rule is required in order to clear the market when the price p is selected as $s_{(m)}$: sellers whose asks are at or below p trade with buyers whose bids are strictly more than p.

as

$$(v-b)\underbrace{\int_{-\infty}^{\infty} f_{x|\mu}(b|\mu) f(v-\mu)\, d\mu}_{f_{x|v}(b|v)} - \underbrace{\int_{-\infty}^{\infty} \Pr[x \le b \le y|\mu] f(v-\mu)\, d\mu}_{\Pr[x\le b\le y|v]} = 0. \tag{3}$$

Here, $f(v-\mu) = f_{\mu|v}(\mu|v)$, the density of state μ conditional on the buyer's value v, which follows from the method of generating values/costs; $f_{x|\mu}(\cdot|\mu)$ denotes the density of x and $\Pr[x \le b \le y|\mu]$ denotes the probability that the focal buyer sets the price, both conditional on μ.

The complexity of the first order condition becomes apparent when $f_{x|\mu}(b|\mu)$ and $\Pr[x \le b \le y|\mu]$ are expressed directly in terms of m, n and F. We illustrate this point by expanding $\Pr[x \le b \le y|\mu]$. Let $v(b)$ denote the inverse strategy $B^{-1}(b)$ for $b \in \mathbb{R}$. Given the state μ, the probability that m of the nonfocal traders bid/ask below b and $n-1$ of them bid/ask above b is

$$\Pr[x \le b \le y|\mu] = \sum_{\substack{i+j=m \\ 0\le i\le m-1 \\ 0\le j\le n}} \binom{m-1}{i}\binom{n}{j} F(v(b)-\mu)^i F(b-\mu)^j\, \overline{F}(v(b)-\mu)^{m-1-i}\, \overline{F}(b-\mu)^{n-j}.$$

Here, i indexes the buyers and j the sellers who bid/ask below b as we sum over the different ways in which a total of exactly $i+j=m$ nonfocal traders bid/ask below b. Conditional on the state μ, a nonfocal buyer bids below b with probability $F(v(b)-\mu)$ and a seller asks below b with probability $F(b-\mu)$. Using the notation $\overline{F} \equiv 1-F$, the formula is then completed by insuring that the other $n-1$ traders bid/ask above b. The formula for the density $f_{x|\mu}(b|\mu)$ of x also sums over the multiple events that result in b equaling the order statistic x, with the added complexity that it involves the derivative $\dot{v}(b)$ of the inverse strategy in representing the probability density of a bid b. Finally, to obtain $\Pr[x \le b \le y|v]$ and $f_{x|v}(b|v)$, we have to integrate over μ as shown in (3), which adds another layer of difficulty.

3.1 Asymptotic Results

For fixed numbers m of buyers and n of sellers, we consider in the remainder of the paper a sequence of markets with ηm buyers and ηn sellers for $\eta \in \mathbb{N}$. Let $B(\cdot;\eta)$ denote a strategy for buyers in the market of size η. The main result of this section identifies the asymptotic distributions of (i) the two order statistics that are relevant in a buyer's decision problem and (ii) the equilibrium price in the BBDA. All three statistics are shown to be consistent, asymptotically unbiased and normal estimators of the REE price for the limit market in each state μ. We begin by restating SWZ (Thms. 4 and 5) for the purposes of this paper.

Theorem 1 (Buyers' Misrepresentation) *If the distribution F satisfies $\lim_{v\to-\infty} \sup F(v)/f(v) < \infty$, then the buyers' equilibrium offset $v - B(v;\eta)$ is $O(1/\eta)$.*

Let $q \equiv m/(m+n)$, the relative size of demand in the market, and $\xi_q \equiv F^{-1}(q)$, the q^{th} quantile of distribution F. The *limit market* in state μ consists of probability masses of measure q

of buyers and measure $1-q$ of sellers with values/costs z, which conditional on μ are i.i.d. according to $F(z-\mu)$. The following theorem establishes the REE price, the BBDA price, and equilibrium behavior in the limit market.

Theorem 2 (Limit Market)

(i) The unique REE price in the limit market in state μ is $p^{\text{REE}} \equiv \mu + \xi_q$.

(ii) All traders report honestly their values/costs in the BBDA in the limit market. This results in the equilibrium price in state μ equaling $\mu + \xi_q$, i.e., the equilibrium price is the REE price.

We next turn to the three order statistics that are of interest in this section. The first is

$$x(\eta) \equiv s_{\eta m:\eta(m+n)-1}, \tag{4}$$

the $(\eta m)^{\text{th}}$ order statistic in the sample of $\eta m - 1$ buyers playing $B(\cdot;\eta)$ and ηn sellers asking truthfully. This is the bid/ask that a focal buyer must outbid in order to trade. The second is

$$y(\eta) \equiv s_{\eta m+1:\eta(m+n)-1}, \tag{5}$$

the $(\eta m+1)^{\text{st}}$ order statistic in the sample of $\eta m - 1$ buyers playing $B(\cdot;\eta)$ and ηn sellers asking truthfully. The focal buyer sets the price at which he trades when his bid falls between $x(\eta)$ and $y(\eta)$. The third is

$$p^{\text{BBDA}}(\eta) \equiv s_{\eta m+1:\eta(m+n)}, \tag{6}$$

the $(\eta m+1)^{\text{st}}$ order statistic in the sample of all buyers playing $B(\cdot;\eta)$ and all sellers asking truthfully. When all buyers use the strategy $B(\cdot;\eta)$, $p^{\text{BBDA}}(\eta)$ is the price in the BBDA in the market of size η.

Theorem 3 states the asymptotic distributions of $x(\eta)$, $y(\eta)$, and $p^{\text{BBDA}}(\eta)$. It is notable that the result is derived using only the assumption that buyers use a strategy $B(\cdot;\eta)$ in the market of size η for which $v - B(v;\eta)$ is $O(1/\eta)$. As stated in Theorem 1, this is in fact true of an equilibrium offset strategy; the point here is that the same asymptotic distributions apply for all strategies $B(\cdot;\eta)$ for which $v - B(v;\eta)$ is $O(1/\eta)$ including those that do not define equilibrium. As we turn to the focal buyer's decision problem in section 4, this implies a robustness to a buyer's choice of his optimal bid, namely, robustness to his knowledge of or hypothesis concerning the strategy of other buyers.[7] It also implies a robustness to nonequilibrium bidding by buyers in the convergence of the market price $p^{\text{BBDA}}(\eta)$ to p^{REE} in each state μ. As explained after the statement of the theorem, this is true because the magnitude of underbidding by the other buyers vanishes sufficiently fast that it is inconsequential in the focal buyer's decision problem for large η: the asymptotic distributions of $x(\eta)$, $y(\eta)$, and $p^{\text{BBDA}}(\eta)$ are the same regardless of whether the sample is equilibrium or nonequilibrium bids/asks.

[7] Theorem 3 in fact does not even require that buyers use the same strategy $B(\cdot;\eta)$. We limit the result to the symmetric case to avoid the notational complexity of dealing with multiple strategies.

Theorem 3 (Asymptotic Distributions) *Assume the use of a strategy B by buyers for which $v - B(v;\eta)$ is $O(1/\eta)$ for all $v \in \mathbb{R}$. In each state $\mu \in \mathbb{R}$ and for its corresponding REE price $p^{\mathrm{REE}} = \mu + \xi_q$ we have*

$$(i) \quad x(\eta), y(\eta) \sim \mathcal{AN}\left(p^{\mathrm{REE}}, \frac{m\,n/(m+n)^2}{[\eta(m+n)-1]\,f^2(\xi_q)}\right),$$
$$(ii) \quad p^{\mathrm{BBDA}}(\eta) \sim \mathcal{AN}\left(p^{\mathrm{REE}}, \frac{m\,n/(m+n)^2}{\eta(m+n)\,f^2(\xi_q)}\right),$$

i.e., $x(\eta)$, $y(\eta)$ and $p^{\mathrm{BBDA}}(\eta)$ are consistent, asymptotically unbiased and normal estimators of the REE price in state μ.

The proof is in Appendix A. To better appreciate this result it is useful to look also at the asymptotic distributions of $z_{\eta m:\eta(m+n)-1}$, $z_{\eta m+1:\eta(m+n)-1}$, and $z_{\eta m+1:\eta(m+n)}$, which are the analogs of the order statistics in Theorem 3 computed in the hypothetical case in which buyers bid their values and sellers ask their costs. From Serfling (1980, Thm. 2.3.3 A and Thm.-Cor. 2.5.2) we have

$$(i) \quad z_{\eta m:\eta(m+n)-1}, z_{\eta m+1:\eta(m+n)-1} \sim \mathcal{AN}\left(p^{\mathrm{REE}}, \frac{m\,n/(m+n)^2}{[\eta(m+n)-1]\,f^2(\xi_q)}\right),$$
$$(ii) \quad z_{\eta m+1:\eta(m+n)} \sim \mathcal{AN}\left(p^{\mathrm{REE}}, \frac{m\,n/(m+n)^2}{\eta(m+n)\,f^2(\xi_q)}\right).$$

The asymptotic distributions of the strategically determined $x(\eta)$, $y(\eta)$ and $p^{\mathrm{BBDA}}(\eta)$ are thus exactly the same as their counterparts in which all traders act as price takers. The intuition is as follows. From SWZ (Thm. 6) we know that the expected absolute error $\mathbb{E}\left[\left|z_{\eta m:\eta(m+n)-1} - p^{\mathrm{REE}}\right| |\mu\right]$ is $\Theta(1/\sqrt{\eta})$, driven by sampling error.[8] Because the offset $v - B(v)$ is $O(1/\eta)$, strategic error is relatively inconsequential compared to sampling error for large markets. Its impact thus disappears completely in the asymptotic analysis.[9] The challenge and the contribution of the proof of Theorem 3 lies in its extension of known results concerning the asymptotic distribution of order statistics of i.i.d. random variables to the case in which the variables (i.e., bids/asks given μ) are independent but not identically distributed.

The REE price in state μ is $\mu + \xi_q$, and so any trader or market observer who knows the distribution F can estimate the state μ from the BBDA's realized price as $p^{\mathrm{BBDA}}(\eta) - \xi_q$. This is a particularly salient issue for traders in a case that is excluded here by our private values assumption: if traders instead observe noisy signals concerning their values/costs, and if their values/costs may vary with the state μ, then estimating μ allows a trader to estimate the gains that he receives by trading. This is an *interdependent values* model, and it has proven difficult to formally analyze. Numerical work in SWZ (Sec. 5.3.3), however, shows that the above theorem concerning the market

[8] For functions $u_1(\eta)$, $u_2(\eta) : \mathbb{N} \to \mathbb{R}_+$, $u_1(\eta) = \Theta(u_2(\eta))$ means that there exist constants $k_1, k_2 \in \mathbb{R}_+$ and $\eta_0 \in \mathbb{N}$ such that $k_1 u_2(\eta) < u_1(\eta) < k_2 u_2(\eta)$ for all $\eta > \eta_0$.

[9] The accuracy of the BBDA's price as an estimate of p^{REE} is illustrated numerically for markets with 2, 4, 8 and 16 traders on each side in Panel B of Table 4 of SWZ.

price $p^{\mathrm{BBDA}}(\eta)$ as an estimator of the p^{REE} extends to the interdependent values case.

4 An Asymptotic Analysis of a Buyer's Decision Problem

The first order condition (3) is difficult to use because of the complexity of the density $f_{x(\eta)|\mu}(b|\mu)$ and the probability $\Pr[x(\eta) \leq b \leq y(\eta)|\mu]$. Applying Theorem 3, we now simplify this equation by identifying the asymptotic values of $f_{x(\eta)|\mu}(b|\mu)$ and $\Pr[x(\eta) \leq b \leq y(\eta)|\mu]$. This produces the AFOC, which we then solve for the asymptotic offset in the case of a particular family of distributions.

The formal asymptotic analysis appears in the proof of the theorem below, which is deferred to Appendix B. For notational simplicity in this section, we omit the dependence of $x(\eta)$ and $y(\eta)$ on the market size η. Let $w \equiv y - x$. Rewrite the focal buyer's first order condition (3) as

$$(v-b)\int_{-\infty}^{\infty} f_{x|\mu}(b|\mu)f(v-\mu)d\mu - \int_{-\infty}^{\infty} \Pr[0 < b - x < w|\mu]f(v-\mu)d\mu \;=\; 0, \tag{7}$$

where we express the event of the focal buyer setting the price in terms of the difference w instead of y. The asymptotic distributions of x and w are now substituted into this formula: (i) Siddiqui (1960) shows that a suitable rescaling of w is asymptotically exponential; (ii) Theorem 3 (i) states that x conditional on μ is asymptotically normal. The AFOC is then

$$\begin{aligned}&(v-b)\int_{-\infty}^{\infty} \tilde{f}_{x|\mu}(b|\mu)f(v-\mu)d\mu - \frac{2\Delta}{\eta}\int_{-\infty}^{\infty} \frac{1}{1-\frac{2\Delta}{\Lambda}\alpha}\,\tilde{f}_{x|\mu}(\alpha+\xi_q+\mu|\mu)\,f(v-b+\xi_q+\alpha)d\alpha \\ &-O\left(\frac{1}{\eta^2}\right)\int_{-\infty}^{\infty} \tilde{f}_{x|\mu}(b|\mu)f(v-\mu)d\mu = 0,\end{aligned} \tag{8}$$

where $\tilde{f}_{x|\mu}$ denotes the asymptotic density of the $(\eta m)^{\text{th}}$ order statistic x conditional on μ. Here, Δ, Λ are constants that depend on m, n, η and ξ_q whose formulas are given by (32) and (29) in Appendix B. Comparing the left sides of (8) and (7), the first term in (8) corresponds to the first term in (7), while the second and third terms in (8) correspond to the second term in (7).

As discussed in section 3.1, the AFOC (8) is derived under the assumption that nonfocal buyers all use a strategy $B(\cdot;\eta)$ in the market of size η for which $v - B(v;\eta)$ is $O(1/\eta)$ for all $v \in \mathbb{R}$. We next derive a formula for the asymptotic offset that solves (8) by making two additional assumptions. First, to compute the integral in closed form we restrict attention to densities f that are mixtures of normals, i.e., $f(t) = \sum_{k=1}^{K} w_k \phi\left(t; m_k, \sigma_k^2\right)$, $t \in \mathbb{R}$, with $w_k > 0$, $\sum_{k=1}^{K} w_k = 1$ and $\phi_k(t) \equiv \phi\left(t; m_k, \sigma_k^2\right)$, the density of a $\mathcal{N}\left(m_k, \sigma_k^2\right)$ random variable. We denote this as $\varepsilon \sim \mathcal{MN}\left(\{w_k, m_k, \sigma_k^2\}_{k=1}^{K}\right)$.[10] Mixtures of normals approximate arbitrarily closely any continuous density in a variety of different norms (McLachlan and Peel (2000)). They also capture the idea of model uncertainty, i.e., a trader may assign probabilities to different specifications of f.

[10]While our derivation of the asymptotic offset does not require symmetry of f, it is used elsewhere in our paper and in SWZ. A necessary condition for a mixture of normals to be symmetric is that $\sum_{k=1}^{K} w_k m_k = 0$. Two sufficient conditions that we use in section 4.1 are (i) $m_k = 0$ for all k and (ii) for K even, $w_{k+1} = w_{K-k}$, $m_{k+1} = -m_{K-k}$, and $\sigma_{k+1} = \sigma_{K-k}$ for $k \in \{0, \ldots, K/2-1\}$.

Second, in the focal buyer's search for his approximately optimal bid, we assume that he restricts attention to strategies $\widetilde{B}(\cdot;\eta)$ in each size of market η such that $v-\widetilde{B}(v;\eta)$ is $O(1/\eta^{\epsilon})$ for *some* choice of $\epsilon>0$. Formally, this assumption allows us to reduce the integral in (8) using a Taylor series approximation; we need some restriction on the strategies to start the proof. It is in fact true in equilibrium; the point in considering $\epsilon<1$ is to restrict as little as possible ex ante the range of strategies that the focal buyer considers in deriving his approximately optimal bid. In other words, we will deduce that his approximately optimal response is to underbid by an amount that is $O(1/\eta)$ and not assume it. Note also that the strategy $\widetilde{B}(\cdot;\eta)$ need not equal $B(\cdot;\eta)$; all that the focal buyer needs to know about the nonfocal buyers' strategy is that $v-B(v;\eta)$ is $O(1/\eta)$. This implies a robustness of the approximate optimality of his asymptotic offset.

These additional assumptions allow the reduction of the AFOC to

$$(v-b)\tilde{f}_{x|v}(b|v)-\left[\frac{1}{(m+n)\eta-1}\frac{1}{f(\xi_q)}+O\left(\frac{1}{\eta^2}\right)\right]\tilde{f}_{x|v}(b|v)=0, \tag{9}$$

where $\tilde{f}_{x|v}$ is the asymptotic density of the order statistic x conditional on the value v,

$$\tilde{f}_{x|v}(b|v) \quad = \quad \int_{-\infty}^{\infty}\tilde{f}_{x|\mu}(b|\mu)f(v-\mu)d\mu.$$

Since $\tilde{f}_{x|v}(b|v)>0$, the unique solution of (9) is the asymptotic offset

$$\tilde{\lambda}(\eta)\equiv v-b=\frac{1}{(m+n)\eta-1}\frac{1}{f(\xi_q)}+O\left(\frac{1}{\eta^2}\right). \tag{10}$$

This is a constant that is independent of v. An offset strategy thus uniquely solves the buyer's asymptotic first order condition, which mirrors the existence and uniqueness of the offset equilibrium in finite markets that is established numerically in SWZ (Sec. 4).

Theorem 4 (Approximate Solution) *A focal buyer faces ηn sellers asking honestly and $\eta m-1$ buyers employing a strategy $B(\cdot;\eta)$ such that $v-B(v;\eta)$ is $O(1/\eta)$ for all $v\in\mathbb{R}$. Suppose that F is the cumulative distribution for $\mathcal{MN}\left(\{w_k,m_k,\sigma_k^2\}_{k=1}^K\right)$, a mixture of normals, and that the focal buyer considers strategies $\widetilde{B}(\cdot;\eta)$ in which his underbidding is $O(1/\eta^{\epsilon})$ for some $\epsilon>0$. Then the unique strategy that solves the focal buyer's asymptotic first order condition (9) is the offset $\tilde{\lambda}(\eta)$ given in (10). Dispensing with the higher order terms, the asymptotic offset $\tilde{\lambda}(\eta)$ is further approximated by*

$$\lambda_{\text{approx}}(\eta)=\frac{1}{(m+n)\eta-1}\frac{1}{f(\xi_q)}, \tag{11}$$

which approximates $\tilde{\lambda}(\eta)$ in the sense that $\lim_{\eta\to\infty}\tilde{\lambda}(\eta)/\lambda_{\text{approx}}(\eta)=1$.

Notably, the approximation (11) depends on the density f only at a single point and does not require exact knowledge of the bidding strategy of the other buyers for its derivation. Nearly optimal bidding by a buyer thus requires considerably less than common knowledge of strategies and beliefs, as required by Bayesian-Nash equilibrium. Equilibrium bidding in the BBDA has been shown to produce a relatively high level of efficiency in the BBDA's allocation.[11] Our observation here thus implies that the strong performance of the BBDA may be more robust in trader bidding behavior than the equilibrium analysis suggests.

Formula (11) facilitates the comparative statics exercise of exploring the dependence of equilibrium upon the distribution F, which is new to the double auctions literature. The offset $\lambda_{\text{approx}}(\eta)$ depends upon f only through its value at the quantile ξ_q. As stated in Theorem 3, this reflects the fact that in each state μ the statistics of interest in a buyer's decision problem are concentrated in a large market at $\mu + \xi_q$, the REE price in this state. Moreover, $\lambda_{\text{approx}}(\eta)$ increases as $f(\xi_q)$ decreases, which suggests that equilibrium buyer underbidding should also vary with $f(\xi_q)$ in this way: if bids/asks are less concentrated at $\mu + \xi_q$, then the focal buyer is more able to influence price in his favor and consequently underbids by more. This prediction is not at all apparent from inspection of the buyer's FOC (3).

It is also notable that $\lambda_{\text{approx}}(\eta)$ depends upon the relative sizes of m and n only through their determination of q and ξ_q; otherwise, the dependence on m and n is limited to the sum $m+n$. Equilibrium bidding depends separately on each variable m and n because the distribution of a buyer's bid is different from the distribution of a seller's ask. This asymmetry, however, becomes inconsequential asymptotically as buyers' bids converge to their true values and all nonfocal traders behave increasingly the same. The asymmetry of behavior thus matters less and less as $\eta \to \infty$ and all that does matter is the sum $m+n$ in its role in determining the total number $(m+n)\eta - 1$ of nonfocal traders. The value of q continues to matter to a buyer as the market increases in size because it determines where the price is selected within the distribution of bids/asks. These observations suggests a second comparative statics exercise. Consider for fixed $k_1, k_2 \in \mathbb{N}$ markets with (i) ηk_1 buyers, ηk_2 sellers and (ii) ηk_2 buyers, ηk_1 sellers. Because of the assumption that f is symmetric about 0, it follows that $f(\xi_q) = f(\xi_{1-q})$ where $q = k_1/(k_1+k_2)$ and ξ_q, ξ_{1-q} are respectively the quantiles of interest in markets (i) and (ii). Consequently, the value of $\lambda_{\text{approx}}(\eta)$ is exactly the same across the two markets, and so equilibrium bidding behavior by buyers should be approximately the same across them.

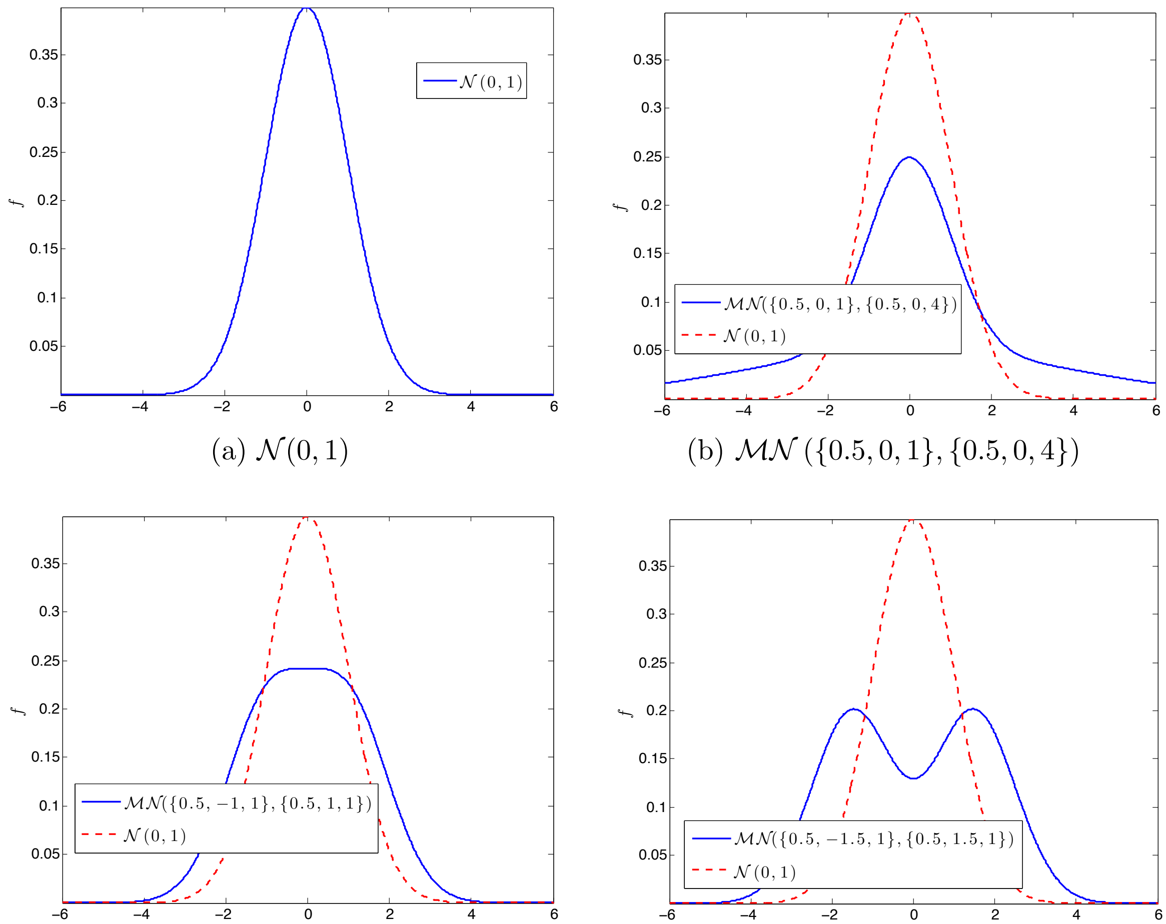
4.1 Numerical Example

Numerical calculations presented below suggest that $\lambda_{\text{approx}}(\eta)$ as given by (11) approximates the equilibrium offset quite well even in small markets and for a rich set of distributions, namely, mixtures of normals. The two comparative statics predictions that are made above are also illustrated by the calculations that follow below.

We begin by considering the four mixtures of normals depicted in Figure 1. In (a) we have the standard normal $\mathcal{N}(0,1)$, in (b) an equal mixture of two normals $\mathcal{MN}(\{0.5,0,1\}, \{0.5,0,4\})$ centered at zero but with different variances, in (c) an equal mixture of two normals $\mathcal{MN}(\{0.5,-1,1\}, \{0.5,1,1\})$ centered at -1 and 1 with equal variances, and in (d) an equal mixture of two normals

[11] See, for instance, Rustichini, Satterthwaite, and Williams (1994, Thm. 3.2), Satterthwaite and Williams (2002, Thm. 2) and SWZ (Thm. 4 and Result 2).

$\mathcal{MN}(\{0.5, -1.5, 1\}\ ,\ \{0.5, 1.5, 1\})$ centered at -1.5 and 1.5 with equal variances, which produces a bimodal distribution.



(a) $\mathcal{N}(0,1)$ (b) $\mathcal{MN}(\{0.5,0,1\},\{0.5,0,4\})$

(c) $\mathcal{MN}(\{0.5,-1,1\},\{0.5,1,1\})$ (d) $\mathcal{MN}(\{0.5,-1.5,1\},\{0.5,1.5,1\})$

Figure 1: The densities f of the mixture of normals that we use for our numerical illustration.

Table 1 concerns the case of $m = n = 1$ and $\eta = 2, 4, 8, 16$. Its four panels A–D correspond to the four distributions (a)–(d) in Figure 1. Column 1 in each panel lists the market size η, column 2 lists the numerically calculated equilibrium offset λ, column 3 lists the approximate solution λ_{approx}, column 4 lists the absolute error in the approximation of equilibrium, and column 5 lists the absolute error as a fraction of the equilibrium offset. The equilibrium offset is computed by solving the FOC (2), with sufficiency then verified by graphing marginal expected utility; see SWZ (Secs. 4.2.1 and 4.3) for further information on the calculation of equilibrium.

Beginning with Panel A in Table 1 we see that the relative error in the approximation diminishes quickly so that the approximate solution performs well even for modest market sizes. This is true also in Panels B and C. The most challenging case is the bimodal distribution in Panel D. The worse performance of the approximate formula is to be expected because for bimodal distributions a larger sample size is required to achieve the same accuracy in convergence (in the sense of the central limit theorem) in comparison to unimodal distributions. Nonetheless, the approximate formula is already quite accurate for 16 traders on each side of the market even in this case. We have also computed these values for the four distributions in the asymmetric case of $m = 1, n = 2$ and obtained qualitatively similar results.

Because $m = n = 1$ in Table 1, we have $q = 1/2$ for each of the four distributions and $\xi_q = 0$. It is clear from the captions to the four panels that the density $f(\xi_q)$ decreases as one moves from distribution (a) to (b) to (c) to (d). As noted above, the approximate solution λ_{approx} therefore increases for each size of market as the distributions change in this way; moreover, the equilibrium offset also increases as the distribution changes, even in the case of $\eta = 2$. The comparative statics prediction that is suggested by the asymptotically derived, approximate formula (11) thus holds in equilibrium for even the smallest of markets.

Table 2 addresses our second comparative statics prediction in the case of the standard normal distribution. We consider both the cases of $m = 1$, $n = 2$ and $m = 2$, $n = 1$, with η again equaling $\eta = 2, 4, 8,$ and 16. It is observed above that buyer underbidding should be approximately the same in these two sequences once η is sufficiently large. Column 1 of the table again lists η, column 2 lists the equilibrium offset $\lambda_{1,2}$ for the sequence of markets with $m = 1$ and $n = 2$, and column 3 lists the equilibrium offset $\lambda_{2,1}$ for the sequence of markets with $m = 2$ and $n = 1$. Notice that the two lists of offsets are very close to one another even in the smallest of markets; they are in fact identical to our level of computational accuracy for $\eta = 16$. Column 4 lists the absolute difference $|\lambda_{1,2} - \lambda_{2,1}|$ of these equilibrium offsets. Column 5 lists this absolute difference as a fraction of the offset $\lambda_{1,2}$, which is done to provide some sense of the scale of the absolute error. Even in the case of $\eta = 2$, this relative error is barely larger than 1%.

5 Conclusion

We analyze the equilibrium price and a buyer's decision problem in the buyer's bid double auction from an asymptotic perspective. The asymptotic distribution of the equilibrium price is determined. It reveals that this price is a consistent, asymptotically unbiased and normal estimator of the REE price. The REE price is thus approximately implemented in a finite market by a strategically determined, market-clearing price. The asymptotic first order condition is determined by identifying the asymptotic probabilities in a buyer's first order condition. For a rich family of distributions, the solution to this equation is a simple formula in the fundamentals of our model that determines a unique offset strategy. Numerical investigation suggests that this strategy closely approximates the equilibrium bidding strategy even in small markets. This simple formula identifies the first order considerations in a buyer's selection of his bid and thereby allows a comparative statics analysis of equilibrium.

Panel A: $\mathcal{N}(0,1)$, $\xi_q = 0$, $f(\xi_q) = 0.3989$.

η	λ	λ_{approx}	$\lvert\lambda_{\text{approx}} - \lambda\rvert$	$\frac{\lvert\lambda_{\text{approx}}-\lambda\rvert}{\lambda}$
2	0.6896	0.8355	0.1459	0.2116
4	0.3398	0.3581	0.0183	0.0539
8	0.1639	0.1671	0.0031	0.0195
16	0.0805	0.0809	0.0004	0.0050

Panel B: $\mathcal{MN}(\{0.5,0,1\},\{0.5,0,4\})$, $\xi_q = 0$, $f(\xi_q) = 0.2992$.

η	λ	λ_{approx}	$\lvert\lambda_{\text{approx}} - \lambda\rvert$	$\frac{\lvert\lambda_{\text{approx}}-\lambda\rvert}{\lambda}$
2	0.9304	1.1141	0.1837	0.1974
4	0.4617	0.4775	0.0158	0.0342
8	0.2215	0.2228	0.0065	0.0293
16	0.1077	0.1078	0.0001	0.0009

Panel C: $\mathcal{MN}(\{0.5,-1,1\},\{0.5,1,1\})$, $\xi_q = 0$, $f(\xi_q) = 0.2420$.

η	λ	λ_{approx}	$\lvert\lambda_{\text{approx}} - \lambda\rvert$	$\frac{\lvert\lambda_{\text{approx}}-\lambda\rvert}{\lambda}$
2	1.0468	1.3776	0.3308	0.3160
4	0.5305	0.5904	0.0599	0.1129
8	0.2610	0.2755	0.0145	0.0556
16	0.1296	0.1333	0.0037	0.0285

Panel D: $\mathcal{MN}(\{0.5,-1.5,1\},\{0.5,1.5,1\})$, $\xi_q = 0$, $f(\xi_q) = 0.1295$.

η	λ	λ_{approx}	$\lvert\lambda_{\text{approx}} - \lambda\rvert$	$\frac{\lvert\lambda_{\text{approx}}-\lambda\rvert}{\lambda}$
2	1.4650	2.5737	1.1087	0.7568
4	0.7626	1.1030	0.3404	0.4464
8	0.3948	0.5147	0.1199	0.3037
16	0.2084	0.2491	0.0407	0.1953

Table 1: For different market sizes η, the equilibrium offset λ is compared to its approximation λ_{approx} in the case of $m = 1$ buyer and $n = 1$ seller and the four distributions depicted in Figure 1.

η	$\lambda_{1,2}$	$\lambda_{2,1}$	$\lvert\lambda_{1,2} - \lambda_{2,1}\rvert$	$\frac{\lvert\lambda_{1,2}-\lambda_{2,1}\rvert}{\lambda_{1,2}}$
2	0.5027	0.5085	0.0058	0.0115
4	0.2433	0.2441	0.0008	0.0033
8	0.1184	0.1185	0.0001	0.0008
16	0.0583	0.0583	0	0

Table 2: For different market sizes η and F standard normal, the equilibrium offset $\lambda_{1,2}$ for the case of $m = 1$ buyer, $n = 2$ sellers is compared to the equilibrium offset $\lambda_{2,1}$ for the case of $m = 2$ buyers, $n = 1$ seller.

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A Asymptotic Distributions of Order Statistics

As in Serfling (1980, Secs. 2.3–2.5) we first prove in Lemmas 1 and 2 the asymptotic normality of the q^{th} quantile in the corresponding sample and then state the results in terms of order statistics in Theorem 3. Before we state the lemmas let us introduce some notation. Let $\widetilde{F}_{\eta(m+n)-1}$ denote the empirical distribution of the sample of bids/asks of the $\eta(m+n)-1$ nonfocal traders,

$$\widetilde{F}_{\eta(m+n)-1}(t) \equiv \frac{1}{\eta(m+n)-1}\sum_{i=1}^{\eta m-1}\mathbb{I}\{b_i \le t\} + \frac{1}{\eta(m+n)-1}\sum_{j=1}^{\eta n}\mathbb{I}\{c_j \le t\}. \tag{12}$$

We also define the q^{th} quantile from this population,

$$\tilde{\xi}_{q[\eta(m+n)-1]} \equiv \inf\{t : \widetilde{F}_{\eta(m+n)-1}(t) \le q\}.$$

Finally, recall that $\xi_q = F^{-1}(q)$. The following result establishes the asymptotic relationship between $\tilde{\xi}_{q[\eta(m+n)-1]}$ and ξ_q conditional on μ.

Lemma 1 *Assume that nonfocal buyers use a strategy $B(\cdot;\eta)$ such that $v - B(v;\eta)$ is $O(1/\eta)$ for all $v \in \mathbb{R}$. Then for all $t \in \mathbb{R}$ and $0 < q < 1$,*

$$\lim_{\eta\to\infty} \Pr\left(\frac{\sqrt{\eta(m+n)-1}\left(\tilde{\xi}_{q[\eta(m+n)-1]} - \mu - \xi_q\right)}{\sqrt{q(1-q)}/f\left(\xi_q\right)} \le t\right) = \Phi(t),$$

where $\Phi(\cdot)$ is the standard normal distribution function. Therefore,

$$\tilde{\xi}_{q[\eta(m+n)-1]} \sim \mathcal{AN}\left(\mu + \xi_q, \frac{q(1-q)}{[\eta(m+n)-1]\, f^2(\xi_q)}\right).$$

The proof follows below after the proof of Theorem 3. Similarly, let $\widetilde{F}_{\eta(m+n)}$ denote the empirical distribution of the entire sample of $\eta(m+n)$ bids/asks,

$$\widetilde{F}_{\eta(m+n)}(t) \equiv \frac{1}{\eta(m+n)}\sum_{i=1}^{\eta m}\mathbb{I}\{b_i \le t\} + \frac{1}{\eta(m+n)}\sum_{j=1}^{\eta n}\mathbb{I}\{c_j \le t\},$$

and let $\tilde{\xi}_{q[\eta(m+n)]}$ denote its q^{th} quantile,

$$\tilde{\xi}_{q[\eta(m+n)]} \equiv \inf\{t : \widetilde{F}_{\eta(m+n)}(t) \le q\}.$$

The following result establishes the asymptotic relationship between $\tilde{\xi}_{q[\eta(m+n)]}$ and ξ_q conditional on μ.

Lemma 2 *Assume that all buyers use a strategy $B(\cdot;\eta)$ such that $v - B(v;\eta)$ is $O(1/\eta)$ for all*

$v \in \mathbb{R}$. *Then for all* $t \in \mathbb{R}$, *and* $0 < q < 1$,

$$\lim_{\eta\to\infty} \Pr\left(\frac{\sqrt{\eta(m+n)}\left(\tilde{\xi}_{q[\eta(m+n)]} - \mu - \xi_q\right)}{\sqrt{q(1-q)}/f\left(\xi_q\right)} \le t\right) = \Phi(t).$$

Therefore,

$$\tilde{\xi}_{q[\eta(m+n)]} \sim \mathcal{AN}\left(\mu + \xi_q, \frac{q(1-q)}{\eta(m+n)\, f^2(\xi_q)}\right).$$

The proof is identical with that of Lemma 1, with the terms $\eta m - 1$, and $\eta(m+n) - 1$ replaced by ηm and $\eta(m+n)$, respectively. We are now ready to prove Theorem 3.

Proof of Theorem 3. Recall the definitions of the order statistics $x(\eta)$, $y(\eta)$, and $p^{\text{BBDA}}(\eta)$ in (4)–(6) along with the samples in which they appear. The ratio between the order of each statistic and the cardinality of its sample size satisfies as $\eta \to \infty$ respectively,

$$\frac{\eta m}{\eta(m+n)-1} = \frac{m}{m+n} + \frac{m}{(m+n)(\eta(m+n)-1)} = \frac{m}{m+n} + o\left(\frac{1}{\eta^{1-\epsilon}}\right), \tag{13}$$

$$\frac{\eta m+1}{\eta(m+n)-1} = \frac{m}{m+n} + \frac{2\,m+n}{(m+n)(\eta(m+n)-1)} = \frac{m}{m+n} + o\left(\frac{1}{\eta^{1-\epsilon}}\right), \tag{14}$$

$$\frac{\eta m+1}{\eta(m+n)} = \frac{m}{m+n} + \frac{1}{\eta(m+n)} = \frac{m}{m+n} + o\left(\frac{1}{\eta^{1-\epsilon}}\right), \tag{15}$$

for any small $\epsilon > 0$.[12] Lemmas 1 and 2 establish the asymptotic distribution of the q^{th} quantile in particular samples. Equations (13)–(15) link the order statistics $x(\eta)$, $y(\eta)$, and $p^{\text{BBDA}}(\eta)$ with the q^{th} quantile. An application of Serfling (1980, Thm. and Cor. 2.5.2) then implies the asymptotic relationships between $x(\eta)$, $y(\eta)$, and $p^{\text{BBDA}}(\eta)$ with ξ_q conditional on μ as stated in the theorem. ■

Proof of Lemma 1. The difference $v(b;\eta) - b$ is $O(1/\eta)$, where $v(b;\eta) \equiv B^{-1}(b;\eta)$ for $b \in \mathbb{R}$ and $\eta \in \mathbb{N}$. This rate implies that $\lim_{\eta\to\infty} \sup_{b\in\mathbb{R}} |v(b;\eta) - b| = 0$ but also that

$$\lim_{\eta\to\infty} \sup_{b\in\mathbb{R}} \{\sqrt{\eta}\,|v(b;\eta) - b|\} = 0. \tag{16}$$

The fact that the buyers' strategy satisfies (16) is the crux of our proof of asymptotic normality. Let $\lambda(b;\eta) \equiv v(b;\eta) - b$, where $\lambda(b;\eta) \ge 0$ for all $b \in \mathbb{R}$ and $\eta \in \mathbb{N}$. Since we do not need to restrict attention to an offset strategy, $\lambda(b;\eta)$ need not be a constant for all b.

Starting from (12), the empirical distribution of the bids/asks of the $\eta(m+n) - 1$ nonfocal

[12] For functions $u_1(\eta)$, $u_2(\eta) : \mathbb{N} \to \mathbb{R}_+$, $u_1(\eta) = o\left(u_2(\eta)\right)$ means that $\lim_{\eta\to\infty} u_1(\eta)/u_2(\eta) = 0$.

traders is for $b \in \mathbb{R}$

$$\widetilde{F}_{\eta(m+n)-1}(b) \quad = \quad \frac{1}{\eta(m+n)-1}\sum_{i=1}^{\eta m-1}\mathbb{I}\{v_i \le b+\lambda(b;\eta)\} + \frac{1}{\eta(m+n)-1}\sum_{j=1}^{\eta n}\mathbb{I}\{c_j \le b\}.$$

This follows from (12) by applying $v(\cdot;\eta)$ to both sides of the inequality inside the first indicator function and then using the definition of $\lambda(b;\eta)$. The q^{th} quantile from this population is denoted as $\tilde{\xi}_{q[\eta(m+n)-1]}$.

In this proof we depart from the notation used in the main text and denote the distribution and density of values/costs conditional on μ as F_μ and f_μ, respectively. Of course, conditional on μ values/costs are i.i.d. with $F_\mu(b) = F(b-\mu)$ and $f_\mu(b) = f(b-\mu)$ for all $(b,\mu) \in \mathbb{R}^2$. As before, $\overline{F}_\mu = 1 - F_\mu$. Also by definition

$$F_\mu(b) = \lim_{\eta\to\infty}\frac{1}{\eta m}\sum_{i=1}^{\eta m}\mathbb{I}\{v_i \le b\}.$$

The q^{th} quantile of distribution F is ξ_q so that $\mu + \xi_q = F_\mu^{-1}(q)$. The proof extends Serfling (1980, Thm. 2.3.3 A) to our model in which bids/asks conditional on μ are independent but not identically distributed.

Let $A > 0$ be a normalizing constant to be specified later. Define

$$\begin{aligned}
G_{\eta(m+n)-1}(t) \quad &\equiv \quad \Pr\left(\frac{\sqrt{\eta(m+n)-1}\left(\tilde{\xi}_{q[\eta(m+n)-1]} - \xi_q - \mu\right)}{A} \le t\right) \\
&= \quad \Pr\left(\tilde{\xi}_{q[\eta(m+n)-1]} \le \mu + \xi_q + t\,A\sqrt{\eta(m+n)-1}^{\,-1}\right) \\
&= \quad \Pr\left(\widetilde{F}_{\eta(m+n)-1}\left(\mu + \xi_q + t\,A\sqrt{\eta(m+n)-1}^{\,-1}\right) \ge q\right),
\end{aligned}$$

where the last line follows from Serfling (1980, Lem. 1.1.4 (iii)). Setting

$$\Delta \quad \equiv \quad \mu + \xi_q + t\,A\sqrt{\eta(m+n)-1}^{\,-1}, \tag{17}$$

$\widetilde{F}_{\eta(m+n)-1}(\Delta)$ is a random variable with mean and variance[13]

$$\mathbb{E}[\widetilde{F}_{\eta(m+n)-1}(\Delta)] \quad = \quad \frac{\eta m-1}{\eta(m+n)-1}F_\mu(\Delta+\lambda(\Delta;\eta)) + \frac{\eta n}{\eta(m+n)-1}F_\mu(\Delta), \tag{18}$$

$$\begin{aligned}
\mathrm{VAR}[\widetilde{F}_{\eta(m+n)-1}(\Delta)] \quad &= \quad \frac{\eta m-1}{(\eta(m+n)-1)^2}F_\mu(\Delta+\lambda(\Delta;\eta))\overline{F}_\mu(\Delta+\lambda(\Delta;\eta)) \\
&\quad + \frac{\eta n}{(\eta(m+n)-1)^2}F_\mu(\Delta)\overline{F}_\mu(\Delta).
\end{aligned} \tag{19}$$

[13] The dependence of Δ on μ, η, and t is suppressed for notational brevity.

Therefore,

$$\begin{aligned}
G_{\eta(m+n)-1}(t) &= \Pr\left(\widetilde{F}_{\eta(m+n)-1}\left(\mu+\xi_q+tA\sqrt{\eta(m+n)-1}^{-1}\right)\geq q\right)\\
&= \Pr\left(\frac{\widetilde{F}_{\eta(m+n)-1}(\Delta)-\mathbb{E}[\widetilde{F}_{\eta(m+n)-1}(\Delta)]}{\sqrt{\mathrm{VAR}[\widetilde{F}_{\eta(m+n)-1}(\Delta)]}}\geq\frac{q-\mathbb{E}[\widetilde{F}_{\eta(m+n)-1}(\Delta)]}{\sqrt{\mathrm{VAR}[\widetilde{F}_{\eta(m+n)-1}(\Delta)]}}\right)\\
&= \Pr\left(\widetilde{F}^*_{\eta(m+n)-1}(\Delta)\geq c(\Delta)\right),
\end{aligned}$$

where

$$\begin{aligned}
\widetilde{F}^*_{\eta(m+n)-1}(\Delta) &\equiv \frac{\widetilde{F}_{\eta(m+n)-1}(\Delta)-\mathbb{E}[\widetilde{F}_{\eta(m+n)-1}(\Delta)]}{\sqrt{\mathrm{VAR}[\widetilde{F}_{\eta(m+n)-1}(\Delta)]}}, \quad \text{and}\\
c(\Delta) &\equiv \frac{q-\mathbb{E}[\widetilde{F}_{\eta(m+n)-1}(\Delta)]}{\sqrt{\mathrm{VAR}[\widetilde{F}_{\eta(m+n)-1}(\Delta)]}}.
\end{aligned} \tag{20}$$

By invoking the Lindeberg-Feller Central Limit Theorem (Serfling (1980, Thm. 1.9.2 A)) for $t=0$ we obtain,

$$\lim_{\eta\to\infty}\Pr[\sqrt{\eta(m+n)-1}(\tilde{\xi}_{q[\eta(m+n)-1]}-\xi_q)\geq 0]=\Phi(0)=\frac{1}{2}.$$

Using the Berry-Esseen Theorem,[14] we have[15]

$$\sup_{x\in\mathbb{R}}\left|\Pr\left(\widetilde{F}^*_{\eta(m+n)-1}(\Delta)\leq x\right)-\Phi(x)\right|\leq K\frac{\beta(\Delta)}{\left[(\eta(m+n)-1)^2\,\mathrm{VAR}[\widetilde{F}_{\eta(m+n)-1}(\Delta)]\right]^3},$$

where K is a universal constant, and

$$\beta(x)\equiv\sum_{i=1}^{\eta m-1}\mathbb{E}\left[|\mathbb{I}\{v_i\leq x+\lambda(x;\eta)\}-F_\mu\left(x+\lambda(x;\eta)\right)|^3\right]+\sum_{j=1}^{\eta n}\mathbb{E}\left[|\mathbb{I}\{c_j\leq x\}-F_\mu(x)|^3\right]. \tag{21}$$

After some algebra we arrive at

$$\begin{aligned}
\mathbb{E}\left[|\mathbb{I}\{v_i\leq x+\lambda(x;\eta)\}-F_\mu\left(x+\lambda(x;\eta)\right)|^3\right] &= R(x;\eta)\,\overline{R}(x;\eta)\left[\overline{R}^2(x;\eta)+R^2(x;\eta)\right],\\
\mathbb{E}\left[|\mathbb{I}\{c_j\leq x\}-F_\mu(x)|^3\right] &= F_\mu(x)\,\overline{F}_\mu(x)\left[\overline{F}^2_\mu(x)+F^2_\mu(x)\right]
\end{aligned}$$

[14] See Serfling (1980, Thm. 1.9.5) for the case of i.i.d. random variables and Batirov, Mavenich, and Nagaev (1977) for the case of independent but not identically distributed random variables, which is relevant here.

[15] For the purposes of this proof, we use x to denote a generic real variable and not a particular order statistic, as it is used elsewhere in the paper. This is done for the sake of aligning our notation with Serfling (1980, Thm. 2.3.3 A) to facilitate the use of this reference.

where

$$\begin{aligned} R(x;\eta) &\equiv F_\mu\left(x+\lambda(x;\eta)\right), \\ \overline{R}(x;\eta) &\equiv \overline{F}_\mu\left(x+\lambda(x;\eta)\right) = 1 - F_\mu\left(x+\lambda(x;\eta)\right). \end{aligned}$$

Therefore

$$\begin{aligned} \left|\Pr\left(\widetilde{F}^*_{\eta(m+n)-1}(\Delta) \geq c(\Delta)\right) - \Phi(t)\right| &\leq K \frac{\beta(\Delta)}{\left[(\eta(m+n)-1)^2 \,\mathrm{VAR}[\widetilde{F}_{\eta(m+n)-1}(\Delta)]\right]^3} \\ &\quad + \left|\Phi(t) - \Phi\left(-c(\Delta)\right)\right|. \end{aligned} \tag{22}$$

We need to show that

$$\lim_{\eta\to\infty} \frac{\beta(\Delta)}{\left[(\eta(m+n)-1)^2 \,\mathrm{VAR}[\widetilde{F}_{\eta(m+n)-1}(\Delta)]\right]^3} = 0. \tag{23}$$

We have

$$\begin{aligned} &\frac{\beta(\Delta)}{\left[(\eta(m+n)-1)^2 \,\mathrm{VAR}[\widetilde{F}_{\eta(m+n)-1}(\Delta)]\right]^3} \\ &= \frac{(\eta m - 1)\, R(\Delta;\eta)\, \overline{R}(\Delta;\eta) \left[\overline{R}^2(\Delta;\eta) + R^2(\Delta;\eta)\right] + \eta n\, F_\mu(\Delta)\, \overline{F}_\mu(\Delta) \left[\overline{F}^2_\mu(\Delta) + F^2_\mu(\Delta)\right]}{\left[(\eta m - 1)\, R(\Delta;\eta)\overline{R}(\Delta;\eta) + \eta n\, F_\mu(\Delta)\overline{F}_\mu(\Delta)\right]^3} \\ &= \frac{(m - 1/\eta)\, R(\Delta;\eta)\, \overline{R}(\Delta;\eta) \left[\overline{R}^2(\Delta;\eta) + R^2(\Delta;\eta)\right] + n\, F_\mu(\Delta)\, \overline{F}_\mu(\Delta) \left[\overline{F}^2_\mu(\Delta) + F^2_\mu(\Delta)\right]}{\left[(\eta^{2/3}\, m - \eta^{-1/3})\, R(\Delta;\eta)\overline{R}(\Delta;\eta) + \eta^{2/3}\, n\, F_\mu(\Delta)\overline{F}_\mu(\Delta)\right]^3}, \end{aligned} \tag{24}$$

where in the second line we substituted for $\beta(\Delta)$ from (21) and $\mathrm{VAR}[\widetilde{F}_{\eta(m+n)-1}(\Delta)]$ from (19). Bearing in mind that

$$\begin{aligned} \lim_{\eta\to\infty} \lambda(\Delta;\eta) &= 0, \\ \lim_{\eta\to\infty} \Delta &= \mu + \xi_q, \\ \lim_{\eta\to\infty} F_\mu(\Delta) &= \lim_{\eta\to\infty} R(\Delta;\eta) = q, \\ \lim_{\eta\to\infty} \overline{F}_\mu(\Delta) &= \lim_{\eta\to\infty} \overline{R}(\Delta;\eta) = 1 - q, \end{aligned}$$

the limit of (24) as $\eta \to \infty$ equals 0, which proves (23). To complete the proof we have to find an appropriate constant A such that

$$\lim_{\eta\to\infty} c(\Delta) = -t, \tag{25}$$

where the dependence of Δ on A can be seen in (17) and $c(\Delta)$ is defined in (20). Substituting for $q = m/(m+n)$ and $\mathbb{E}[\widetilde{F}_{\eta(m+n)-1}(\Delta)]$, $\mathrm{VAR}[\widetilde{F}_{\eta(m+n)-1}(\Delta)]$ from (18)–(19), we rewrite $c(\Delta)$ as

$$\begin{aligned}
c(\Delta) &= \frac{q - \mathbb{E}[\widetilde{F}_{\eta(m+n)-1}(\Delta)]}{\sqrt{\mathrm{VAR}[\widetilde{F}_{\eta(m+n)-1}(\Delta)]}} \\
&= -\frac{\mathbb{E}[\widetilde{F}_{\eta(m+n)-1}(\Delta)] - F_\mu(\mu+\xi_q)}{\sqrt{\mathrm{VAR}[\widetilde{F}_{\eta(m+n)-1}(\Delta)]}} \\
&= -\frac{\frac{\eta m-1}{\eta(m+n)-1} F_\mu(\Delta+\lambda(\Delta;\eta)) + \frac{\eta n}{\eta(m+n)-1} F_\mu(\Delta) - F_\mu(\mu+\xi_q)}{\sqrt{\frac{\eta m-1}{(\eta(m+n)-1)^2} F_\mu(\Delta+\lambda(\Delta;\eta))\overline{F}_\mu(\Delta+\lambda(\Delta;\eta)) + \frac{\eta n}{(\eta(m+n)-1)^2} F_\mu(\Delta)\overline{F}_\mu(\Delta)}} \\
&= -\frac{\Delta+\lambda(\Delta;\eta)-\xi_q-\mu}{\sqrt{\frac{\eta m-1}{(\eta(m+n)-1)^2} F_\mu(\Delta+\lambda(\Delta;\eta))\overline{F}_\mu(\Delta+\lambda(\Delta;\eta)) + \frac{\eta n}{(\eta(m+n)-1)^2} F_\mu(\Delta)\overline{F}_\mu(\Delta)}} \\
&\quad \times \frac{\frac{\eta m-1}{\eta(m+n)-1} F_\mu(\Delta+\lambda(\Delta;\eta)) + \frac{\eta n}{\eta(m+n)-1} F_\mu(\Delta) - F_\mu(\mu+\xi_q)}{\Delta+\lambda(\Delta;\eta)-\xi_q-\mu},
\end{aligned}$$

where in the last equality we multiplied and divided by $\Delta + \lambda(\Delta;\eta) - \xi_q - \mu$. Multiplying and dividing by $(\eta(m+n)-1)/\sqrt{\eta}$ produces

$$\begin{aligned}
c(\Delta) &= -\frac{\frac{\eta(m+n)-1}{\sqrt{\eta}}\,(\Delta+\lambda(\Delta;\eta)-\xi_q-\mu)}{\sqrt{\frac{\eta\, m-1}{\eta} F_\mu(\Delta+\lambda(\Delta;\eta))\overline{F}_\mu(\Delta+\lambda(\Delta;\eta)) + n\, F_\mu(\Delta)\overline{F}_\mu(\Delta)}} \\
&\quad \times \frac{\frac{\eta m-1}{\eta(m+n)-1} F_\mu(\Delta+\lambda(\Delta;\eta)) + \frac{\eta n}{\eta(m+n)-1} F_\mu(\Delta) - F_\mu(\mu+\xi_q)}{\Delta+\lambda(\Delta;\eta)-\xi_q-\mu} \\
&= -\frac{\frac{\eta(m+n)-1}{\sqrt{\eta}}\,(\Delta+\lambda(\Delta;\eta)-\xi_q-\mu)}{\sqrt{\frac{\eta\, m-1}{\eta} F_\mu(\Delta+\lambda(\Delta;\eta))\overline{F}_\mu(\Delta+\lambda(\Delta;\eta)) + n\, F_\mu(\Delta)\overline{F}_\mu(\Delta)}} \\
&\quad \times \left\{ \frac{\eta m-1}{\eta(m+n)-1} \frac{F_\mu(\Delta+\lambda(\Delta;\eta)-\xi_q-\mu+\mu+\xi_q) - F_\mu(\mu+\xi_q)}{\Delta+\lambda(\Delta;\eta)-\xi_q-\mu} \right. \\
&\quad \left. + \frac{\eta n}{\eta(m+n)-1} \frac{F_\mu(\Delta-\xi_q-\mu+\xi_q+\mu) - F_\mu(\mu+\xi_q)}{\Delta-\xi_q-\mu} \times \frac{\Delta-\xi_q-\mu}{\Delta+\lambda(\Delta;\eta)-\xi_q-\mu} \right\},
\end{aligned} \tag{26}$$

where the last equality follows from separating the second term in the first equality into two terms. In the second term we add and subtract $\mu + \xi_q$ from the argument of $F_\mu(\Delta)$ and then multiply and divide the whole term by $\Delta - \xi_q - \mu$. Note that

$$\begin{aligned}
\lim_{\eta\to\infty} \Delta + \lambda(\Delta;\eta) - \xi_q - \mu &= \lim_{\eta\to\infty} t\,A\sqrt{\eta(m+n)-1}^{-1} + \lambda(\Delta;\eta) = 0,\\
\lim_{\eta\to\infty} \Delta - \xi_q - \mu &= \lim_{\eta\to\infty} t\,A\sqrt{\eta(m+n)-1}^{-1} = 0,
\end{aligned}$$

so that

$$\begin{aligned}
\lim_{\eta\to\infty} \frac{\eta\,m-1}{\eta(m+n)-1}\frac{F_\mu(\Delta+\lambda(\Delta;\eta)-\xi_q-\mu+\mu+\xi_q)-F_\mu(\mu+\xi_q)}{\Delta+\lambda(\Delta;\eta)-\xi_q-\mu} &= \frac{m}{m+n}F'(\xi_q) = \frac{m}{m+n}f(\xi_q),\\
\lim_{\eta\to\infty} \frac{\eta n}{\eta(m+n)-1}\frac{F_\mu(\Delta-\xi_q-\mu+\mu+\xi_q)-F_\mu(\mu+\xi_q)}{\Delta-\xi_q-\mu} &= \frac{n}{m+n}F'(\xi_q) = \frac{n}{m+n}f(\xi_q).
\end{aligned}$$

Also,

$$\lim_{\eta\to\infty} \sqrt{\frac{\eta\,m-1}{\eta}F_\mu(\Delta+\lambda(\Delta;\eta))\overline{F}_\mu(\Delta+\lambda(\Delta;\eta)) + n\,F_\mu(\Delta)\overline{F}_\mu(\Delta)} = \sqrt{(m+n)q(1-q)}.$$

Recall that $\lambda(\Delta;\eta)$ is $O(1/\eta)$ and thus satisfies (16). Therefore,

$$\begin{aligned}
\lim_{\eta\to\infty} \frac{\eta(m+n)-1}{\sqrt{\eta}}\left(\Delta+\lambda(\Delta;\eta)-\xi_q-\mu\right) &= \lim_{\eta\to\infty} \frac{\eta(m+n)-1}{\sqrt{\eta}}\left(t\,A\sqrt{\eta(m+n)-1}^{-1} + O(1/\eta)\right)\\
&= \lim_{\eta\to\infty} \left(\sqrt{\frac{\eta(m+n)-1}{\eta}}t\,A + O\left(\frac{1}{\sqrt{\eta}}\right)\right)\\
&= \sqrt{m+n}\,t\,A,
\end{aligned}$$

and

$$\begin{aligned}
\lim_{\eta\to\infty} \frac{\Delta-\xi_q-\mu}{\Delta+\lambda(\Delta;\eta)-\xi_q-\mu} &= \lim_{\eta\to\infty} \frac{t\,A\sqrt{\eta(m+n)-1}^{-1}}{t\,A\sqrt{\eta(m+n)-1}^{-1}+\lambda(\Delta;\eta)}\\
&= \lim_{\eta\to\infty} \frac{t\,A}{t\,A + O\left(\frac{1}{\sqrt{\eta}}\right)}\\
&= 1.
\end{aligned}$$

We now take the limit of $c(\Delta)$, as given by (26), as $\eta \to \infty$ using the above limits to obtain

$$\lim_{\eta\to\infty} c(\Delta) = -\frac{t\,A}{\sqrt{q(1-q)}}f(\xi_q).$$

To produce (25) we choose

$$A = \frac{\sqrt{q(1-q)}}{f(\xi_q)}.$$

Using (23) and (25) we obtain from (22) that

$$\lim_{\eta\to\infty}\left|\Pr\left(\widetilde{F}^{*}_{\eta(m+n)-1}(\Delta)\geq c(\Delta)\right)-\Phi(t)\right|=0,$$

which establishes the result. ■

B Approximate Formula

Proof of Theorem 4. We want to express the probability of the focal buyer setting the price conditional on his value v for a market of size η in terms of the distribution of w and the density of x. Our notational convention for density functions is that, e.g., $f_{xw|v}(x,w|v;\eta)$ denotes the joint density of x and w computed at (x,w) conditional on v for a market size η. Given this we have

$$\begin{aligned}\Pr[x<b<y|v;\eta] &= \Pr[0<b-x<y-x|v;\eta]\\ &= \Pr[0<b-x<w|v;\eta]\\ &= \int_{x=-\infty}^{b}\int_{w=b-x}^{\infty} f_{xw|v}\left(x,w|v;\eta\right)dwdx\\ &= \int_{\mu=-\infty}^{\infty}\int_{x=-\infty}^{b}\int_{w=b-x}^{\infty} f_{xw\mu|v}\left(x,w,\mu|v;\eta\right)dwdxd\mu\\ &= \int_{\mu=-\infty}^{\infty}\int_{x=-\infty}^{b}\int_{w=b-x}^{\infty} f_{xw|\mu,v}\left(x,w|\mu,v;\eta\right)f_{\mu|v}\left(\mu|v\right)dwdxd\mu,\end{aligned}$$

where in the second equality we used the definition of w, and in the fourth equality we introduced μ into the marginal by integrating over all $\mu\in\mathbb{R}$. Conditional on μ all values/costs are independent, and hence also the order statistics of others' bids/asks are independent from the value v of the focal buyer. We can therefore write $f_{xw|\mu,v}\left(x,w|\mu,v;\eta\right)=f_{xw|\mu}\left(x,w|\mu;\eta\right)$ and

$$\begin{aligned}\Pr[x<b<y|v;\eta] &= \int_{\mu=-\infty}^{\infty}\int_{x=-\infty}^{b}\int_{w=b-x}^{\infty} f_{xw|\mu}\left(x,w|\mu;\eta\right)dwdxf_{\mu|v}\left(\mu|v\right)d\mu\\ &= \int_{\mu=-\infty}^{\infty}\int_{x=-\infty}^{b}\int_{w=b-x}^{\infty} f_{w|x,\mu}\left(w|x,\mu;\eta\right)f_{x|\mu}\left(x|\mu;\eta\right)dwdxf_{\mu|v}\left(\mu|v\right)d\mu\\ &= \int_{\mu=-\infty}^{\infty}\underbrace{\int_{x=-\infty}^{b}\overline{F}_{w|x,\mu}\left(b-x|x,\mu;\eta\right)f_{x|\mu}\left(x|\mu;\eta\right)dx}_{\Pr[x<b<y|\mu;\eta]}f_{\mu|v}\left(\mu|v\right)d\mu, \qquad (27)\end{aligned}$$

where the last line follows using the right-hand distribution function of w, $\overline{F}_{w|x,\mu}(\cdot|x,\mu;\eta)\equiv 1-F_{w|x,\mu}\left(\cdot|x,\mu;\eta\right)$. Also, as annotated above, the inner integral in (27) is equal to the probability that the focal buyer sets the price conditional on the state μ for market size η.

Recall that we are interested in the q^{th} quantile where $q=m/(m+n)$ and $\xi_q=F^{-1}(q)$. We write

$$\begin{aligned}
\overline{F}_{w|x,\mu}\left(b-x|x,\mu;\eta\right) &= \Pr\left[w>b-x|x,\mu;\eta\right]\\
&= \Pr\left[2\left[\eta\left(m+n\right)-1\right]f\left(\xi_q\right)w>2\left[\eta\left(m+n\right)-1\right]f\left(\xi_q\right)\left(b-x\right)|x,\mu;\eta\right]\\
&= \Pr\left[\widehat{w}>2\left[\eta(m+n)-1\right]f\left(\xi_q\right)\left(b-x\right)|x,\mu;\eta\right],
\end{aligned}$$

where in the second equality we multiplied both sides in the probability event by $2\left[\eta\left(m+n\right)-1\right]f\left(\xi_q\right)$ and then in the third equality introduced $\widehat{w}\equiv 2\left[\eta\left(m+n\right)-1\right]f\left(\xi_q\right)w$.

The variable $\widehat{w}$ is the difference $w=y-x$ scaled by the number of nonfocal traders $\eta(m+n)-1$ and two times the density at the quantile of interest $f\left(\xi_q\right)$. This scaling allows us to use the asymptotic result of Siddiqui (1960) that $\widehat{w}$ is χ^2 with two degrees of freedom and independent of x.[16] Letting asymptotic distributions/densities be denoted by tildes, we have

$$\tilde{\overline{F}}_{\widehat{w}|x,\mu}\left(t|x,\mu;\eta\right) = \exp\left(-\frac{t}{2}\right), t\in\mathbb{R}^+.$$

From Theorem 3, we know that x conditional on μ is asymptotically normal with mean $\mu+\xi_q$ and variance Λ/η, i.e.,

$$\tilde{f}_{x|\mu}\left(t|\mu;\eta\right) = \frac{1}{\sqrt{2\pi\frac{\Lambda}{\eta}}}\exp\left(-\frac{\left(t-\xi_q-\mu\right)^2}{2\frac{\Lambda}{\eta}}\right), t\in\mathbb{R}, \tag{28}$$

$$\text{where } \Lambda \equiv \frac{mn/(m+n)^2}{(m+n)-1/\eta}\frac{1}{f^2\left(\xi_q\right)}. \tag{29}$$

Of course, as in the main text, using the asymptotic density of x given μ we write the asymptotic density of x given the value v of the focal buyer as

$$\tilde{f}_{x|v}(t|v;\eta) = \int_{\mu=-\infty}^{\infty}\tilde{f}_{x|\mu}\left(t|\mu;\eta\right)f_{\mu|v}(\mu|v)d\mu. \tag{30}$$

Using the asymptotic distribution of $\widehat{w}$ and the asymptotic density of x, we can write the asymptotic probability that the focal buyer sets the price conditional on μ—as given by the inner

[16] This result applies even in the presence of two populations because our Theorem 3 on the asymptotic normality of x with two populations can be applied within the proof in Siddiqui (1960).

integral in (27)—also denoted by a tilde,

$$\begin{aligned}
& \widetilde{\Pr}[x < b < y|\mu;\eta] \\
= & \int_{x=-\infty}^{b} \overline{\overline{F}}_{\widehat{w}|x,\mu}\left(2\left[\eta(m+n)-1\right] f\left(\xi_q\right)(b-x)\,|x,\mu;\eta\right) \tilde{f}_{x|\mu}\left(x|\mu;\eta\right) dx \\
= & \frac{1}{2\left[(m+n)-1/\eta\right] f\left(\xi_q\right)} \int_{t=0}^{\infty} \overline{\overline{F}}_{\widehat{w}|x,\mu}\left(\eta\, t|x,\mu;\eta\right) \tilde{f}_{x|\mu}\left(b-\frac{t}{2\left[(m+n)-1/\eta\right] f\left(\xi_q\right)}\middle|\,\mu;\eta\right) dt \\
= & \Delta \int_{t=0}^{\infty} \overline{\overline{F}}_{\widehat{w}|x,\mu}\left(\eta\, t|x,\mu;\eta\right) \tilde{f}_{x|\mu}\left(b-\Delta\cdot t|\mu;\eta\right) dt \\
= & \Delta\sqrt{\frac{\eta}{2\pi\Lambda}} \int_{t=0}^{\infty} \left[\exp\left(-\frac{t}{2}\right)\exp\left(-\frac{(b-\Delta\cdot t-\xi_q-\mu)^2}{2\Lambda}\right)\right]^{\eta} dt,
\end{aligned} \tag{31}$$

where in the second equality we changed the variable of integration to $t = (b-x)/\Delta$ with

$$\Delta \equiv \frac{1}{2\left[(m+n)-1/\eta\right] f\left(\xi_q\right)}. \tag{32}$$

The integral in (31) is equal to

$$\exp\left(-\frac{\eta}{8}\frac{4\Delta(b-\xi_q-\mu)-\Lambda}{\Delta^2}\right)\Phi\left(\frac{\sqrt{\eta}}{2}\frac{2\Delta(b-\xi_q-\mu)-\Lambda}{\sqrt{\Lambda}\Delta}\right).$$

To get a simpler expression, however, we apply Fibich and Gavious (2010, Lem. 2) to approximate the integral in (31) and write[17]

$$\begin{aligned}
& \widetilde{\Pr}[x < b < y|\mu;\eta] \\
= & \Delta\sqrt{\frac{\eta}{2\pi\Lambda}} \left\{\int_{t=0}^{\infty} \left[\exp\left(-\frac{t}{2}\right)\exp\left(-\frac{(b-\Delta\cdot t-\xi_q-\mu)^2}{2\Lambda}\right)\right]^{\eta} dt\right\} \\
= & \Delta\sqrt{\frac{\eta}{2\pi\Lambda}} \left\{\frac{1}{\eta} 2 \frac{\exp\left[-\frac{(b-\xi_q-\mu)^2}{2\frac{\Lambda}{\eta}}\right]}{1-\frac{2\Delta}{\Lambda}(b-\xi_q-\mu)}\left[1+O\left(\frac{1}{\eta}\right)\right]\right\} \\
= & \frac{2\Delta}{\eta}\frac{1}{1-\frac{2\Delta}{\Lambda}(b-\xi_q-\mu)} \tilde{f}_{x|\mu}\left(b|\mu;\eta\right)\left[1+O\left(\frac{1}{\eta}\right)\right] \\
= & \left[\frac{2\Delta}{\eta}\frac{1}{1-\frac{2\Delta}{\Lambda}(b-\xi_q-\mu)}+O\left(\frac{1}{\eta^2}\right)\right] \tilde{f}_{x|\mu}\left(b|\mu;\eta\right),
\end{aligned} \tag{33}$$

where the third equality follows by using the definition of the asymptotic density of x conditional on μ given by (28).

[17] For $R(t)$ equal to the term in square brackets in (31), this lemma implies

$$\int_{t=0}^{\infty} R^{\eta}(t)dt = -\frac{1}{\eta}\frac{R^{\eta+1}(0)}{R'(0)}\left[1+O\left(\frac{1}{\eta}\right)\right].$$

Substituting the expression for $\widetilde{\Pr}[x < b < y|\mu;\eta]$ given by (33) in (27) produces the asymptotic probability that the focal buyer sets the price conditional on v:

$$\begin{aligned}
\widetilde{\Pr}[x < b < y|v;\eta] &= \int_{\mu=-\infty}^{\infty} \widetilde{\Pr}[x < b < y|\mu;\eta] f_{\mu|v}(\mu|v)d\mu \\
&= \int_{\mu=-\infty}^{\infty} \frac{2\Delta}{\eta} \frac{1}{1-\frac{2\Delta}{\Lambda}(b-\xi_q-\mu)} \tilde{f}_{x|\mu}\left(b|\mu;\eta\right) f_{\mu|v}(\mu|v)d\mu \\
&\quad + O\left(\frac{1}{\eta^2}\right) \int_{\mu=-\infty}^{\infty} \tilde{f}_{x|\mu}\left(b|\mu;\eta\right) f_{\mu|v}(\mu|v)d\mu \\
&= \frac{2\Delta}{\eta} \int_{\mu=-\infty}^{\infty} \frac{1}{1-\frac{2\Delta}{\Lambda}(b-\xi_q-\mu)} \tilde{f}_{x|\mu}\left(b|\mu;\eta\right) f_{\mu|v}(\mu|v)d\mu \\
&\quad + O\left(\frac{1}{\eta^2}\right) \tilde{f}_{x|v}(b|v;\eta).
\end{aligned} \tag{34}$$

We used (30) in the last line. Recall that due to the uniform improper prior assumption on μ, $f_{\mu|v}(\mu|v) = f(v-\mu)$. By also substituting for $\tilde{f}_x\left(b|\mu;\eta\right)$ from (28), the integral in (34) becomes

$$\begin{aligned}
&\int_{\mu=-\infty}^{\infty} \frac{1}{1-\frac{2\Delta}{\Lambda}(b-\xi_q-\mu)} \frac{1}{\sqrt{2\pi\frac{\Lambda}{\eta}}} \exp\left(-\frac{(b-\xi_q-\mu)^2}{2\frac{\Lambda}{\eta}}\right) f(v-\mu)d\mu \\
= &\int_{\alpha=-\infty}^{\infty} \frac{1}{1-\frac{2\Delta}{\Lambda}\alpha} \frac{1}{\sqrt{2\pi\frac{\Lambda}{\eta}}} \exp\left(-\frac{\alpha^2}{2\frac{\Lambda}{\eta}}\right) f(\lambda+\xi_q+\alpha)d\alpha,
\end{aligned} \tag{35}$$

where in the last line we change the variable of integration to $\alpha = b-\xi_q-\mu = b-(\mu+\xi_q)$. We also use λ to denote the amount by which the focal buyer underbids (i.e., $\lambda \equiv v-b$) without assuming that it is constant for all v.

Even for a particular choice of the density f, the integral in (35) is still not computable in closed form due to the term $\left[1-\frac{2\Delta}{\Lambda}\alpha\right]^{-1}$. In order to proceed we take a Taylor's series expansion of $\left[1-\frac{2\Delta}{\Lambda}\alpha\right]^{-1}$ around zero

$$\frac{1}{1-\frac{2\Delta}{\Lambda}\alpha} = 1 + \sum_{i=1}^{\infty}\left(\frac{2\Delta}{\Lambda}\alpha\right)^i.$$

Then by substituting in (35) we have

$$\begin{aligned}
\int_{\alpha=-\infty}^{\infty} \frac{1}{1-\frac{2\Delta}{\Lambda}\alpha} \frac{1}{\sqrt{2\pi\frac{\Lambda}{\eta}}} \exp\left(-\frac{\alpha^2}{2\frac{\Lambda}{\eta}}\right) f(\lambda+\xi_q+\alpha)d\alpha &= \\
\int_{\alpha=-\infty}^{\infty} \frac{1}{\sqrt{2\pi\frac{\Lambda}{\eta}}} \exp\left(-\frac{\alpha^2}{2\frac{\Lambda}{\eta}}\right) f(\lambda+\xi_q+\alpha)d\alpha &+ \\
\int_{\alpha=-\infty}^{\infty} \sum_{i=1}^{\infty}\left(\frac{2\Delta}{\Lambda}\alpha\right)^i \frac{1}{\sqrt{2\pi\frac{\Lambda}{\eta}}} \exp\left(-\frac{\alpha^2}{2\frac{\Lambda}{\eta}}\right) f(\lambda+\xi_q+\alpha)d\alpha.&
\end{aligned}$$

Again using (30), the first term in the above sum is $\tilde{f}_{x|v}(b|v;\eta)$. Substituting back in (34) we get the following expression for the asymptotic probability that the focal buyer sets the price conditional on v:

$$\begin{aligned}\widetilde{\Pr}[x < b < y|v;\eta] &= \left[\frac{2\Delta}{\eta} + O\left(\frac{1}{\eta^2}\right)\right]\tilde{f}_{x|v}(b|v;\eta) \\ &+ \frac{2\Delta}{\eta}\int_{\alpha=-\infty}^{\infty}\sum_{i=1}^{\infty}\left(\frac{2\Delta}{\Lambda}\alpha\right)^i \underbrace{\frac{1}{\sqrt{2\pi\frac{\Delta}{\eta}}}\exp\left(-\frac{\alpha^2}{2\frac{\Delta}{\eta}}\right)}_{\tilde{f}_{x|\mu}(b|\mu;\eta)} f(\lambda+\xi_q+\alpha)d\alpha. \quad (36)\end{aligned}$$

Our goal in what follows is to show that the integral in (36) is $O(1/\eta)\tilde{f}_{x|v}(b|v;\eta)$. This will lead to a simple expression for the asymptotic probability and subsequently an approximate formula for the equilibrium strategy of the focal buyer. To see this note that if

$$\int_{\alpha=-\infty}^{\infty}\sum_{i=1}^{\infty}\left(\frac{2\Delta}{\Lambda}\alpha\right)^i \frac{1}{\sqrt{2\pi\frac{\Delta}{\eta}}}\exp\left(-\frac{\alpha^2}{2\frac{\Delta}{\eta}}\right) f(\lambda+\xi_q+\alpha)d\alpha = O\left(\frac{1}{\eta}\right)\tilde{f}_{x|v}(b|v;\eta), \quad (37)$$

then the asymptotic probability (36) that the focal buyer sets the price becomes

$$\begin{aligned}\widetilde{\Pr}[x < b < y|v;\eta] &= \left[\frac{2\Delta}{\eta} + O\left(\frac{1}{\eta^2}\right)\right]\tilde{f}_{x|v}(b|v;\eta) + \frac{2\Delta}{\eta}O\left(\frac{1}{\eta}\right)\tilde{f}_{x|v}(b|v;\eta) \\ &= \left[\frac{2\Delta}{\eta} + O\left(\frac{1}{\eta^2}\right)\right]\tilde{f}_{x|v}(b|v;\eta).\end{aligned}$$

Substituting the above probability into the focal buyer's FOC (2), we obtain the AFOC (9). This yields a unique solution for the difference $\lambda = v - b$ that does not depend on v; it is the offset solution (10), where we substitute for Δ from (32). Therefore, in the case where (37) holds we get the approximate formula (11) for the focal buyer's optimal offset strategy.

In what follows we first prove that (37) holds for F normal and then extend it to the case of mixture of normals. The normal and mixtures of normals allow us to compute the integral in (36) in closed form and thus makes it possible to establish (37). Recall that we also assume that the focal buyer considers strategies $\widetilde{B}(\cdot;\eta)$ in which his underbidding λ is $O\left(1/\eta^{\epsilon}\right)$ for some $\epsilon > 0$.

Normal distribution case. Start by assuming that $f = \phi_k(x)$. Substituting in the integral in (36) yields

$$\int_{\alpha=-\infty}^{\infty}\sum_{i=1}^{\infty}\left(\frac{2\Delta}{\Lambda}\alpha\right)^i \frac{1}{\sqrt{2\pi\frac{\Delta}{\eta}}}\exp\left(-\frac{\alpha^2}{2\frac{\Delta}{\eta}}\right)\frac{1}{\sqrt{2\pi\sigma_k^2}}\exp\left(-\frac{(\lambda+\xi_q+\alpha-m_k)^2}{2\sigma_k^2}\right)d\alpha. \quad (38)$$

Focusing on the first three terms of the infinite sum, we get

$$\underbrace{\int_{\alpha=-\infty}^{\infty} \frac{2\Delta}{\Lambda}\alpha \, \frac{1}{\sqrt{2\pi\frac{\Lambda}{\eta}}} \exp\left(-\frac{\alpha^2}{2\frac{\Lambda}{\eta}}\right) \frac{1}{\sqrt{2\pi\sigma_k^2}} \exp\left(-\frac{(\lambda+\xi_q+\alpha-m_k)^2}{2\sigma_k^2}\right) d\alpha}_{I_1} \tag{39}$$

$$+ \underbrace{\int_{\alpha=-\infty}^{\infty} \left[\frac{2\Delta}{\Lambda}\alpha\right]^2 \frac{1}{\sqrt{2\pi\frac{\Lambda}{\eta}}} \exp\left(-\frac{\alpha^2}{2\frac{\Lambda}{\eta}}\right) \frac{1}{\sqrt{2\pi\sigma_k^2}} \exp\left(-\frac{(\lambda+\xi_q+\alpha-m_k)^2}{2\sigma_k^2}\right) d\alpha}_{I_2} \tag{40}$$

$$+ \underbrace{\int_{\alpha=-\infty}^{\infty} \left[\frac{2\Delta}{\Lambda}\alpha\right]^3 \frac{1}{\sqrt{2\pi\frac{\Lambda}{\eta}}} \exp\left(-\frac{\alpha^2}{2\frac{\Lambda}{\eta}}\right) \frac{1}{\sqrt{2\pi\sigma_k^2}} \exp\left(-\frac{(\lambda+\xi_q+\alpha-m_k)^2}{2\sigma_k^2}\right) d\alpha}_{I_3} . \tag{41}$$

Also, observe that when $f = \phi_k$ equation (30) implies

$$\begin{aligned}
\tilde{f}_{x|v}(b|v;\eta) &= \int_{\mu=-\infty}^{\infty} \frac{1}{\sqrt{2\pi\frac{\Lambda}{\eta}}} \exp\left(-\frac{(b-\xi_q-\mu)^2}{2\frac{\Lambda}{\eta}}\right) \frac{1}{\sqrt{2\pi\sigma_k^2}} \exp\left(-\frac{(\mu-v-m_k)^2}{2\sigma_k^2}\right) d\mu \\
&= \frac{1}{\sqrt{2\pi\left(\sigma_k^2+\frac{\Lambda}{\eta}\right)}} \exp\left(-\frac{(v-b+\xi_q-m_k)^2}{2\left(\sigma_k^2+\frac{\Lambda}{\eta}\right)}\right) \\
&= \frac{1}{\sqrt{2\pi\left(\sigma_k^2+\frac{\Lambda}{\eta}\right)}} \exp\left(-\frac{(\lambda+\xi_q-m_k)^2}{2\left(\sigma_k^2+\frac{\Lambda}{\eta}\right)}\right), \qquad (42)
\end{aligned}$$

i.e., the asymptotic density of x given the value v is also normal with mean $m_k - v - \xi_q$ and variance $\sigma_k^2 + \Lambda/\eta$. In the last equality above we simply substituted for $\lambda = v - b$.

The integrals I_1, I_2, I_3 as annotated in (39)–(41) are easily computed in closed form. Using formula (42) for $\tilde{f}_{x|v}(b|v;\eta)$ in the normal case we have that

$$\begin{aligned}
I_1 &= -\tilde{f}_{x|v}(b|v;\eta)\frac{2\Delta}{\Lambda}\frac{\frac{\Lambda}{\eta}}{\sigma_k^2+\frac{\Lambda}{\eta}}(\lambda+\xi_q-m_k), \qquad (43)\\
I_2 &= \tilde{f}_{x|v}(b|v;\eta)\left(\frac{2\Delta}{\Lambda}\right)^2 \frac{\frac{\Lambda}{\eta}}{\left(\sigma_k^2+\frac{\Lambda}{\eta}\right)^2}\left(1+\frac{\Lambda}{\eta}\left(1+(\lambda+\xi_q-m_k)^2\right)\right), \qquad (44)\\
I_3 &= -\tilde{f}_{x|v}(b|v;\eta)\left(\frac{2\Delta}{\Lambda}\right)^3 \frac{\left(\frac{\Lambda}{\eta}\right)^2}{\left(\sigma_k^2+\frac{\Lambda}{\eta}\right)^3}\left(3+\frac{\Lambda}{\eta}\left(3+(\lambda+\xi_q-m_k)^2\right)\right)(\lambda+\xi_q-m_k). \qquad (45)
\end{aligned}$$

Notice that $\tilde{f}_x(b|v;\eta)$ is a common factor in I_1, I_2, I_3. Those terms differ in their exponents on the factors $2\Delta/\Lambda$, Λ/η and $\lambda + \xi_q - m_k$. To derive a simple formula for the focal buyer's strategy, we focus on the dependence of these terms on η. Using (29) and (32) we have that

$$\frac{2\Delta}{\Lambda} = \frac{f(\xi_q)(m+n)^2}{mn},$$

i.e., it is a constant that does not depend on η. Furthermore, again from (29),

$$\frac{\Lambda}{\eta} = \frac{mn}{(m+n)^2 f^2(\xi_q)} \frac{1}{(m+n)\eta - 1},$$

and so it is $O(1/\eta)$. We also have that $\frac{\Lambda}{\eta}/\left(1+\frac{\Lambda}{\eta}\right)^2$ is $O(1/\eta)$ and $\left(\frac{\Lambda}{\eta}\right)^2/\left(1+\frac{\Lambda}{\eta}\right)^3$ is $O(1/\eta^2)$.

We have assumed that the focal buyer restricts attention to λ that is $O(1/\eta^\epsilon)$ for some $\epsilon > 0$. It is then easy to see from (43)–(45) for I_1, I_2, I_3, that[18]

$$\begin{aligned} I_1 &= -\tilde{f}_{x|v}(b|v;\eta) O\left(\frac{1}{\eta}\right), \\ I_2 &= \tilde{f}_{x|v}(b|v;\eta) O\left(\frac{1}{\eta}\right), \\ I_3 &= -\tilde{f}_{x|v}(b|v;\eta) O\left(\frac{1}{\eta^2}\right). \end{aligned}$$

Similarly, if we compute the integrals corresponding to higher terms (i.e., I_n for $n > 3$) from the series expansion of $\left[1 - \frac{2\Delta}{\Lambda}\alpha\right]^{-1}$ we get expressions of the form $\tilde{f}_{x|v}(b|v;\eta) O\left(1/\eta^\kappa\right)$ for $\kappa \geq 2$. Hence, summing the I_n's the integral in (38) is[19]

$$I_1 + I_2 + I_3 + \ldots = O\left(\frac{1}{\eta}\right) \tilde{f}_{x|v}(b|v;\eta).$$

This means that (37) is satisfied in the normal distribution case, and the asymptotic offset (10) as well as the approximate formula (11) hold simply by substituting ϕ_k for f.

Mixture of normals. Now consider a nondegenerate mixture, i.e., $f = \sum_{k=1}^K w_k \phi_k$. We substitute in the integral in (36)

$$f(\lambda + \xi_q + \alpha) = \sum_{k=1}^K w_k \phi_k(\lambda + \xi_q + \alpha).$$

[18] Strictly speaking, we should treat the case $\xi_q = m_k$ separately because then the order term in I_1 is $O(1/\eta^{1+\epsilon})$ and the order term in I_3 is $O(1/\eta^{2+\epsilon})$. However, since the order term in I_2 remains $O(1/\eta)$, this does not affect the end result.

[19] Note that $O(1/\eta) - O(1/\eta) = O(1/\eta)$.

Above we proved that (37) holds for each ϕ_k, i.e.,

$$
\begin{aligned}
& \int_{\alpha=-\infty}^{\infty} \sum_{i=1}^{\infty} \left(\frac{2\Delta}{\Lambda}\alpha\right)^i \frac{1}{\sqrt{2\pi\frac{\Lambda}{\eta}}} \exp\left(-\frac{\alpha^2}{2\frac{\Lambda}{\eta}}\right) \frac{1}{\sqrt{2\pi\sigma_k^2}} \exp\left(-\frac{(\lambda+\xi_q+\alpha-m_k)^2}{2\sigma_k^2}\right) d\alpha \\
= \; & O\left(\frac{1}{\eta}\right) \frac{1}{\sqrt{2\pi\left(\sigma_k^2+\frac{\Lambda}{\eta}\right)}} \exp\left(-\frac{(\lambda+\xi_q-m_k)^2}{2\left(\sigma_k^2+\frac{\Lambda}{\eta}\right)}\right). \qquad (46)
\end{aligned}
$$

Furthermore, for the mixture of normals case from (30),

$$
\begin{aligned}
\tilde{f}_{x|v}(b|v;\eta) &= \int_{\mu=-\infty}^{\infty} \frac{1}{\sqrt{2\pi\frac{\Lambda}{\eta}}} \exp\left(-\frac{(b-\xi_q-\mu)^2}{2\frac{\Lambda}{\eta}}\right) \sum_{k=1}^{K} w_k \frac{1}{\sqrt{2\pi\sigma_k^2}} \exp\left(-\frac{(\mu-v-m_k)^2}{2\sigma_k^2}\right) d\mu \\
&= \sum_{k=1}^{K} w_k \int_{\mu=-\infty}^{\infty} \frac{1}{\sqrt{2\pi\frac{\Lambda}{\eta}}} \exp\left(-\frac{(b-\xi_q-\mu)^2}{2\frac{\Lambda}{\eta}}\right) \frac{1}{\sqrt{2\pi\sigma_k^2}} \exp\left(-\frac{(\mu-v-m_k)^2}{2\sigma_k^2}\right) d\mu \\
&= \sum_{k=1}^{K} w_k \frac{1}{\sqrt{2\pi\left(\sigma_k^2+\frac{\Lambda}{\eta}\right)}} \exp\left(-\frac{(\lambda+\xi_q-m_k)^2}{2\left(\sigma_k^2+\frac{\Lambda}{\eta}\right)}\right). \qquad (47)
\end{aligned}
$$

Multiplying both sides of (46) by w_k and summing across $k = \{1, \ldots, K\}$ using (47) shows that (37) is satisfied for mixture of normals. The asymptotic offset (10) and the approximate formula (11) thus hold. ■