Price Discovery Using a Double Auction

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Abstract

We investigate equilibrium in the buyer's bid double auction (BBDA) in a model with correlated signals and either private or interdependent values. Using a combination of theorems and numerical experiments, we demonstrate that simple equilibria exist even in small markets. Moreover, we bound traders' strategic behavior as a function of market size and derive rates of convergence to zero of (i) inefficiency in the allocation caused by strategic behavior and (ii) the error in the market price as an estimate of the rational expectations price. These rates together with numerical experiments suggest that strategic behavior is inconsequential even in small markets in its effect on allocational efficiency and information aggregation. The BBDA thus simultaneously accomplishes both the informational and allocational goals that markets ideally fulfill; it does this perfectly in large markets and approximately in small markets, with the error attributable mainly to the smallness itself and not the strategic behavior of traders.

1 Introduction

A market can have both allocational and informational purposes. The allocational purpose is to redistribute goods among traders so as to achieve gains from trade. The informational purpose is to aggregate the private information of traders into a meaningful price that individuals both inside and outside the market can use to make better consumption and investment decisions. In markets with a limited number of traders—a small market for short—difficulties that may interfere with the market generating an efficient allocation and an informative price include both traders' efforts to influence the market price in their favor and the randomness of who participates in the market at any given moment.

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We investigate here how a market's smallness affects its performance with respect to both of these purposes. Specifically, we evaluate the performance of a double auction, which is a simple model of a call market.[1] Three issues guide our inquiry. First, we characterize equilibrium in markets of arbitrary size. Second, we calculate equilibria in small markets and gain insight into their properties. Third, we measure the impact of strategic behavior upon the allocative efficiency of the market and the meaningfulness of its price. With respect to this last issue, we evaluate the properties of the strategically determined market price as an estimate of the *rational expectations equilibrium (REE) price*, as defined by Radner (1979).

The Model. There are m buyers, each of whom wishes to buy at most one item, and n sellers, each of whom has a single item to sell. The call market that we analyze is the *buyer's bid double auction* (BBDA).[2] After collecting bids from buyers and asks from sellers, the BBDA sorts them from lowest to highest and selects as the price the upper endpoint of the interval of possible market-clearing prices. Trade occurs at this price between buyers whose bids are at or above the price and sellers whose offers are below the price. We study this price-setting rule because it simplifies the behavior of traders on one side of the market. Specifically, it implies that if a seller's ask results in the sale of his item, then it did not set the market price. A seller thus cannot affect the terms at which he trades; he therefore sets his ask equal to an estimate of his cost in a way that protects him from a winner's curse. We call this his *price-taking ask*. A buyer calculates his *price-taking bid* in a way similar to sellers. A buyer, however, does not submit it as his bid because his bid may set the price. This gives him an incentive to bid less than his price-taking bid so that in expectation it nudges the terms of trade in his favor. He therefore computes a *strategic term* that is the amount by which he shades his bid below his price-taking bid.

Each trader's utility is quasilinear in his value/cost and money. We import from Bayesian statistical decision theory a simple process for generating traders' signals and values/costs. The market's *state* $\mu \in \mathbb{R}$ is drawn from the uniform improper prior, which may be thought of informally as the "uniform distribution over the entire real line".[3] For each trader, an idiosyncratic preference term is independently drawn from a proper distribution on $\mathbb{R}$ and then added to the state to determine his value/cost. If each trader observes his value/cost, then the environment is *correlated private values* (CPV). Alternatively, suppose each trader observes a noisy signal of his value/cost that is generated by adding to it an independent draw from a second proper distribution

[1]Traders submit bids and asks in a call market until a pre-announced closing time whereupon trades are consummated at a market-clearing price. Most stock and futures markets (e.g., the NYSE) open their trading sessions with call markets. These markets also use these procedures to restart trading following a halt. Some national treasuries use call markets to sell their bills. For further discussion see O'Hara (1997, p. 10), Harris (2002, p. 90-92 and 120-132), and Biais, Glosten, and Spatt (2005, sec. 3.1.3).

[2]The name for this particular double auction originates in the bilateral case where the buyer's bid is the price when trade occurs. However, this need not be the case in the multilateral BBDA.

[3]Improper prior distributions have played an important role in statistical decision theory. See DeGroot (1970, secs. 10.1–10.4) and Pratt, Raiffa, and Schlaifer (1995, secs. 10.3.4, 11.4.4, and 16.3.2). The uniform improper prior (also referred to in the literature as the "uninformative" or "diffuse" prior) has previously been used in the case of one-sided auctions by Wilson (1998) and Klemperer (1999) and in the theory of global games by Morris and Shin (2003).

on $\mathbb{R}$. The environment is then *correlated interdependent values* (CIV). Interdependence exists because each trader i's signal carries information concerning the state μ; if a trader's signal were observable to other traders, that information would help them to estimate their own values/costs. The state μ creates correlation among traders' values/costs and signals despite the independence of the preference and noise terms.

The uniform improper prior is a limiting representation of extreme ex ante uncertainty about the state μ. DeGroot (1980, p. 190) motivates it as a model of a decision maker who has little information ex ante concerning future random events but who will receive a valuable signal at the interim stage on which he can update his probabilistic beliefs. It may not be worthwhile for the decision maker to spend time and effort in properly specifying his ex ante beliefs if he takes an action at the interim stage conditional on his informative signal. There is an additional motivation for its use in our paper. The uniform improper prior models a situation in which no trader ex ante has any idea of what the market price will be. Call markets are commonly used to discover a price when none currently exists (e.g., at the start of a trading session or following a trading halt). The uniform improper prior is thus an appropriate test case for a call market. It can be seen as maximally challenging the market with respect to the twin issues of allocating the items and discovering a meaningful price.

The main value of the uniform improper prior in our paper, however, is that it makes our analysis tractable while allowing the important features of correlation and interdependence. The tractability stems from an *invariance property*: with a uniform improper prior, every draw of a trader's signal provides him with identical information as to where other traders' signals are likely to be relative to his own draw. Thus the fundamentals of trader i's decision problem are invariant with respect to his signal. Invariance is used throughout the paper to reduce a trader's decision problem at any possible value of his signal to the decision problem at a specific value of his signal.

We study such a simple trading environment because it provides insights into trading that are not found within richer models such as general equilibrium theory or the theory of large markets. A goal of the double auction literature is to understand strategic behavior in explicit models of price formation, with smallness of the market a necessary component in creating opportunities for strategic behavior. It is not enough to consider double auctions with arbitrarily large numbers of traders, for economic theory already has rich models of large markets; a necessary complement to asymptotic results on double auctions is to determine whether or not they actually describe smaller, finite markets, or alternatively, how large a number of traders are required for the results to be observed. As part of our investigation of small markets, the simplicity of our model facilitates the computation of equilibrium in both the CPV and CIV cases. This is new to the double auction literature for to our knowledge no examples of double auction equilibria have previously been computed for these cases. Solving numerically for equilibria is important both for demonstrating that results and theorems actually describe small markets but also for the purpose of experimental and empirical testing of the theory.

Theorems and Numerical Results. Our solution concept is Bayesian Nash equilibrium. A trader chooses his bid/ask as a function of his privately observed signal. Conceivably, the difference between a trader's signal and his bid/ask may vary with his signal. This seems implausible, however, because a trader' beliefs about the signals of others remains invariant as described above. It is therefore reasonable to conjecture that this difference is a constant rather than a function of the signal. This extends the invariance with respect to signals to bids/asks. Consequently, we study symmetric *offset equilibria* in which each trader on a side of the market adds the same constant to each of his possible signals to determine his bid/ask. Computational evidence we present suggests that restricting attention to offset equilibria is without loss of generality in the sense that no other increasing, differentiable symmetric equilibrium exists.

Our informational structure permits us to obtain a sequence of theorems and numerical results. A *theorem*, of course, is a statement that follows by the rules of logic from the assumptions of the model. The difficulty that this paper confronts is that theorists have been unable to address important conjectures about small double auction markets with correlated or interdependent values. A *numerical result* in this paper is a statement that we have been unable to prove but for which we are able to provide convincing numerical evidence. For example, within the context of our model, graphing a trader's marginal expected utility and checking that the computed offset is the maximizer is not a proof of optimality but is nevertheless good evidence. A theorem is certainly better than a result, but a result is much better than either simple ignorance or informally argued conjecture. For the most part, numerical results are stated in the general CIV case with additional restrictions then specified to enable the proofs of theorems in the simpler CPV case.

For fixed m and n, we consider sequences of markets with ηm buyers and ηn sellers, where $\eta \in \mathbb{N}$ is the size of the market. Our theorems/results concerning strategic behavior and the inefficiency that it causes are summarized as follows:

- Offset equilibria exist in each size of market, are uniquely determined, and are straightforward to compute. They are the only increasing and differentiable strategies that define symmetric equilibrium in the BBDA. The strategic term of a buyer (i.e., the difference between his equilibrium offset bid and his price-taking bid) is $O(1/\eta)$.[4]

- The expected gains from trade that are lost to the traders due to the strategic efforts of buyers to influence price in their favor as a fraction of the ex ante potential gains from trade when traders act as price takers is $O(1/\eta^2)$.

We turn next to the impact of strategic behavior upon the market price as an estimate of the REE price in state μ. The REE price is defined by its two properties of (i) revealing the state μ and (ii) clearing the limiting continuum market determined by m, n and the distributions of preference and signal noise terms. Identifying μ in both the CPV and CIV cases is valuable to anyone planning to trade in the future. In the CIV case it is also useful for current participants in

[4]For functions $u_1(\eta)$, $u_2(\eta) : \mathbb{N} \to \mathbb{R}_+$, $u_1(\eta) = O(u_2(\eta))$ means that there exist constants $k \in \mathbb{R}_+$ and $\eta_0 \in \mathbb{N}$ such that $u_1(\eta) < ku_2(\eta)$, for all $\eta > \eta_0$. The notation $u_1(\eta) = \Theta(u_2(\eta))$ means that there exist constants $k_1, k_2 \in \mathbb{R}_+$ and $\eta_0 \in \mathbb{N}$ such that $k_1 u_2(\eta) < u_1(\eta) < k_2 u_2(\eta)$, for all $\eta > \eta_0$.

the market as it allows them to better estimate their ex post gains from trading after the market closes. We find that:

- The error in the equilibrium price as an estimate of the REE price that is attributable to the strategic behavior of buyers is $O(1/\eta)$. The expected error in the equilibrium price as an estimate that is attributable to the finite size of the market and the noise in trader signals is $\Theta(1/\sqrt{\eta})$; consequently, the impact of strategic behavior on average is of lower order and, except in small markets, is inconsequential in its effect on the market price as an estimate of the REE price.

This last point supports the view that the strategic use by traders of their private information does not impede a market's ability to meaningfully aggregate dispersed information in its market price. Moreover, expected sampling error of order $\Theta(1/\sqrt{\eta})$ is inherent in the estimation of the REE price by any market mechanism, whether incentive compatible or not; the rate of convergence to zero at which the expected total error in the BBDA's price as an estimate of the rational expectations price is thus the fastest rate that is possible. Trade at a market-clearing price in the BBDA is in this sense *asymptotically optimal* as an algorithm for estimating the REE price, which it accomplishes while respecting the privacy of information and the strategic behavior that privacy permits.[5]

We present numerical experiments as follows:

- In the CIV case in which preference and noise terms are normally distributed, section 4.2 explores how the traders' equilibrium offsets depend upon the numbers of traders m and n and the variances of these two distributions. Three of these four variables are held constant while the fourth is varied in a comparative statics analysis. The results in an intuitive fashion demonstrate the interplay between the incentive for a buyer to influence price in his favor together with the necessity for every trader of protecting himself from a winner's curse. This is the first work in the double auction literature that explores the relationship between the beliefs of traders and the bidding behavior of traders.

- All numerical results and theorems discussed above are illustrated in section 5.4 with computed examples of equilibrium in a range of small market sizes ($m = n = 2$, 4, 8, 16) and preference and noise terms that are standard normal. All of these examples are replicated for the Cauchy and Laplace distributions in online Appendices L and M. The rates of convergence are thus shown to describe the smallest of markets and not just large markets.

Finally, our paper presents in Appendices K and L two robustness checks on our model:

- To investigate the robustness of our use of the uniform improper prior, we consider in online Appendix K the CIV case in which preference and noise terms are standard normal and

[5] As shown by Satterthwaite and Williams (2002) in an independent private values model, the BBDA is also worst case asymptotically optimal among all plausible market mechanisms as an algorithm for maximizing the expected gains from trading.

the state is drawn from a proper normal distribution whose variance is now treated as a variable. The uniform improper prior is interpreted as the limit of the proper normal as its variance goes to infinity. Calculations suggest that: (i) equilibria for these proper priors exist and (ii) the equilibria for moderate values of the proper prior's standard deviation (e.g., as small as 1.732) are virtually indistinguishable from the offset equilibrium of the limiting case. This suggests that our results are representative of what is true in a model with a proper distribution of the state; the tractability obtained by using the uniform improper prior does not come at the cost of misleading results.[6]

- The Cauchy distribution does not satisfy the regularity conditions that we impose on the distribution of preference terms in the CPV case to prove our convergence theorems. It is commonly used to test the robustness of results in statistics because its density resembles that of a normal distribution. It has "fat tails", however, which implies that moments of all orders fail to exist for this distribution. The importance of fat tails in the field of finance suggests testing our model with the Cauchy distribution. In online Appendix L we first show that the fat downward tail implies nonexistence of equilibrium in the bilateral case, as the expected benefit to the buyer from lowering his bid and the price that he pays when he trades always exceeds the expected loss of a profitable trade. Nevertheless in the multilateral case ($m, n \geq 2$) equilibria exist and have all the good properties stated in our theorems. The effectiveness of the BBDA as a market mechanism is thus more robust than our theorems indicate.

Related Work. This paper contributes to the development of an explicit theory of how trading among rational, noncooperative traders with private information can, as their numbers increase, lead to increasingly efficient allocations at a price that more accurately reveals the market's underlying fundamentals. This is a rich topic of research; we thus limit our discussion here to investigations of these issues using double auction models. Wilson (1985) and Gresik and Satterthwaite (1989) initiated the theoretical study of multilateral double auctions. Within an independent private values (IPV) environment Satterthwaite and Williams (1989b), Williams (1991), and Rustichini, Satterthwaite, and Williams (1994) established the linear rate of convergence to price-taking behavior and the quadratic rate of convergence to efficiency in the k-double auction. Experimental tests of these results include Kagel and Vogt (1993) and Cason and Friedman (1997). Relying on Jackson and Swinkels (2005) for existence of equilibrium, Cripps and Swinkels (2006) studied the efficiency of large double auctions and show within a general CPV environment that efficiency is approached at the same quadratic rate as in the IPV environment. Notably, however, Cripps and Swinkels (2006) includes no examples of equilibrium and thus fails to demonstrate either that this rate is descriptive of small markets or how many traders are required for the rate to emerge.

[6]Note also that a nonzero mean for the state's distribution linearly translates equilibria and does not change their properties. Choosing a large mean for the state's proper distribution can make the likelihood of negative values/costs arbitrarily small; if they are troubling for reasons of modeling, then they can in this way be made inconsequential.

Reny and Perry (2006) investigated the existence and efficiency of double auction equilibria in a multilateral CIV environment that is more general than the CIV environment we study. They prove that if the number of traders is sufficiently large, then an equilibrium in which each trader's strategy is increasing with respect to his private signal exists, the resulting allocation is nearly efficient and the realized market price closely identifies the REE price. They, however, did not investigate small markets, did not establish the rates at which equilibria converge to efficiency and the REE price, and did not provide insight as to what equilibrium strategies look like. Our numerical results for CIV environments, albeit in a less general model, speak to these limitations by showing: offset equilibria exist; convergence to efficiency is quadratic; convergence to the REE price is $\Theta(1/\sqrt{\eta})$ and is driven by sampling error, not strategic behavior; all of these properties are exhibited in even the smallest of markets.

2 Model

2.1 Values, Costs, and Signals

A state μ is drawn from the uniform improper prior on $\mathbb{R}$. Given μ, an idiosyncratic preference term $\varepsilon_i \sim G_\varepsilon$ is independently drawn for each trader i to determine his value/cost $z_i = \mu + \varepsilon_i$, which he observes in the CPV case. In the CIV case, a trader observes a noisy signal $\sigma_i = z_i + \delta_i$ of his value/cost, where $\delta_i \sim G_\delta$. We assume:

A1: G_ε and G_δ are absolutely continuous with finite first moments and positive densities g_ε and g_δ on $\mathbb{R}$ that are symmetric about 0.

Symmetry implies that the mean and the median of each distribution equal zero. It also implies that a trader's signal σ is an unbiased estimator of his value/cost z and also of the market's state:

$$\mathbb{E}[\mu|\sigma] = \mathbb{E}[z|\sigma] = \sigma.$$

In the CIV case, the presence of noise in each trader's signal implies that values/costs and signals are interdependent, i.e., $\mathbb{E}[z|\sigma, \sigma_j]$ varies with another trader's signal σ_j as well as his own signal σ because knowledge of σ_j enables the focal trader to update his estimate of μ, which in turn enables him to revise his expectation of his own value/cost z.

If a buyer with value v trades at price p, then his ex post utility is $v - p$, and if a seller with cost c trades at price p, then his ex post utility is $p - c$. Those who do not trade receive zero utility.

2.2 Trading Mechanism

We now formalize the BBDA in terms of order statistics. Buyers and sellers simultaneously announce their bids and asks that are then sorted in increasing order:[7]

$$s_{(1)} \leq s_{(2)} \leq \cdots \leq s_{(m)} \leq s_{(m+1)} \leq \cdots \leq s_{(m+n)}.$$

[7]We denote with $s_{(k)}$ the k^{th} order statistic in a specified sample of bids and asks.

The BBDA selects $s_{(m+1)}$ as the market price, with buyers whose bids are at least this price acquiring units from sellers whose asks are strictly less than this price.[8] Those buyers who acquire units pay the price $p = s_{(m+1)}$ and those sellers who give up units receive this price. At the time traders submit their bids/asks, each only knows his own signal. A strategy for buyer i is therefore a function $B_i : \mathbb{R} \to \mathbb{R}$ from his signal σ_i to his bid b_i and a strategy for seller j is a function $S_j : \mathbb{R} \to \mathbb{R}$ from his signal σ_j to his ask a_j.

All traders share common knowledge both of the stochastic structure by which signals and values/costs are generated and of the strategies B_i and S_j that each trader is playing. We restrict strategies as follows:

A2: Each buyer i uses the same strategy B and each seller j uses the same strategy S, where B and S are strictly increasing and differentiable.

Given A2, ties between two or more traders' bids/asks are measure zero events that we ignore.

The literature on auctions with interdependent values, starting with Milgrom and Weber (1982), proves that equilibrium strategies are increasing by assuming affiliation between values/costs and signals. This approach, however, is ineffective in the context of double auctions, which is why we assume increasing strategies in A2. Affiliation guarantees this property if all traders play identical strategies using a mechanism that treats each of them symmetrically. This is clearly not the case in a double auction where different allocation and transfer rules along with different utilities cause buyers and sellers to behave differently.[9]

2.3 Posterior Beliefs

In this section we introduce some basic formulas that concern how a trader updates his probabilistic beliefs upon learning his signal. Consider first the CPV case in which a trader's signal equals his value/cost. Select a focal trader with value/cost $z = \mu + \varepsilon$. The pdf of z conditional on μ is $f_{z|\mu}(z|\mu) = g_\varepsilon(z - \mu)$.[10] Less obviously, Bayes' Rule and μ's uniform improper density g_μ imply

$$f_{\mu|z}(\mu|z) = \frac{f_{z|\mu}(z|\mu)\, g_\mu(\mu)}{\int f_{z|\mu}(z|\mu) g_\mu(\mu)\, d\mu} = g_\varepsilon(z - \mu) = f_{z|\mu}(z|\mu) \tag{1}$$

[8] As shown in Satterthwaite and Williams (1989b, p. 480–81), if $s_{(m+1)} > s_{(m)}$, then the number of bids at or above $s_{(m+1)}$ equals the numbers of asks that are strictly below this value and so the market clears at the price $s_{(m+1)}$. If $s_{(m+1)} = s_{(m)}$, then there may be excess demand at the price $s_{(m+1)}$. Priority in receiving units is assigned in this case first to buyers whose bids strictly exceed $s_{(m+1)}$, with any remaining supply then allocated using a fair lottery among buyers who bid $s_{(m+1)}$.

[9] Examples of the failure of affiliation in double auctions can be found in Reny and Perry (2006, p. 1246–48) and our online Appendix F. Reny and Zamir (2004, sec. 3) also provide two revealing examples concerning this difficulty in a first price auction with asymmetric buyers.

[10] This formula illustrates how we denote conditional pdfs throughout the paper, e.g., $f_{\mu|z}$ denotes the conditional pdf of μ given the the focal trader's value/cost z. Subscripted g denotes a primative density of our model and subscripted f denotes a conditional density.

is the posterior pdf of μ given z.[11] Due to the symmetry of g_ε about 0, each trader's posterior of μ is centered on his value/cost, which reflects the invariance property of our model that is fundamental in our analysis. This formula can be derived by assuming that μ is distributed uniformly on the interval $(-r, r)$ for $r > 0$ and letting $r \to \infty$ so as to approach the uniform improper prior.

In the CIV case, a trader observes a noisy signal $\sigma = z + \delta = \mu + \varepsilon + \delta$. Let $g_{\varepsilon+\delta}$ denote the density of the sum $\varepsilon + \delta$ and $G_{\varepsilon+\delta}$ its distribution. Because ε and δ are independent, $g_{\varepsilon+\delta}$ is given by the convolution

$$g_{\varepsilon+\delta}(t) = \int g_\varepsilon(s)\, g_\delta(t-s) ds. \tag{2}$$

Because g_ε and g_δ are symmetric about 0, it is straightforward to verify that $g_{\varepsilon+\delta}$ is also symmetric about 0. Conditional on μ, a trader's signal σ has pdf

$$f_{\sigma|\mu}(\sigma|\mu) = \int f_{\sigma|z}(\sigma|z)\, f_{z|\mu}(z|\mu)\, dz = \int g_\delta(\sigma - z)\, g_\varepsilon(z-\mu)\, dz = g_{\varepsilon+\delta}(\sigma - \mu). \tag{3}$$

As with z and μ, $f_{\sigma|\mu}(\sigma|\mu) = f_{\mu|\sigma}(\mu|\sigma)$ and the trader's posterior of μ is centered on his signal σ.

The focal trader's posterior on the signal σ_j of some other trader j given his signal σ is

$$f_{\sigma_j|\sigma}(\sigma_j|\sigma) = \int f_{\sigma_j|\mu}(\sigma_j|\mu)\, f_{\mu|\sigma}(\mu|\sigma)\, d\mu = \int g_{\varepsilon+\delta}(\sigma_j - \mu)\, g_{\varepsilon+\delta}(\sigma - \mu)\, d\mu.$$

Since $g_{\varepsilon+\delta}$ is symmetric about 0, $f_{\sigma_j|\sigma}(\sigma_j|\sigma)$ is also symmetric about 0 so that trader j's posterior on signal σ_j is centered around his signal σ. Hence, $f_{\sigma_j|\sigma}$ is a function only of $|\sigma_j - \sigma|$. This implies

$$\Pr[\sigma_j \in [\sigma' + k_1, \sigma' + k_2]|\sigma'] = \Pr[\sigma_j \in [\sigma'' + k_1, \sigma'' + k_2]|\sigma'']$$

for all constants $k_1 < k_2$ and any pair of signals σ', σ'', again reflecting the invariance of our model.

Finally, we assume:

A3: $\mathbb{E}[z|\mu, \sigma]$ is strictly increasing in σ.

This is a strict version of first order stochastic dominance. It is satisfied by the normal, Laplace, and Cauchy distributions. Our demonstration in section 5.3.1 that a REE price exists depends on this plausible assumption.

3 A First Order Approach

A necessary condition for strategies B, S satisfying A2 to define an equilibrium is that they solve the linked differential equations that the buyers' and sellers' first order conditions (FOCs) imply. In this section we derive these FOCs, which are the foundation for both our formal and computational analysis of equilibrium. We then resolve a buyer's FOC into a strategic term and a price-taking term. The strategic term captures the incentive of a buyer to influence the price in his favor while

[11] In the interest of notational simplicity, we omit the limits of integration whenever the integral is defined over the entire real line.

the price-taking term reflects his effort to protect himself from a winner's curse. A seller's FOC consists only of a price-taking term because he cannot influence the price at which he trades. These terms are essential for our analysis of convergence to price-taking behavior in section 5.

We begin with the FOC derived from a buyer's decision problem. Pick a focal buyer. Fix the $m-1$ nonfocal buyers' strategies at B and the n sellers' strategies at S. These strategies determine an ordered random $(n+m-1)$-vector of bids/asks against which the focal buyer with signal σ chooses his bid b to maximize his expected utility. Let x denote $s_{(m)}$ and y denote $s_{(m+1)}$ in this ordered vector. Given the focal buyer's signal σ and his choice of bid b, one of three events occurs:

E1: If $x < y < b$, then the price is $p = y$, the buyer trades, and his utility is $\mathbb{E}[v|\sigma, p = y] - y$.

E2: If $x < b < y$, then the price is $p = b$, the buyer trades, and his utility is $\mathbb{E}[v|\sigma, p = b] - b$.

E3: If $b < x < y$, then the price is $p = x$, the buyer does not trade, and his utility is 0.

The random variables x and y are thus critical to the focal buyer's choice of his bid. Conditional on the focal buyer's signal σ, the strategy S of the n sellers, and the strategy B of the other $m-1$ buyers, let $f^B_{x|\sigma}(x|\sigma)$ be the pdf of x.

Suppose the buyer decides to increase his bid from b to $b+\Delta b$ where $\Delta b > 0$ is small. This can have two effects on his utility. First, if his bid b places him in E2, then (for Δb sufficiently small) $x < b < b+\Delta b < y$, he continues to trade, but the price he pays increases by Δb. This decreases his utility by Δb. Second, if the bid b places him in E3 and if, in addition, $b < x < b+\Delta b < y$, then he jumps over x, his new bid $b+\Delta b$ becomes the $(m+1)^{\text{st}}$ smallest value in the entire sample of $m+n$ bids/asks, and he trades at the price $b+\Delta b$. His utility therefore increases from 0 to $\mathbb{E}[v|\sigma, b < x < b+\Delta b] - (b+\Delta b)$.

Given the small $\Delta b > 0$, the probability of the first event is $\Pr[x < b < y|\sigma]$ and the probability of the second is approximately $f^B_{x|\sigma}(b|\sigma)\Delta b$. Therefore, the change in the focal buyer's expected utility from increasing his bid by Δb is

$$\pi^B_b(b|\sigma)\,\Delta b \approx \Big\{(\mathbb{E}[v|\sigma, b < x < b+\Delta b] - (b+\Delta b))\, f^B_{x|\sigma}(b|\sigma) - \Pr[x < b < y|\sigma]\Big\}\,\Delta b,$$

where $\pi^B_b(b|\sigma)$ is the focal buyer's marginal utility conditional on his signal σ and bid b. Taking the limit as $\Delta b \to 0$ his FOC is therefore

$$\pi_b(b|\sigma_B) = (\mathbb{E}[v|\sigma, x = b] - b)\, f^B_{x|\sigma}(b|\sigma) - \Pr[x < b < y|\sigma] = 0 \tag{4}$$

or

$$b = \mathbb{E}[v|\sigma, x = b] - \frac{\Pr[x < b < y|\sigma]}{f^B_{x|\sigma}(b|\sigma)}. \tag{5}$$

The negative term in (5) captures from the first order perspective the focal buyer's ability to influence the price at which he trades. We refer to it as the *strategic term.* The other term, $\mathbb{E}[v|\sigma, x = b]$, is the *price-taking term.* In the CPV case, it simply equals the buyer's value v

because $\sigma = v$. In the CIV case, generally $\mathbb{E}[v|\sigma, x = b] \neq \sigma$; examples of equilibria in Section 4.4 show how this term adjusts the signal to protect the buyer from a winner's curse.

We next turn to the first order condition derived from a seller's decision problem. Select a focal seller and let x denote the m^{th} order statistic and y the $(m+1)^{\text{st}}$ order statistic in the ordered vector of bids and asks from the m buyers and the $n-1$ other sellers. Given a realization of other traders' signals, the focal seller's signal σ and his ask a, one of three events occurs:

E1′: If $x < y < a$, then the price is $p = y$, the seller does not trade, and his utility is 0.

E2′: If $x < a < y$, then the price is $p = a$, the seller does not trade, and his utility is 0.

E3′: If $a < x < y$, then the price is $p = x$, the seller trades, and his utility is $x - \mathbb{E}[c|\sigma, x = p]$.

Observe that even though the focal seller sets the price in event E2′, he never simultaneously sets the price and trades. Consequently he has no incentive to influence the price.

Suppose the seller decides to increase his ask from a to $a + \Delta a$ where $\Delta a > 0$ is small. This only affects his utility if the ask a is in E3′ and increasing it to $a + \Delta a$ jumps him over x and places him in E2′, thereby going from trading to not trading. His utility then decreases by $x - \mathbb{E}[c|\sigma, a < x < a + \Delta a]$. The probability of jumping over x is approximately $f^S_{x|\sigma}(a|\sigma)\,\Delta a$, where $f^S_{x|\sigma}(\cdot|\sigma) \neq f^B_{x|\sigma}(\cdot|\sigma)$ because the order statistic x faced by a focal seller is defined for a different ordered vector of bids/asks than the order statistic x faced by a focal buyer. The focal seller's marginal utility from increasing his ask by Δa is approximately

$$\pi^S_a(a|\sigma)\,\Delta a \approx -\left(a - \mathbb{E}[c|\sigma_S, a < x < a + \Delta a]\right) f^S_{x|\sigma}(a|\sigma_S)\,\Delta a.$$

As $\Delta a \to 0$, this implies the FOC

$$\pi^S_a(a|\sigma_S) = a - \mathbb{E}[c|\sigma_S, x = a] = 0 \tag{6}$$

or

$$a = \mathbb{E}[c|\sigma_S, x = a] \tag{7}$$

because $f^S_{x|\sigma}$ has full support on $\mathbb{R}$. The focal seller's FOC thus consists only of a price-taking term. Its interpretation parallels that of the buyer's: in the CPV case, $c = \mathbb{E}[c|\sigma, x = a]$ because $\sigma = c$; in the CIV case, generally $\mathbb{E}[c|\sigma_S, x = a] \neq c$ as the seller protects himself from a winner's curse.

Finally, (5) and (7) are intuitive and are therefore useful for the text of our paper. Appendix A derives expanded versions of the FOCs, (22) and (29), that use the conditional independence of signals and values/costs upon the state μ to produce formulas that are useful for computation and in some proofs. These alternative formulas are used in all numerical experiments in the paper.

4 Existence and Uniqueness of Offset Equilibria

An *offset strategy* has the form $B(\sigma_B) = \sigma_B + \lambda_B$ for buyers and $S(\sigma_S) = \sigma_S + \lambda_S$ for sellers, where $\lambda_B, \lambda_S \in \mathbb{R}$. We establish the following numerical result that, as with all of our numerical results, holds in both the CIV and CPV cases:

Numerical Result I *Consider either the CPV or the CIV case. There exists a unique pair of offset strategies that solve the FOCS (5) and (7). This pair of offset strategies defines an equilibrium. No other pair of strategies satisfying A2 (either offset or not) exists that defines an equilibrium.*

For the CPV case Theorem 1 proves the existence part of Numerical Result I. Its proof is in Appendix C.

Theorem 1 *Consider the CPV case. If sellers play their dominant strategy $S(c) = c$, then a negative constant $\lambda(G_\varepsilon, n, m)$ exists such that the strategy $B(v) = v + \lambda(G_\varepsilon, n, m)$ satisfies the FOC (5) at all $v \in \mathbb{R}$.*

We begin this section by discussing how equilibrium is computed in the paper. Next we present numerical examples in the CPV and CIV cases that typify what we have found to hold in general. These examples are used to explain the observations and reasoning behind Numerical Result I. We then discuss the sufficiency of the first order approach, addressing first its numerical demonstration and then the difficulties specific to double auctions that thwart a formal proof of sufficiency. Finally, we conclude with a comparative statics exercise that explores the dependence of the offset equilibrium upon m, n and the variances of G_ε and G_δ when each is a normal distribution.

4.1 Computation of Equilibrium

Expansion of FOCs (5) and (7) reveals that each is a differential equation in B^{-1} and S^{-1}. We follow a methodology developed in Satterthwaite and Williams (1989a) in the case of independent private values to solve these equations. For $\beta \in \mathbb{R}$, let $\sigma_B(\beta)$ and $\sigma_S(\beta)$ denote the signals of a buyer and a seller at which they bid/ask β, i.e., $\sigma_B(\beta) \equiv B^{-1}(\beta)$ and $\sigma_S(\beta) \equiv S^{-1}(\beta)$. The FOCs can be represented as a linear system of equations whose solution is a vector field

$$\overrightarrow{\mathcal{V}}(\sigma_B, \beta, \sigma_S) \equiv \left(\dot{\sigma}_B, \frac{d\beta}{d\beta} = 1, \dot{\sigma}_S\right)$$

in $\mathbb{R}^3$. A standard result in differential equations (e.g., Arnold (1973, Thm. 7.1)) states that a solution to the FOCs (5) and (7) can be traced out by following the vector field $\overrightarrow{\mathcal{V}}$ using any $(\sigma_B, \beta, \sigma_S) \in \mathbb{R}^3$ at which $\dot{\sigma}_B$ and $\dot{\sigma}_S$ are finite as an initial point. The projection of such a solution curve into the (σ_B, β)-plane produces a graph of the strategy B while its projection into the (σ_S, β)-plane produces a graph of the strategy S. As discussed in section 4.2 below, the representation of the FOCs as a vector field is important because the geometric properties of this vector field underlie the uniqueness conclusions of Numerical Result I.

The computation of equilibrium is further simplified by a property of this vector field that arises from the invariance of the model, namely, $\overrightarrow{\mathcal{V}}$ is invariant with respect to translations along the 45° diagonal of $\mathbb{R}^3$. The property both formally and intuitively guarantees equilibria in offset strategies.

Lemma 1 *Consider the CIV case. For any* $(\sigma_B, \beta, \sigma_S) \in \mathbb{R}^3$ *and any* $\rho \in \mathbb{R}$*, the vector field* $\overrightarrow{\mathcal{V}}$ *satisfies*

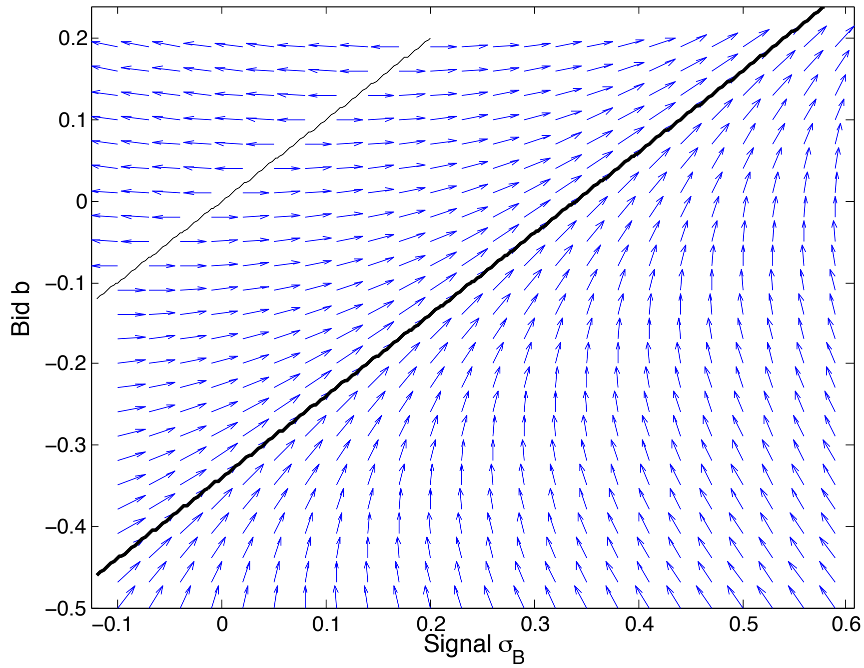
$$\overrightarrow{\mathcal{V}}(\sigma_B, \beta, \sigma_S) = \overrightarrow{\mathcal{V}}(\sigma_B + \rho, \beta + \rho, \sigma_S + \rho).$$

The lemma is proven in Appendix B by showing that the coefficients of the linear system that defines $\overrightarrow{\mathcal{V}}$ are constant to translations along the 45° diagonal of $\mathbb{R}^3$.

Lemma 1 implies that if a point $\omega^* = (\sigma_B^*, \beta^*, \sigma_S^*)$ can be found at which $\overrightarrow{\mathcal{V}}(\omega^*) = (1,1,1)$, then the line $(\sigma_B^* + \rho, \beta^* + \rho, \sigma_S^* + \rho)_{\rho\in\mathbb{R}}$ is a solution curve for the vector field. The solution curve defines the pair of offset strategies $B(\sigma_B) = \sigma_B + \lambda_B$ and $S(\sigma_S) = \sigma_S + \lambda_S$ that solve the FOCs for equilibrium for the constants $\lambda_B = \beta^* - \sigma_B^*$ and $\lambda_S = \beta^* - \sigma_S^*$. This suggests a method of computing equilibrium: fixing σ_B^*=0 without loss of generality, we solve for (β^*, σ_S^*) at which $\dot{\sigma}_B = \dot{\sigma}_S = 1$. The problem of solving a differential equation for a solution curve in $\mathbb{R}^3$ is in this way replaced by the simpler problem of solving two equations in two real variables.

4.2 Numerical Example: The Vector Field $\overrightarrow{\mathcal{V}}$

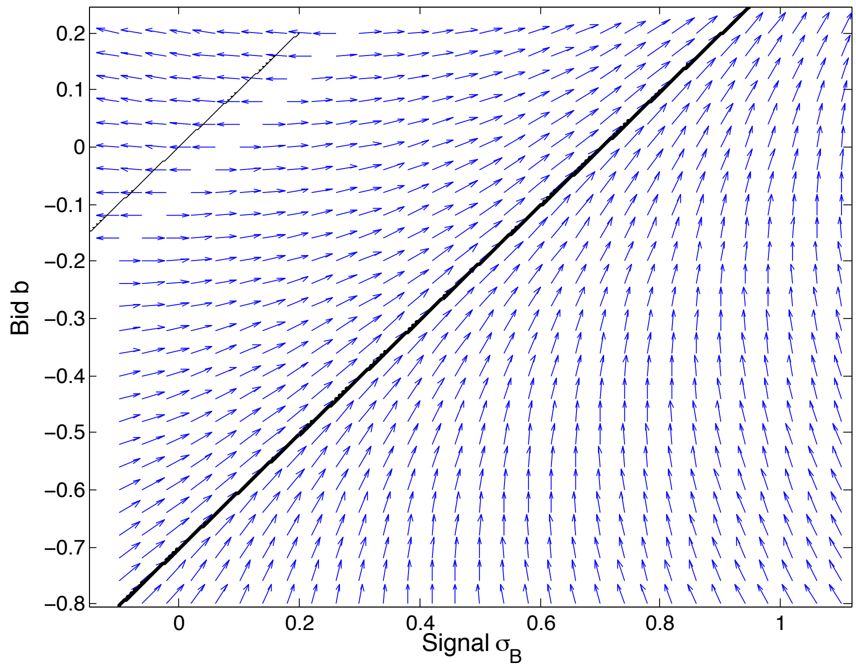
We explore numerically in this section the properties of the vector field $\overrightarrow{\mathcal{V}}$. We begin in the CPV case wherein the normalized vector field can be depicted in $\mathbb{R}^2$ due to the dominant strategy of each seller of setting his ask equal to his cost. Figure 1 depicts $\overrightarrow{\mathcal{V}}$ in the case of $m = n = 4$ and G_ε standard normal. The light 45° line in the figure is $v = b$, i.e., bidding one's value, while the heavier line below it is an offset solution to the buyers' FOC. We note two properties of this figure. First, there exists a unique offset solution to the buyers' FOC. Second, any other solution curve to the buyers' FOC fails to define a strategy that is increasing and differentiable on the entire real line: tracing in the direction of decreasing v, a solution curve through a point above the offset solution terminates at the line $v = b$, while a solution curve through a point below the offset solution curls back on itself. These graphical properties underlie our claim of existence and uniqueness of equilibrium satisfying A2. These properties of $\overrightarrow{\mathcal{V}}$ carry over to the three dimensional CIV case. Suppose now that the preference term ε and noise term δ of each trader are each drawn from the standard normal distribution. The two panels of Figure 2 show projections of the vector field when $m = n = 4$. With foresight as to what the equilibrium is, assume that the sellers play the offset strategy $S(\sigma_S) = \sigma_S + 0.2172$. The left-hand panel depicts the vector field $(\dot{\sigma}_B, 1)$ in the plane $(\sigma_B, \beta, \beta - 0.2172)$ parameterized by $(\sigma_B, \beta) \in \mathbb{R}^2$ within the space $(\sigma_B, \beta, \sigma_S) \in \mathbb{R}^3$, where $\sigma_S = \beta - 0.2172$ is a seller's signal at which he asks β. The invariance of the vector field with respect to translations in the $\beta = \sigma_B$ direction is apparent in the figure. Inspection of the graph shows that along the line $\sigma_B = \beta + 0.7036$ the slope is unity, $\dot{\sigma}_B(\beta + 0.7036, \beta, \beta - 0.2172) = 1$. The right-hand panel repeats this construction from the viewpoint of a seller, assuming that the buyers play the offset strategy $B(\sigma_B) = \sigma_B - 0.7036$. Inspection of the figure shows that the inverse offset strategy that solves the seller's FOC is the $\sigma_S = \beta - 0.2172$.



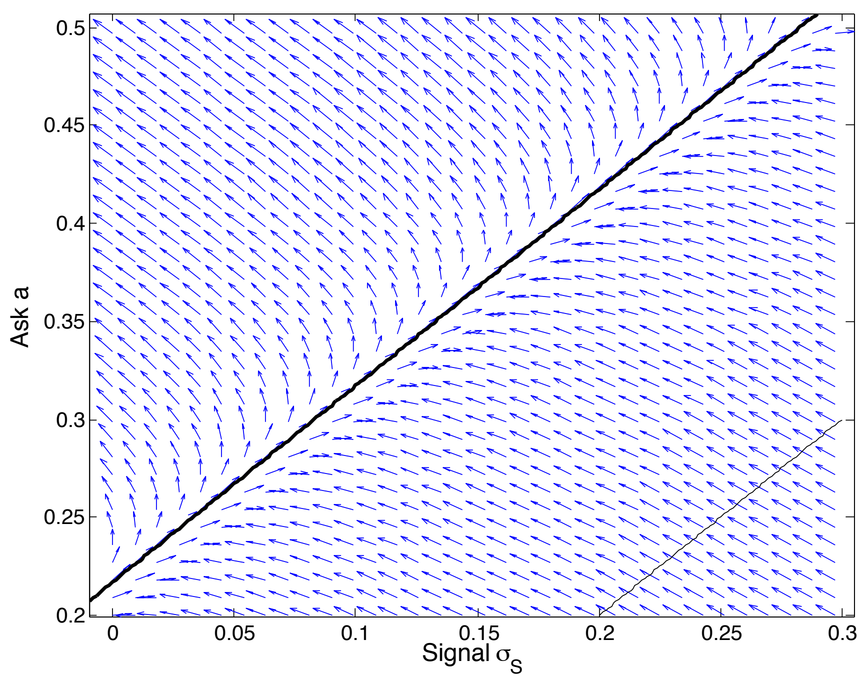
Buyer

Figure 1: The normalized vector field $\overrightarrow{\mathcal{V}}$ for buyers in the CPV case ($m = n = 4$, G_ε standard normal). The thick line signifies the solution to the buyer's FOC (5). The thin line is the 45° diagonal.

We make three observations concerning Figure 2. Consider first the 45° line (the thin black line) that corresponds to traders submitting bids/asks equal to their signals. It can be shown in the case of $m = n$ that if buyers are not strategic and use only their price-taking terms in selecting their bids (as sellers do), then the unique offset solution is $\lambda_B = \lambda_S = 0$, i.e., traders bid/ask their signals. The gap between the 45° line and the offset solution for $m = n = 4$ in Figure 2 (thick black line) is therefore a measure of buyers' strategic behavior (which of course also affects a seller's calculation of his ask). Second, in both panels there are 45° lines across which the relevant $\dot{\sigma}$ changes from positive to negative. This appears as an "empty region" in the buyer's panel just below the line $\sigma_B = b$ and it lies just below the offset solution in the seller's figure. Consider a point below this "line of singularity" $\dot{\sigma}_B = 0$ in the buyer's panel and increase the value of b. As derived in Appendix B, the singularity is the point at which the marginal expected gain from trading at the price b changes from positive to negative when conditioned on the following event: b equals the m^{th} smallest bid/ask x of the other traders when it is the bid of some other buyer. As depicted in Figure 1, this singularity occurs in the CPV case on the 45° line $v - b = 0$ on which the buyer's value v equals his bid b; as b increases, his marginal gain $v - b$ from trading at the price b changes from positive to negative along this line. It occurs in the CIV case along a different line from the one that equates a buyer's signal with his bid because his marginal expected gain from trading is conditioned on the event specified above. Unlike the CPV case, the line of singularity in



(a) Buyer



(b) Seller

Figure 2: The normalized vector field $\overrightarrow{\mathcal{V}}$ defined by the FOCs ($m = n = 4$, G_ε, G_δ standard normal). The thick line in each figure is the offset solution to the FOCs (5) and (7) for each side of the market. The thin line is the 45° diagonal.

the CIV case thus varies with m, n and the choice of the distributions. A similar analysis applies to the seller's figure.

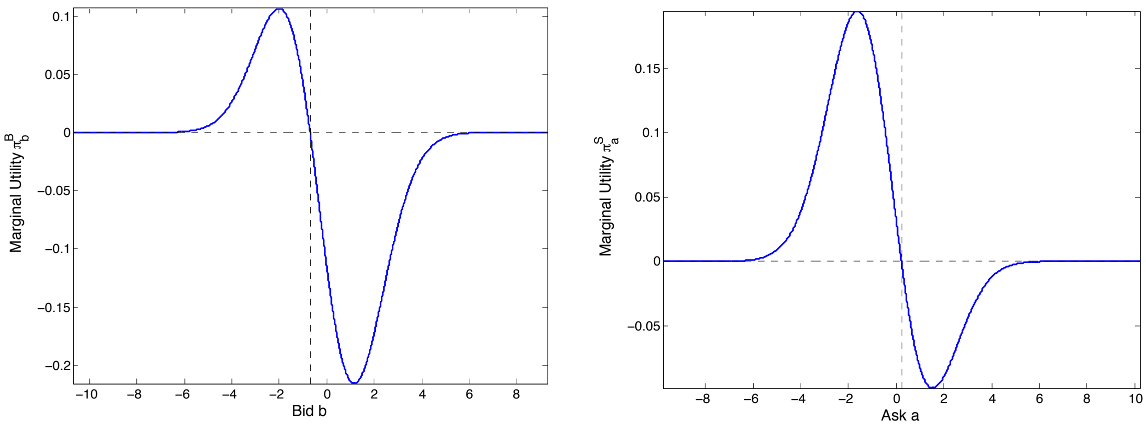
Third, while a solution curve exists through every point in the plane (σ_B, β) in the top panel at which $\dot{\sigma}_B$ is nonzero, the offset solution is the only solution curve that defines an increasing strategy for all $\sigma_B \in \mathbb{R}$. As in the CPV case, solution curves above the offset solution line terminate along the line $\dot{\sigma}_B = 0$ while solution curves below the offset solution turn back on themselves and fail to define increasing strategies. It is this observation that leads to the statement in Numerical Result I that offset solutions to the FOCs are unique and determine the only solutions to the FOCs satisfying A2 that can define equilibrium. In summary, if the sellers play the offset strategy $S(\sigma_S) = \sigma_S + 0.2172$, then the unique solution to a buyer's FOC that defines an increasing strategy for all $\sigma_B \in \mathbb{R}$ is given by the offset strategy $B(\sigma_B) = \beta = \sigma_B - 0.7036$. Similarly, given the buyers' use of this strategy, the offset solution for sellers $\sigma_S = \beta - 0.2172$ in the right panel is the only solution to the seller's FOC that defines an increasing strategy for all $\sigma_S \in \mathbb{R}$. The pair $(\lambda_B, \lambda_S) = (-0.7036, 0.2172)$ thus simultaneously solves the FOCs and is therefore a candidate for equilibrium.

These figures typify all of our calculations of the vector field in both the CPV and the CIV cases for various G_ε, G_δ and sizes of market in the following three respects: (i) there exists a unique pair of constants (λ_B, λ_S) that define offset strategies forming a solution curve to the vector field; (ii) this offset pair is straightforward to calculate using the method discussed above; (iii) all other solution curves fail to define increasing and differentiable strategies across the entire real line. Only the issue of sufficiency of the first order approach remains to be addressed in support of Numerical Result I.

4.3 Sufficiency of the First Order Approach

Graphing marginal expected utility is effective for verifying that offset strategies that satisfy the FOCs define an equilibrium. Suppose offsets (λ_B, λ_S) solve the FOCs (5) and (7). Pick a focal buyer and let his signal be $\sigma_B = 0$. Graph his marginal expected utility as he deviates from bidding λ_B. If this graph transitions from positive to negative at λ_B, then λ_B is a best response. Invariance and the use of offset strategies imply that if λ_B is a best response at $\sigma_B = 0$, then $\sigma_B + \lambda_B$ is a best response for any $\sigma_B \in \mathbb{R}$. For a focal seller check, in the same way, that λ_S is a best response at $\sigma_S = 0$. If the model did not possess invariance, verification by graphing would be harder because it would be necessary to check that the offset pair (λ_B, λ_S) is a best response at a sufficiently fine grid of points $(\sigma_B, \sigma_S) \in \mathbb{R}^2$.

All offsets solutions to the FOCs throughout this paper are shown to define equilibria by graphing marginal expected utility of the focal trader as a function of his bid/ask. Figure 3 in the case of $m = n = 4$ and G_ε, G_δ standard normal is representative of these graphs. The focal trader's signal is fixed at 0 in these calculations. Figure 3 (a) depicts a focal buyer's marginal expected utility while (b) depicts a focal seller's, both under the assumption that the other traders are using the offset solutions $(\lambda_B, \lambda_S) = (-0.7036, 0.2172)$ from the CIV example in section 4.2. Both (a) and (b) depict the appropriate change in sign of marginal expected utility at the offset value, which ver-



(a) Buyer

(b) Seller

Figure 3: Marginal expected utility for focal traders ($m = n = 4$, G_ε, G_δ standard normal)). The vertical dashed line indicates the offset solution to the focal trader's FOC.

ifies sufficiency. It is worth noting, however, that neither marginal utility is monotone decreasing, as each converges to zero as the bid/ask becomes large in magnitude and its impact upon expected utility becomes neglible. Unlike other problems in mechanism design, restricting the distributions to insure decreasing marginal utility is thus not a promising approach in this setting.

We next discuss the difficulties of finding a useful analytical condition for guaranteeing the sufficiency of the first order approach. A candidate is provided by Theorem 2.

Theorem 2 *Consider the CIV case. Suppose that for all $(\sigma_B, \sigma_S) \in \mathbb{R}^2$ the strategies $\langle B, S\rangle$ solve the buyers' and sellers' FOCs (5) and (7) and satisfy assumption A2. The following are sufficient conditions for $\langle B, S\rangle$ to be an equilibrium:*

1. *for all $b \in \mathbb{R}$, the function*

$$\mathbb{E}[v|\sigma_B, x = b] - \frac{\Pr[x < b < y|\sigma_B]}{f^B_{x|\sigma}(b|\sigma_B)} \tag{8}$$

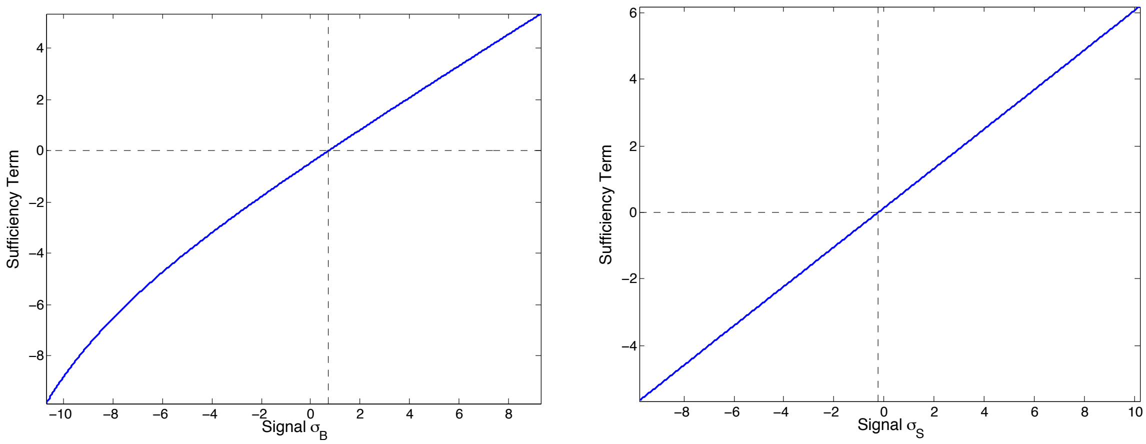
   is increasing in σ_B where the order statistics x and y are from the perspective of a focal buyer;

2. *for all $a \in \mathbb{R}$, the function*

$$\mathbb{E}[c|\sigma_S, x = a] \tag{9}$$

   is increasing in σ_S where the order statistic x is from the perspective of a focal seller.

The proof follows an argument of Milgrom and Weber (1982, Thm. 14) and is in online Appendix H. The monotonicity requirement in (8) serves the same purpose for the sufficiency of the first order approach in the multilateral BBDA in the CIV case as the condition that Kadan (2007, A.2) uses to prove existence of equilibrium in the bilateral BBDA in the CPV case. As in Milgrom and Weber (1982), our condition (8) for the multilateral CIV case as well as the condition of Kadan



(a) Buyer (b) Seller

Figure 4: The terms (8) and (9) for focal traders ($m = n = 4$, F, G standard normal). The vertical dashed line indicates the offset solution to the focal trader's FOC.

(2007) for the bilateral CPV case all involve the valuation of the marginal buyer conditional on trading minus a fraction. The numerator in the fraction is the marginal expected cost to a buyer from increasing his bid and thereby driving up the price that he pays and the denominator is the marginal probability of acquiring an item by increasing his bid.

The difficulty of conditions (8) and (9) in the multilateral CIV case is that they state sufficient conditions on complicated functions of the distributions G_ε and G_δ and the strategies B and S, not directly on the distributions that are the fundamentals of our model. The complexity has two aspects. First, the fact that buyers and sellers behave differently creates asymmetry in the sample of bids/asks, which complicates the distributions of the order statistics x and y. This complexity is typically avoided in symmetric auction models. Second, the focus in the BBDA is upon the "interior" order statistics $x = s_{(m)}$ and $y = s_{(m+1)}$ in a sample of $n + m - 1$ bids/asks. Except for cases in which there is a single trader on one side of the market, these order statistics are considerably more complicated to work with than the extremal order statistics that are focal in the theory of auctions.

Numerically, Theorem 2 works in establishing sufficiency equally as well as graphing expected marginal utility. Figure 4 graphs the terms (8) and (9) in Theorem 2 for the pair of offset solutions $(\lambda_B, \lambda_S) = (-0.7036, 0.2172)$ from the CIV example in section 4.2. It shows that these terms are increasing with respect to the trader's signal and therefore establishes that the offsets are an equilibrium. Even though we cannot prove that our testbed distributions satisfy (8) and (9) for all m and n, our numerical work suggests that these terms behave similarly to the analogous terms from auction theory.

$m \backslash n$	2	4	8	16
2	$-1.3404, 0.4124$	$-0.8372, 0.4912$	$-0.3642, 0.6508$	$0.0361, 0.8546$
4	$-1.2189, 0.1332$	$-0.7036, 0.2172$	$-0.2657, 0.3948$	$0.1128, 0.6192$
8	$-1.2084, -0.1712$	$-0.7431, -0.0787$	$-0.3417, 0.1091$	$0.0212, 0.3494$
16	$-1.3011, -0.4677$	$-0.8853, -0.3756$	$-0.5175, -0.1886$	$-0.1754, 0.0614$

Table 1: Equilibrium offsets λ_B, λ_S for different values of m and n in the case of G_ε, G_δ standard normal.

τ_δ	λ_B, λ_S
2	$-0.5135, 0.0959$
4	$-0.4246, 0.0443$
8	$-0.3817, 0.0211$
16	$-0.3606, 0.0103$
∞	$-0.3398, 0.$

(a) $\tau_\varepsilon = 1$

τ_ε	λ_B, λ_S
2	$-0.8017, 0.3673$
4	$-1.1058, 0.6694$
8	$-1.8134, 1.3536$
16	$-2.6416, 2.3328$
∞	$-\infty, \infty$

(b) $\tau_\delta = 1$

k	λ_B, λ_S
2	$-0.4976, 0.1536$
8	$-0.2488, 0.0768$
32	$-0.1244, 0.0384$
128	$-0.0622, 0.0192$
512	$-0.0311, 0.0096$

(c) $\tau_\varepsilon = \tau_\delta = k$

Table 2: Results for buyers' and sellers' offsets λ_B, λ_S for $m = n = 4$ and different values of τ_ε and τ_δ (G_ε, G_δ normal).

4.4 Numerical Example: A Comparative Statics Exercise in m, n, G_ε, and G_δ

This example concerns the CIV case in which G_ε and G_δ are both normal. We explore here the dependence of the offset solutions on the numbers of traders m and n and the variances of these two distributions. Turning first to the number of traders, Table 1 presents the offset solutions for values of m and n between 2 and 16 when both G_ε and G_δ are standard normal. Recall that a focal buyer weighs two effects in choosing his offset: (i) the possibility of affecting price in his favor (strategic); (ii) the estimation of his value given his signal and the event that his bid sets the price (price-taking). Holding n constant and increasing the number m of buyers, effect (i) diminishes as the likelihood that a buyer influences price goes to zero. Effect (ii), however, increases because a buyer who trades and sets the price knows that an increasing number m of bids/asks are below his. As strategies are increasing, this means that the focal buyer receives increasing evidence as m increases that his signal is relatively high in the sample. As we go down any column in Table 1 we therefore observe a buyer's offset first increases due to effect (i) and then decreases as effect (ii) dominates. This later effect is the classic response of the focal buyer protecting himself from a winner's curse. On the other side of the market, a seller's offset increases monotonically for fixed m as n increases due purely to effect (ii), as there is no strategic term in his FOC.

To avoid confusion with our notation for signals, we now use τ_ε, τ_δ to denote the respective *precisions* of G_ε and G_δ, i.e., τ_ε, τ_δ are the reciprocals of their variances. Table 2 explores the dependence of the offset solutions on τ_ε and τ_δ for $m = n = 4$. In Panel (a) we fix $\tau_\varepsilon = 1$ and vary the noisiness of a trader's signal by changing τ_δ. For $\tau_\delta = \infty$, a trader's signal equals his value and hence this is the CPV case. Sellers in this case report their true costs/signals and so their offset is zero while buyers underbid strategically. As τ_δ decreases, both buyers' and sellers' offsets increase in magnitude because the more unsure a trader is of his value/cost, the more aggressively he acts to protect himself from a winner's curse. In Panel (b) we explore the dependence of the offset solutions as we vary τ_ε, holding $\tau_\delta = 1$. For $\tau_\varepsilon = \infty$ a trader's value equals μ; this is a common value environment with no gains from trade and so the offsets tend to infinity. The more gains from trade there are (i.e., as τ_ε decreases), the more the offsets decrease as the winner's curse is mitigated. Finally in Panel (c) we scale both noises by the same amount and, as expected, this scales the offsets proportionally. Doubling the standard deviation results in a doubling of the offsets; this is simply a rescaling of the problem. Because precision is the reciprocal of variance, multiplying both precisions by 4 therefore causes the offsets to fall by half.

5 Convergence

We investigate in this section the convergence properties of the BBDA as the market increases in size and a buyer's ability to influence price diminishes. The three topics explored are: (i) the amount by which a buyer strategically underbids in order to influence price in his favor; (ii) the inefficiency in the market allocation this underbidding causes; (iii) the error in the realized market price as an estimate of p^{REE}, the REE price in the limit market, for each value of the state μ.

5.1 Convergence to Price-Taking Behavior

The offset solutions (λ_B, λ_S) to the FOCs (5) and (7) satisfy the equations

$$\sigma_B + \lambda_B = \underbrace{\mathbb{E}[v|\sigma_B, x = \sigma_B + \lambda_B]}_{\text{price-taking term}} - \underbrace{\frac{\Pr[x < \sigma_B + \lambda_B < y|\sigma_B]}{f_x^B(\sigma_B + \lambda_B|\sigma_B)}}_{\text{strategic term}}, \tag{10}$$

$$\sigma_S + \lambda_S = \underbrace{\mathbb{E}[c|\sigma_S, x = \sigma_S + \lambda_S]}_{\text{price-taking term}}, \tag{11}$$

where, as in section 2, x and y are the m^{th} and $(m+1)^{\text{st}}$ order statistics of other traders' bids and asks from the perspective of the focal trader. Terms in (10) and (11) are labeled as in section 3; when (λ_B, λ_S) satisfies (10) and (11) in equilibrium, we refer to these terms respectively as the *equilibrium price-taking* and *strategic* terms. As demonstrated in the example of section 4.4, the equilibrium price-taking term of each side of the market may or may not converge to zero as the market size η increases; depending upon the relative sizes of m and n of the two sides of the market, the necessity of protecting oneself from a winner's curse can grow or decrease for a particular side of the market as η increases. The likelihood that a buyer can influence price in his favor, however, diminishes as η increases. Convergence to price-taking behavior thus means convergence to zero of the buyers' equilibrium strategic term as the market size η increases. We have the following numerical result:

Numerical Result II *Consider the CIV case for distributions G_ε and G_δ, fixed values of m and* *n, and market size $\eta \in \mathbb{N}$. A buyer's equilibrium strategic term is $O(1/\eta)$ in the market with ηm buyers and ηn sellers, i.e., there exists a constant $K_1(m, n, G_\varepsilon, G_\delta)$ such that, for the equilibrium offsets (λ_B, λ_S) and all values σ_B of a buyer's signal,*

$$\frac{\Pr[x < \sigma_B + \lambda_B < y|\sigma_B]}{f_x^B(\sigma_B + \lambda_B|\sigma_B)} \leq \frac{K_1(m, n, G_\varepsilon, G_\delta)}{\min(m, n)}.$$

In the CPV case a buyer's equilibrium strategic term is λ_B, the equilibrium offset. The equilibrium price-taking term is zero for both buyers and sellers. Numerical Result II can be proven in the CPV case with a rather weak restriction on G_ε:

Theorem 3 *Numerical Result II holds as a theorem in the CPV case if the distribution of preference terms G_ε satisfies*

$$\lim_{x \to -\infty} \sup \frac{G_\varepsilon(x)}{g_\varepsilon(x)} < \infty. \tag{12}$$

The proof is in Appendix D. Condition (12) is strictly weaker than the assumption that the hazard rate $g_\varepsilon(x)/G_\varepsilon(x)$ is strictly decreasing, which is a common assumption in mechanism design. Of our testbed distributions, the normal and Laplace distributions satisfy (12) but the Cauchy distribution does not.

5.2 Convergence to Allocative Efficiency

We next address the diminishing impact of strategic behavior on allocative efficiency as the market size increases. The allocative performance of the BBDA is measured by its realized gains from trade. For a sample of signals in the market with m buyers and n sellers, let $(s_1, ..., s_{m+n})$ be the equilibrium bids/asks where s_i is the equilibrium bid of buyer i for $i \in \{1, .., m\}$ and s_j is the equilibrium ask of seller $j - m$ for $j \in \{m+1, ..., m+n\}$. The order statistic $s_{(m+1)}$ is the price. The equilibrium gains from trade are then

$$GFT^{\text{eq}} = \sum_{\substack{s_i \geq s_{(m+1)} \\ i \in \{1,...,m\}}} v_i - \sum_{\substack{s_j < s_{(m+1)} \\ j \in \{m+1,...,m+n\}}} c_j.$$

Let $\overline{GFT}^{\text{eq}}$ denote the ex ante equilibrium gains from trade conditional on the state μ:

$$\overline{GFT}^{\text{eq}} = \mathbb{E}\left[GFT^{\text{eq}} \,|\mu\right].$$

Invariance and the use of offset strategies implies that this quantity does not vary with the state μ.

Our benchmark for evaluating the BBDA's equilibrium performance is its realized gains from trade if buyers were not to act strategically. In this way we isolate the cost of strategic behavior on the performance of the mechanism. Suppose each buyer decides to ignore the expected gains from shading his bid downwards but continues to protect himself from a winner's curse. This decision to not act strategically makes his decision problem identical to a seller's decision problem. Consequently the FOC for all buyers and sellers are

$$\sigma_B + \lambda_B = \mathbb{E}[v|\sigma_B, x = \sigma_B + \lambda_B], \tag{13}$$

$$\sigma_S + \lambda_S = \mathbb{E}[c|\sigma_S, x = \sigma_S + \lambda_S], \tag{14}$$

which are identical as equations and hence reduce to the single equation in the *price-taking offset* λ^{pt},

$$\sigma + \lambda^{\mathrm{pt}} = \mathbb{E}[z|\sigma, x = \sigma + \lambda^{\mathrm{pt}}]. \tag{15}$$

Let $(t_1, ..., t_{m+n})$ denote a vector of price-taking bids/asks when all traders offset their signals by λ^{pt} in choosing their bids/asks, where $i \in \{1, .., m\}$ and $j - m$ for $j \in \{m+1, ..., m+n\}$ again identify buyers and sellers, respectively. The BBDA's price is $t_{(m+1)}$ and the ex post gains from trade from these price-taking bids and asks equals

$$GFT^{\mathrm{pt}} = \sum_{\substack{t_i \geq t_{(m+1)} \\ i \in \{1,...,m\}}} v_i - \sum_{\substack{t_j < t_{(m+1)} \\ j \in \{m+1,...,m+n\}}} c_j.$$

Let $\overline{GFT}^{\mathrm{pt}}$ denote the ex ante potential gains from trade conditional on μ:

$$\overline{GFT}^{\mathrm{pt}} = \mathbb{E}\left[GFT^{\mathrm{pt}}\,|\mu\right].$$

Again because of invariance and the fact that the price-taking bids/asks are all constant offsets, $\overline{GFT}^{\mathrm{pt}}$ has the same value in all states μ.

The measure of allocative efficiency we use is *relative inefficiency*, which calculates the fraction of the ex ante potential gains from trade that are not achieved in the market due to the strategic behavior of buyers:

$$\frac{\overline{GFT}^{\mathrm{pt}} - \overline{GFT}^{\mathrm{eq}}}{\overline{GFT}^{\mathrm{pt}}}. \tag{16}$$

We have the following numerical result.

Numerical Result III *Consider the CIV case for distributions G_ε and G_δ, fixed values of m and n, and market size $\eta \in \mathbb{N}$. Relative inefficiency is $O(1/\eta^2)$, i.e., there exists a constant $K_2(m, n, G_\varepsilon, G_\delta)$ such that*

$$\frac{\overline{GFT}^{\mathrm{pt}} - \overline{GFT}^{\mathrm{eq}}}{\overline{GFT}^{\mathrm{pt}}} \leq \frac{K_2(m, n, G_\varepsilon, G_\delta)}{\eta^2}$$

in each state μ.

In the CPV case, the price-taking offset λ^{pt} equals zero, each trader's price-taking bid/ask is simply his value/cost, and the benchmark GFT^{pt} is the ex post potential gains from trade. We have:

Theorem 4 *Numerical Result III holds as a theorem in the CPV case if the distribution of preference terms G_ε satisfies (12).*

The tail condition (12) is assumed here so that Theorem 3 can be applied to bound strategic behavior by buyers. The proof adapts an argument of Rustichini, Satterthwaite, and Williams (1994, Thm. 3.2) to the model of this paper and can be found in online Appendix I.

5.3 Convergence to Rational Expectations Equilibrium

This section contrasts two theoretical approaches to market equilibrium. In a REE approach, traders are assumed to see a state-revealing, market-clearing price and then make decisions to buy or sell that confirms that price as market clearing. In the approach of this paper, traders submit bids and asks into the BBDA based upon what they know at that moment and then a market-clearing price is calculated based upon these bids/asks. Theorem 5 below develops for our model a result of Reny and Perry (2006) that in the limit market the two approaches determine the same price. Our contribution is to demonstrate how this result extends to small markets by (i) establishing the rate at which the BBDA price converges to the REE price as the market becomes large and (ii) numerically showing that the BBDA's price is a meaningful estimate of the REE price in small markets. In this way we show how well the BBDA discovers the REE price in small markets and how fast that estimate improves as the market becomes larger.

5.3.1 The Limit Market and REE

Consider the CIV case. The *limit market* is the market in each state μ with m times a unit mass of buyers and n times a unit mass of sellers with values/costs and signals generated using the distributions G_ε, G_δ as in our finite model. Let $V(\sigma) \equiv \mathbb{E}[z|0,\sigma]$ be the expected value/cost of a trader conditional on the state $\mu = 0$ and his signal σ. Assumption A3 implies that V is strictly increasing and so invertible: given $\mu = 0$, $V^{-1}(\cdot)$ maps the trader's expected value/cost back to the signal that generates it.

The REE function $P^{\text{REE}} : \mathbb{R} \to \mathbb{R}$ determines the REE price in the limit market for each state μ. It is defined by two properties. First, it is invertible. Let Λ denote the function that recovers the state μ from the REE price, $\Lambda\left(p^{\text{REE}}\right) = \mu$. A REE price p^{REE} is thus *fully revealing* in the sense that a trader who observes p^{REE} can infer the state μ. Second, $P^{\text{REE}}(\mu) = p^{\text{REE}}$ clears the limit market in the state μ. Specifically, each trader learns his private signal σ, observes p^{REE}, and calculates his expected value/cost $\mathbb{E}\left[z|\Lambda\left(p^{\text{REE}}\right), \sigma\right]$. If he is a buyer, he buys one unit if and only if $\mathbb{E}\left[z|\Lambda\left(p^{\text{REE}}\right), \sigma\right] \geq p^{\text{REE}}$. If he is a seller, he sells his one unit if and only if $\mathbb{E}\left[z|\Lambda\left(p^{\text{REE}}\right), \sigma\right] \leq p^{\text{REE}}$. At price p^{REE} supply equals demand.

Part (i) of the following theorem solves for the REE price function $P^{\text{REE}}(\cdot)$ for the limit market. The BBDA also equates supply with demand in the limit market but with each trader determining his demand/supply based on his private signal, the market fundamentals, and the event in which his bid/ask equals the price. Part (ii) characterizes the offset equilibrium of the BBDA in the limit market and shows that its equilibrium price is the REE price.

Let

$$q \equiv \frac{m}{m+n}$$

denote the proportion of all traders who are buyers, and let $\xi_q^{\varepsilon+\delta} \equiv G_{\varepsilon+\delta}^{-1}(q)$ denote the q^{th} population quantile of $G_{\varepsilon+\delta}$.

Theorem 5 *Consider the CIV case. For fixed m and n, consider the limit market. Then:*

(i) The unique REE price in state μ is

$$p^{\text{REE}} \equiv \mu + V\left(\xi_q^{\varepsilon+\delta}\right). \tag{17}$$

The one-to-one mapping from the REE price to the state is $\Lambda\left(p^{\text{REE}}\right) = p^{\text{REE}} - V\left(\xi_q^{\varepsilon+\delta}\right)$.

(ii) In the BBDA, all traders play the equilibrium offset $\lambda_B = \lambda_S = V\left(\xi_q^{\varepsilon+\delta}\right) - \xi_q^{\varepsilon+\delta}$. This results in the equilibrium price being $\mu + V\left(\xi_q^{\varepsilon+\delta}\right)$, i.e., the equilibrium price is the REE price.

The proof is in online Appendix J and follows an argument of Reny and Perry (2006, sec. 3). Finally, the proof is easily specialized to the CPV case: (i) $V\left(\xi_q^{\varepsilon+\delta}\right)$ in (17) is replaced by ξ_q^{ε}, the q^{th} population quantile of G_ϵ, as $V(z) = z$ in this case; (ii) in the limit market, a trader's equilibrium strategy in the BBDA is to report his value/cost.

5.3.2 The Market Price as an Estimator of the REE Price

For fixed m, n, we again consider sequences of markets with ηm buyers and ηn sellers. Given the state μ, we evaluate in this section the error with which the BBDA's price estimates the REE price $P^{\text{REE}}(\mu) = p^{\text{REE}}$ for the limit market. We bound above the absolute error in this estimation by

$$\left|s_{(\eta m+1)} - p^{\text{REE}}\right| \leq \left|s_{(\eta m+1)} - t_{(\eta m+1)}\right| + \left|t_{(\eta m+1)} - p^{\text{REE}}\right|,$$

where $s_{(\eta m+1)}$ is the BBDA's equilibrium price and $t_{(\eta m+1)}$ is the price-taking price as defined in section 5.2. The term $\left|s_{(\eta m+1)} - t_{(\eta m+1)}\right|$ is the *strategic error*, which captures the effect of strategic underbidding by buyers upon the estimation of p^{REE}. The term $\left|t_{(\eta m+1)} - p^{\text{REE}}\right|$ is attributable to the fact that a sample of $m+n$ values/costs does not perfectly reflect the population together with the error that is attributable to the noise in trader signals. We refer to this term simply as *sampling error*. Numerical Result II implies that strategic error is $O(1/\eta)$. Sampling error in contrast is a random variable distributed on the half line $[0, \infty)$; it can be arbitrarily small or large. The following numerical result, states that the expected value of the sampling error is of strictly larger order than strategic error. Asymptotically, then, the effect of strategic error upon the estimation of p^{REE} vanishes completely because of the dominating effect of the sampling error.

Numerical Result IV *Consider the CIV case for distributions G_ε and G_δ, fixed values of m and n, and market size $\eta \in \mathbb{N}$. The strategic error in the BBDA's price $s_{(\eta m+1)}$ as an estimate of* *the REE price is*

$$\left|s_{(\eta m+1)} - t_{(\eta m+1)}\right| = O(1/\eta).$$

In any state μ, the expected sampling error is

$$\mathbb{E}\left[\left|t_{(\eta m+1)} - p^{\text{REE}}\right| \,\middle|\, \mu\right] = \Theta(1/\eta),$$

and the expected total error is therefore

$$\mathbb{E}\left[\left|s_{(\eta m+1)} - p^{\text{REE}}\right| \,\middle|\, \mu\right] = \Theta(1/\eta).$$

If G_ε satisfies condition (12) in Theorem 3 and, in addition, satisfies the tail condition,

$$\lim_{x\to\infty} \inf \frac{-\log(1 - G_\varepsilon(x))}{\log x} > 0, \tag{18}$$

then Numerical Result IV can be proven in the CPV case. The normal, Laplace, and Cauchy distributions all satisfy (18).[12]

Theorem 6 *Numerical Result IV holds as a theorem in the CPV case if G_ε satisfies (12) and (18).*

The proof is in Appendix E.

5.4 Numerical Examples: Convergence

5.4.1 The CPV Case

We illustrate our convergence results in the CPV case with $m = n = 1$ and G_ε standard normal. In this case $q = 1/2$, $\xi_q^\varepsilon = 0$ and $p^{\text{REE}} = \mu$. Recall that each trader's price-taking bid/ask t in the CPV case is identical to his true value/cost z. Convergence to price-taking behavior and to allocative efficiency are illustrated in Panel A of Table 3.[13] Column 1 lists the number η of traders on each side of the market. Column 2 contains the equilibrium offsets of buyers. As Theorem 3 states, the offsets decrease at the rate of $O(1/\eta)$, i.e., as η doubles the offset is cut in half. The ex ante potential gains from trade $\overline{GFT}^{\text{pt}}$ and the ex ante equilibrium gains from trade $\overline{GFT}^{\text{eq}}$ in any state μ are presented in columns 3 and 4. Relative inefficiency is tabulated in column 5. Consistent with Theorem 4, it vanishes at a rate of $O\left(1/\eta^2\right)$, i.e., as η doubles relative inefficiency drops by a factor of 4.

[12] It is straightforward to verify that each of these distributions satisfies

$$\lim_{x\to\infty} \frac{x g_\varepsilon'(x)}{-g_\varepsilon(x)} > 1.$$

Two applications of L'Hôpital's Rule reduce (18) to a form in which it is clear that this bound is sufficient to insure that (18) holds.

[13] The numbers in column 2 of Panel A in Table 3 and column 2 of Panel A in Table 4 are computed using numerical integration. All other numbers in these tables are calculated using Monte Carlo simulations.

Table 3: Convergence in the CPV case ($m = n = 1$, G_ε standard normal). In all panels, the size of market η is reported in column 1. **Panel A:** Column 2 is the equilibrium offset; columns 3 and 4 are respectively (i) the expected gains from trade when traders act as price takers and (ii) in the equilibrium of column 2 ; column 5 is the relative inefficiency (16). **Panel B:** Columns 2 and 3 are the variances of the errors in the price-taking price and the equilibrium price as estimates of the REE price; column 4 is the variance of their common asymptotic limit. **Panel C:** Columns 2 and 3 are the expected absolute errors in the price-taking price and the equilibrium price as estimates of the REE price; column 4 is the expected absolute difference between the price-taking price and the equilibrium price.

Panel A

η	λ_B	$\overline{GFT}^{\text{pt}}$	$\overline{GFT}^{\text{eq}}$	$(\overline{GFT}^{\text{pt}} - \overline{GFT}^{\text{eq}})/\overline{GFT}^{\text{pt}}$
2	-0.6896	1.3265	1.2221	0.0795
4	-0.3398	2.9008	2.8535	0.0163
8	-0.1639	6.0812	6.0653	0.0026
16	-0.0805	12.4604	12.4516	0.0007

Panel B

η	$\text{VAR}\left(t_{(\eta m+1)} - p^{\text{REE}} \vert \mu\right)$	$\text{VAR}\left(s_{(\eta m+1)} - p^{\text{REE}} \vert \mu\right)$	$\frac{1}{8\eta\phi^2(0)}$
2	0.3646	0.3834	0.3927
4	0.1887	0.1901	0.1963
8	0.0954	0.0958	0.0981
16	0.0482	0.0483	0.0491

Panel C

η	Exp. Sampling Error $\mathbb{E}\left[\left\vert t_{(\eta m+1)} - p^{\text{REE}}\right\vert \vert \mu\right]$	Exp. Total Error $\mathbb{E}\left[\left\vert s_{(\eta m+1)} - p^{\text{REE}}\right\vert \vert \mu\right]$	Exp. Strategic Error $\mathbb{E}\left[\left\vert s_{(\eta m+1)} - t_{(\eta m+1)}\right\vert \vert \mu\right]$
2	0.5329	0.4926	0.3198
4	0.3644	0.3457	0.1679
8	0.2549	0.2474	0.0819
16	0.1782	0.1756	0.0381

Panels B and C of Table 3 concern the BBDA's market price as an estimate of the REE price. Recall that $t_{(\eta m+1)}$ is the price in the BBDA when buyers use their price-taking bids while $s_{(\eta m+1)}$ is the price when buyers play their equilibrium offset strategy. Satterthwaite, Williams, and Zachariadis (2014, Thm. 2) establishes that $t_{(\eta m+1)} - p^{\text{REE}}$ and $s_{(\eta m+1)} - p^{\text{REE}}$ are asymptotically normal with mean zero and variance $\left(8\eta\phi^2(0)\right)^{-1}$, where ϕ is the density of the standard normal distribution. Panel B of Table 3 compares the variances of $t_{(\eta m+1)} - p^{\text{REE}}$ and $s_{(\eta m+1)} - p^{\text{REE}}$ with this asymptotic variance. It is notable how close the values of the variances are to their asymptotic limiting value even for these small values of η. This suggests that the asymptotic limit is meaningful for approximations involving $t_{(\eta m+1)}$ or $s_{(\eta m+1)}$ even in very small markets.

Panel C of Table 3 presents the expected values of total error, sampling error and strategic error

Table 4: Convergence in the CIV case ($m = n = 1$, G_ε, G_δ standard normal). In all panels, the size of market η is reported in column 1. **Panel A:** Column 2 is the buyer's strategic term computed at the equilibrium offsets; column 3 is the expected gains from trade $\overline{GFT}^{\text{eq}}$ in equilibrium and column 4 is the expected gains from trade $\overline{GFT}^{\text{pt}}$ when traders submit their price-taking bids/asks; column 5 is the relative inefficiency. **Panel B:** Columns 2 and 3 are the variances of the errors in the price-taking price and the equilibrium price as estimates of the REE price. **Panel C:** Columns 2 and 3 are the expected absolute errors of the price-taking price and the equilibrium price as estimates of the REE price; column 4 is the expected absolute difference between these two prices.

Panel A

η	$\frac{\Pr[x<\lambda_B<y\|\sigma_B]}{f_x^B(\lambda_B\|\sigma_B)}$	$\overline{GFT}^{\text{pt}}$	$\overline{GFT}^{\text{eq}}$	$(\overline{GFT}^{\text{pt}} - \overline{GFT}^{\text{eq}})/\overline{GFT}^{\text{pt}}$
2	0.9279	0.9395	0.7151	0.2389
4	0.4864	2.075	1.9354	0.0594
8	0.2326	4.3011	4.2434	0.0134
16	0.1139	8.8093	8.776	0.0037

Panel B

η	$\text{VAR}\left(t_{(\eta m+1)} - p^{\text{REE}}\|\mu\right)$	$\text{VAR}\left(s_{(\eta m+1)} - p^{\text{REE}}\|\mu\right)$
2	0.7216	0.8421
4	0.3741	0.3884
8	0.1915	0.1936
16	0.0972	0.0974

Panel C

η	Exp. Sampling Error $\mathbb{E}\left[\left\|t_{(\eta m+1)} - p^{\text{REE}}\right\| \|\mu\right]$	Exp. Total Error $\mathbb{E}\left[\left\|s_{(\eta m+1)} - p^{\text{REE}}\right\| \|\mu\right]$	Exp. Strategic Error $\mathbb{E}\left[\left\|s_{(\eta m+1)} - t_{(\eta m+1)}\right\| \|\mu\right]$
2	0.7546	0.7327	0.5895
4	0.5174	0.4968	0.3354
8	0.3597	0.3509	0.1682
16	0.2526	0.2491	0.0871

for different values of η. As η doubles, the expected total and sampling errors decrease by a factor of approximately $\sqrt{2}$ while the expected strategic error decreases by a factor of approximately 2, which reflects their respective rates of convergence. As Theorem 6 suggests, Panel C shows that expected strategic error rapidly becomes small relative to expected sampling error as the market becomes larger.

5.4.2 The CIV Case

Consider the case in which G_ε and G_δ are standard normal and $m = n = 1$, so that $q = 1/2$, $\xi_q^{\varepsilon+\delta} = 0 = V\left(\xi_q^{\varepsilon+\delta}\right)$ and $p^{\text{REE}} = \mu$. Panel A in Table 4 concerns the strategic term of buyers and allocational efficiency of the BBDA. The second column presents the strategic term for different sizes of market η. Consistent with Numerical Result II, it falls by 1/2 as η doubles, reflecting its $O(1/\eta)$ rate of convergence. Turning next to allocative efficiency, columns 3 and 4 present $\overline{GFT}^{\mathrm{e}}$ and $\overline{GFT}^{\mathrm{pt}}$, which are then used to calculate relative inefficiency in the final column. Relative inefficiency defined in this way falls by 1/4 as η doubles, reflecting its $O(1/\eta^2)$ rate of convergence that is stated in Numerical Result III.

Panels B and C concern the price-taking price $t_{(\eta m+1)}$ and the equilibrium price $s_{(\eta m+1)}$ as estimates of p^{REE} in the CIV case. For $m = n$, both prices $t_{(\eta m+1)}$ and $s_{(\eta m+1)}$ share the same asymptotic limit $p^{\mathrm{REE}} = \mu$. The variances of the errors $t_{(\eta m+1)} - p^{\mathrm{REE}}$ and $s_{(\eta m+1)} - p^{\mathrm{REE}}$ for these estimates are presented in Panel B. The variances diminish at the rate $O(1/\eta)$ and are virtually indistinguishable except in the $\eta = 2$ case, which again illustrates the relative insignificance of strategic behavior upon information aggregation by the market. Panel C presents the expected total error in the equilibrium price, the expected sampling error compounded with signal noise and the expected strategic error. Consistent with Numerical Result III, expected sampling error is $\Theta(1/\sqrt{\eta})$ and expected strategic error is $O(1/\eta)$, which causes expected total error to be $O(1/\sqrt{\eta})$. Notice that expected total error is marginally less than expected sampling error, which is attributable to the fact that strategic behavior can cancel some sampling error in determining total error.

6 Conclusion

This paper develops an informational environment for a uniform price double auction that is simple enough to permit formal analysis, computational work, and the display of equilibrium while rich enough to include the correlated private and interdependent value cases. Previous work in these cases has mainly focused on the asymptotic properties of large markets. Our model is not as general as the models in this earlier work but its restrictiveness allows both formal and computational analysis of finite markets, thus demonstrating that the asymptotic results are meaningful in a finite world. We trade generality for a more thorough understanding of finite markets, which we believe is a worthwhile exchange. The main conclusion of our analysis is that private information and the strategic behavior that it generates only marginally affects the market's performance relative to price formation, allocative efficiency, and the estimation of the REE price. Except in the smallest of markets, the BBDA discovers price extremely well.

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Appendices

A Derivation of Marginal Utilities and FOCs

We now derive expanded forms of the FOCs that are used in our computational work and in some proofs. A proof that these FOCs are equivalent to (5) and (7) can be found in online Appendix G.

A.1 Buyer's FOC

Pick a focal buyer who has received signal σ_B concerning his unknown value v. Assume the other buyers and sellers use strategies B and S that satisfy A2. Let x, y be the m^{th} and $(m+1)^{\text{st}}$ order statistics of the other traders' bids/asks. The focal buyer's expected utility from bidding b is

$$\pi^B(\sigma_B, b; B, S) = \int \int_{-\infty}^{y} \int \left[(v-y)\mathbb{I}\{x < y < b\} + (v-b)\mathbb{I}\{x < b \leq y\}\right] f^B_{vxy|\sigma}(v, x, y|\sigma_B) dv dx dy \tag{19}$$

where $\mathbb{I}\{\cdot\}$ is the indicator function. The density may be rewritten as

$$\begin{aligned} f^B_{vxy|\sigma}(v, x, y|\sigma_B) &= \int f^B_{vxy\mu|\sigma}(v, x, y, \mu|\sigma_B) d\mu = \int f^B_{xy|\mu,v,\sigma}(x, y|\mu, v, \sigma_B) f_{\mu|v,\sigma}(\mu|v, \sigma_B) f_{v|\sigma}(v|\sigma_B) d\mu \\ &= \int f^B_{xy|\mu}(x, y|\mu) f_{\mu|v}(\mu|v) f_{v|\sigma}(v|\sigma_B) d\mu = \int f^B_{xy|\mu}(x, y|\mu) g_\varepsilon(v-\mu) g_\delta(\sigma_B - v) d\mu. \end{aligned}$$

The third equality follows from observing that (i), conditional on μ, the order statistics x and y are independent of v and σ_B and (ii), conditional on v, σ_B does not add any more information about μ since the signal σ_B is just a noisy version of v. In the last equality we use $f_{\mu|v}(\mu|v) = g_\varepsilon(v-\mu)$ as derived in section 2.3 and $f_{v|\sigma}(v|\sigma_B) = g_\delta(\sigma_B - v)$, which can be derived similarly.

Crucial to what follows is a formula for $\mathbb{E}[v|\mu, \sigma_B]$. For $r > 0$, let the uniform improper prior be approximated by the uniform proper prior with density $g^r_\mu = 1/2r$ in $(-r, r)$ and 0 otherwise. The joint density of v, μ, and σ_B is

$$f^r_{\mu,v,\sigma}(\mu, v, \sigma_B) = g^r_\mu(\mu) f_{v|\mu}(v|\mu) f_{\sigma|v}(\sigma_B|v) = \frac{g_\varepsilon(v-\mu) g_\delta(\sigma_B - v)}{2r}.$$

The marginal density of μ and σ_B is

$$f^r_{\mu,\sigma}(\mu, \sigma_B) = \int f^r_{\mu,v,\sigma}(\mu, v, \sigma_B) dv = \frac{1}{2r} \int g_\varepsilon(v-\mu) g_\delta(\sigma_B - v) dv.$$

The conditional density of μ and σ_B given v is

$$f_{\mu,\sigma|v}(\mu, \sigma_B|v) = \frac{f^r_{\mu,v,\sigma}(\mu, v, \sigma_B)}{g^r_v(v)} = \frac{\frac{g_\varepsilon(v-\mu) g_\delta(\sigma_B - v)}{2r}}{\left(\frac{1}{2r}\right)} = g_\varepsilon(v-\mu) g_\delta(\sigma_B - v)$$

where g^r_v is the marginal density of v obtained by integrating out μ and σ_B from $f^r_{\mu,v,\sigma}(\mu, v, \sigma_B)$.

Therefore

$$f_{v|\mu,\sigma}\left(v|\mu,\sigma_B\right)=\frac{f_{\mu,\sigma|v}(\mu,\sigma_B|v)g_v^r\left(v\right)}{f_{\mu,\sigma}\left(\mu,\sigma_B\right)}=\frac{g_\varepsilon(v-\mu)g_\delta(\sigma_B-v)(\frac{1}{2r})}{\frac{1}{2r}\int g_\varepsilon(v-\mu)g_\delta(\sigma_B-v)dv,}$$
$$=\frac{g_\varepsilon(v-\mu)g_\delta(\sigma_B-v)}{\int g_\varepsilon(v-\mu)g_\delta(\sigma_B-v)dv}=\frac{g_\varepsilon(v-\mu)g_\delta(\sigma_B-v)}{g_{\varepsilon+\delta}\left(\sigma_B-\mu\right)}$$

and

$$\mathbb{E}[v|\mu,\sigma_B]=\int vf_{v|\mu,\sigma}\left(v|\mu,\sigma_B\right)dv=\frac{\int vg_\varepsilon(v-\mu)g_\delta(\sigma_B-v)dv}{g_{\varepsilon+\delta}\left(\sigma_B-\mu\right)} \tag{20}$$

where from (3) we have $f_{\mu|\sigma}\left(\mu|\sigma_B\right)=f_{\sigma|\mu}(\sigma_B|\mu)=g_{\varepsilon+\delta}\left(\sigma_B-\mu\right)$.

We rewrite the first half of (19) as

$$\begin{aligned}
&\int_{-\infty}^{b}\int_{-\infty}^{y}\int(v-y)f^B_{vxy|\sigma}(v,x,y|\sigma_B)dvdxdy\\
&=\int_{-\infty}^{b}\int_{-\infty}^{y}\int(v-y)\left[\int f^B_{xy|\mu}(x,y|\mu)g_\varepsilon(v-\mu)g_\delta(\sigma_B-v)d\mu\right]dvdxdy\\
&=\int\int_{-\infty}^{b}\int_{-\infty}^{y}f^B_{xy|\mu}(x,y|\mu)\left[\int(v-y)g_\varepsilon(v-\mu)g_\delta(\sigma_B-v)dv\right]dxdyd\mu\\
&=\int\int_{-\infty}^{b}\int_{-\infty}^{y}f^B_{xy|\mu|}(x,y|\mu)\left[\int vg_\varepsilon(v-\mu)g_\delta(\sigma_B-v)dv-yf_{\mu|\sigma}(\mu|\sigma_B)\right]dxdyd\mu\\
&=\int\int_{-\infty}^{b}\left[\mathbb{E}[v|\mu,\sigma_B]-y\right]\left[\int_{-\infty}^{y}f^B_{xy|\mu}(x,y|\mu)dx\right]f_{\mu|\sigma}(\mu|\sigma_B)dyd\mu\\
&=\int\int_{-\infty}^{b}\left[\mathbb{E}[v|\mu,\sigma_B]-y\right]\left[\int f^B_{xy|\mu}(x,y|\mu)dx\right]f_{\mu|\sigma}(\mu|\sigma_B)dyd\mu\\
&=\int\int_{-\infty}^{b}\left[\mathbb{E}[v|\mu,\sigma_B]-y\right]f^B_{y|\mu}(y|\mu)f_{\mu|\sigma}(\mu|\sigma_B)dyd\mu
\end{aligned}$$

where the fourth equality follows from factoring out $f_{\mu|\sigma}(\mu|\sigma_B)$ and using (20). The next-to-last equality follows from observing that by definition $x<y$ and therefore $f^B_{xy}\left(x,y|\mu\right)=0$ for all $x>y$. The second half of the equation simplifies in much the same way:

$$\begin{aligned}
&\int_{b}^{\infty}\int_{-\infty}^{b}\int(v-b)f^B_{vxy|\sigma}(v,x,y|\sigma_B)dvdxdy\\
&=\int\left[\int_{-\infty}^{b}\int_{-\infty}^{y}\left[\mathbb{E}[v|\mu,\sigma_B]-b\right]f^B_{xy|\mu}(x,y|\mu)\right]f_{\mu|\sigma}(\mu|\sigma_B)dxdyd\mu.
\end{aligned}$$

Combining the two parts, the focal buyer's expected utility is

$$\begin{aligned}
\pi^B(b|\sigma_B,B,S) \;=\; &\int\left\{\int_{-\infty}^{b}\left(\mathbb{E}[v|\mu,\sigma_B]-y\right)f^B_{y|\mu}(y|\mu)dy\right.\\
+\;&\left.\int_{-\infty}^{b}\int_{b}^{\infty}\left(\mathbb{E}[v|\mu,\sigma_B]-b\right)f^B_{xy|\mu}(x,y|\mu)dydx\right\}f_{\mu|\sigma}(\mu|\sigma_B)d\mu \iff
\end{aligned}$$

$$\pi^B(b|\sigma_B, B, S) = \int \left\{ \underbrace{\int_{-\infty}^{b} (\mathbb{E}[v|\mu,\sigma_B] - y)\, f^B_{y|\mu}(y|\mu)dy}_{I_1} \right. \tag{21}$$

$$\left. + \underbrace{(\mathbb{E}[v|\mu,\sigma_B] - b)}_{I_2} \underbrace{\int_{-\infty}^{b} \overbrace{\int_b^{\infty} f^B_{xy|\mu}(x,y|\mu)dy}^{\widehat{I}_3(x,b)}\, dx}_{I_3} \right\} f_{\mu|\sigma}(\mu|\sigma_B)d\mu.$$

Differentiating with respect to b we have in turn:

$$\frac{\partial I_1}{\partial b} = (\mathbb{E}[v|\mu,\sigma_B] - b)\, f^B_{y|\mu}(b|\mu), \quad \frac{\partial I_2}{\partial b} = -1,$$

and

$$\begin{aligned}\frac{\partial I_3}{\partial b} &= \widehat{I}_3(b,b) + \int_{-\infty}^{b} \frac{\partial \widehat{I}_3(x,b)}{\partial b}dx = \int_b^{\infty} f^B_{xy|\mu}(b,y|\mu)dy - \int_{-\infty}^{b} f^B_{xy|\mu}(x,b|\mu)dx \\ &= f^B_{x|\mu}(b|\mu) - f^B_{y|\mu}(b|\mu)\end{aligned}$$

where the terms I_1, I_2, I_3 and $\widehat{I}_3(x,b)$ are defined in (21) and the last equality follows from the fact that by definition $x < y$. Combining all the terms yields the buyer's marginal utility:

$$\begin{aligned}\pi^B_b(b|\sigma_B, B, S) &= \int \left\{ (\mathbb{E}[v|\mu,\sigma_B] - b) \left(f^B_{y|\mu}(b|\mu) + f^B_{x|\mu}(b|\mu) - f^B_{y|\mu}(b|\mu) \right) \right. \\ &\quad \left. - \int_{-\infty}^{b} \int_b^{\infty} f^B_{xy|\mu}(x,y|\mu)dydx \right\} f_{\mu|\sigma}(\mu|\sigma_B)d\mu \\ &= \int \left\{ (\mathbb{E}[v|\mu,\sigma_B] - b)\, f^B_{x|\mu}(b|\mu) - \int_{-\infty}^{b} \int_b^{\infty} f^B_{xy|\mu}(x,y|\mu)dydx \right\} f_{\mu|\sigma}(\mu|\sigma_B)d\mu.\end{aligned}$$

Hence, the FOC for a buyer is

$$\int \left\{ (\mathbb{E}[v|\mu,\sigma_B] - b)\, f^B_{x|\mu}(b|\mu) - \int_{-\infty}^{b} \int_b^{\infty} f^B_{xy|\mu}(x,y|\mu)dydx \right\} f_{\mu|\sigma}(\mu|\sigma_B)d\mu = 0 \tag{22}$$

where σ_B is the focal buyer's signal, the distributions $f^B_{x|\mu}$ and $f^B_{xy|\mu}$ are generated by the strategies B and S that the other traders employ, and $\int_{-\infty}^{b} \int_b^{\infty} f^B_{xy|\mu}(x,y|\mu)dydx = \Pr[x < b \leq y|\mu, B, S]$.

In order to evaluate this FOC numerically we must have formulas for the densities $f^B_{x|\mu}$ and $f^B_{xy|\mu}$, where superscript B here signifies that these are the x and y order statistics from the perspective of a buyer. The next two paragraphs develop the notation for these formulas. Fix the focal buyer's signal σ_B, his bid b, and the state μ. Given μ, every other trader's signal σ is independently distributed with density $g_{\varepsilon+\delta}(\sigma-\mu)$ and corresponding distribution $G_{\varepsilon+\delta}(\sigma-\mu)$. Assumption A2 on the strategies implies that $B^{-1}(b) = \sigma_B(b)$ and $S^{-1}(c) = \sigma_S(c)$ are well-defined and differentiable.

Denote their derivatives by $\dot{\sigma}_B(b)$ and $\dot{\sigma}_S(c)$. Then

$$\begin{aligned}
\Pr[B(\sigma) \leq b|\mu] &= \Pr[\sigma \leq \sigma_B(b)|\mu] = G_{\varepsilon+\delta}(\sigma_B(b) - \mu), \\
\Pr[S(\sigma) \leq b|\mu] &= \Pr[\sigma \leq \sigma_S(b)|\mu] = G_{\varepsilon+\delta}(\sigma_S(b) - \mu), \\
\Pr[b \leq B(\sigma) \leq b + \Delta b|\mu] &= G_{\varepsilon+\delta}(\sigma_B(b+\Delta b) - \mu) - G_{\varepsilon+\delta}(\sigma_B(b) - \mu) \\
&\approx g_{\varepsilon+\delta}(\sigma_B(b) - \mu)[\sigma_B(b+\Delta b) - \sigma_B(b)] \\
&\approx g_{\varepsilon+\delta}(\sigma_B(b) - \mu)\dot{\sigma}_B(b)\Delta b, \\
\Pr[b \leq S(\sigma) \leq b + \Delta b|\mu] &\approx g_{\varepsilon+\delta}(\sigma_S(b) - \mu)\dot{\sigma}_S(b)\Delta b.
\end{aligned} \tag{23}$$

In addition to Δb being small, derivation of the third formula depends on three observations: (i) $G_{\varepsilon+\delta}(\sigma + \Delta x - \mu) - G_{\varepsilon+\delta}(\sigma - \mu) \approx g_{\varepsilon+\delta}(\sigma - \mu)\Delta x$; (ii) $B[\sigma_B(b)] = b$ implies that $\dot{\sigma}_B(b) = 1/B'[\sigma_B(b)]$; (iii) $\sigma_B(b+\Delta b) \approx \sigma_B(b) + \dot{\sigma}_B(b)\Delta b$.

$M^B_{m,n}(b|B,S,\mu) = \Pr[x \leq b \leq y|\mu]$ is the probability that the focal buyer's bid b lies between $s_{(m)}$ and $s_{(m+1)}$ in a sample of $m-1$ buyers' bids and n sellers' asks:

$$M^B_{m,n} = \sum_{\substack{k+j=m \\ 0\leq k\leq m-1 \\ 0\leq j\leq n}} \binom{m-1}{k}\binom{n}{j}\left\{\begin{array}{c} G_{\varepsilon+\delta}(\sigma_B(b) - \mu)^k G_{\varepsilon+\delta}(\sigma_S(b) - \mu)^j \\ \times\overline{G}_{\varepsilon+\delta}(\sigma_B(b) - \mu)^{m-1-k}\,\overline{G}_{\varepsilon+\delta}(\sigma_S(b) - \mu)^{n-j} \end{array}\right\}. \tag{24}$$

Here k indexes buyers, j indexes sellers, and $\overline{G}_{\varepsilon+\delta}(\cdot) \equiv 1 - G_{\varepsilon+\delta}(\cdot)$. $K^B_{m,n}(b|B,S,\mu)$ is the probability that the focal buyer's bid b lies between $s_{(m-1)}$ and $s_{(m)}$ in a sample of $m-1$ buyers' bids and $n-1$ sellers' asks:

$$K^B_{m,n} = \sum_{\substack{k+j=m-1 \\ 0\leq k\leq m-1 \\ 0\leq j\leq n-1}} \binom{m-1}{k}\binom{n-1}{j}\left\{\begin{array}{c} G_{\varepsilon+\delta}(\sigma_B(b) - \mu)^k G_{\varepsilon+\delta}(\sigma_S(b) - \mu)^j \\ \times\overline{G}_{\varepsilon+\delta}(\sigma_B(b) - \mu)^{m-1-k}\,\overline{G}_{\varepsilon+\delta}(\sigma_S(b) - \mu)^{n-1-j} \end{array}\right\}. \tag{25}$$

Finally, $L^B_{m,n}(b|B,S,\mu)$ is the probability that the focal buyer's bid b lies between $s_{(m-1)}$ and $s_{(m)}$ in a sample of $m-2$ buyers' bids and n sellers' asks:

$$L^B_{m,n} = \sum_{\substack{k+j=m-1 \\ 0\leq k\leq m-2 \\ 0\leq j\leq n}} \binom{m-2}{k}\binom{n}{j}\left\{\begin{array}{c} G_{\varepsilon+\delta}(\sigma_B(b) - \mu)^k G_{\varepsilon+\delta}(\sigma_S(b) - \mu)^j \\ \times\overline{G}_{\varepsilon+\delta}(\sigma_B(b) - \mu)^{m-2-k}\,\overline{G}_{\varepsilon+\delta}(\sigma_S(b) - \mu)^{n-j} \end{array}\right\}. \tag{26}$$

Conditional on the focal buyer's signal being σ_B, his bid being b, each other buyer using B, and each seller using S, his marginal utility is

$$\pi^B_b(b|\sigma_B,B,S) = \int\left\{(\mathbb{E}(v|\mu,\sigma_B) - b)f_{x|\mu}(b|\mu) - \Pr[x \leq b \leq y|\mu]\right\} f_{\mu|\sigma}(\mu|\sigma_B)d\mu \iff$$

$$
\begin{aligned}
\pi_b^B\left(b|\sigma_B,B,S\right) &= \int (\mathbb{E}(v|\mu,\sigma_B)-b)\dot{\sigma}_S\left(b\right)\overbrace{ng_{\varepsilon+\delta}\left(\sigma_S(b)-\mu\right)K_{m,n}^B\left(b|B,S,\mu\right)}^{A_1^B(b|B,S,\mu)}g_{\varepsilon+\delta}\left(\sigma_B-\mu\right)d\mu \\
&+ \int (\mathbb{E}(v|\mu,\sigma_B)-b)\dot{\sigma}_B\left(b\right)\overbrace{\left(\begin{array}{c}(m-1)g_{\varepsilon+\delta}\left(\sigma_B(b)-\mu\right)\\ \times L_{m,n}^B\left(b|B,S,\mu\right)\end{array}\right)}^{A_2^B(b|B,S,\mu)}g_{\varepsilon+\delta}\left(\sigma_B-\mu\right)d\mu \\
&- \int M_{m,n}^B\left(b|B,S,\mu\right)g_{\varepsilon+\delta}\left(\sigma_B-\mu\right)d\mu. \qquad (27)
\end{aligned}
$$

Using the definitions of A_1^B, A_2^B above,

$$
f_{x|\mu}^B(b|\mu)=\dot{\sigma}_S\left(b\right)A_1^B\left(b|B,S,\mu\right)+\dot{\sigma}_B\left(b\right)A_2^B\left(b|B,S,\mu\right) \qquad (28)
$$

and $\Pr[x<b\leq y|\mu,B,S]=M_{m,n}^B\left(b|B,S,\mu\right)$, as given by (24). We also substituted for $f_{\mu|\sigma}(\mu|\sigma_B)=g_{\varepsilon+\delta}\left(\sigma_B-\mu\right)$ from (3).

To understand formula (27), suppose the focal buyer with signal σ_B increases his bid from b to $b+\Delta b$. Conditional on μ, consider in order the three terms on its right side:

1. Pick a seller k with signal σ_k distributed with density $g_{\varepsilon+\delta}\left(\sigma_k-\mu\right)$ and ask $a_k=S\left(\sigma_k\right)$. The term $\dot{\sigma}_S\left(b\right)g_{\varepsilon+\delta}\left(\sigma_S(b)-\mu\right)K_{m,n}^B\left(b|B,S,\mu\right)\times\Delta b$ is the probability that the buyer's increase Δb in his bid causes him to jump over a_k and go from not trading at b to trading at $b+\Delta b$. This switch to trading requires that $a_k=s_{(m)}$ in the sample of $m+n-1$ bids/asks the focal buyer faces. Because the focal buyer jumps over a_k, $\dot{\sigma}_S(b)g_{\varepsilon+\delta}\left(\sigma_S(b)-\mu\right)\Delta b$ is the probability that $a_k\in(b,b+\Delta b)$. Since Δb is small, $a_k\approx b$ whenever i jumps over a_k. $K_{m,n}^B(b|B,S,\mu)$ therefore is the probability that, excluding both seller k and the focal buyer i, the bids/asks of the other $m-1$ buyers and $n-1$ sellers are arranged so that a_k is the m^{th} order statistic that the focal buyer must jump to trade. Finally, the probability that a_k is both marginal and jumped over is multiplied by n because there are n sellers who could be the marginal seller k.

2. Pick a buyer k. The term $\dot{\sigma}_B\left(b\right)g_{\varepsilon+\delta}\left(\sigma_B(b)-\mu\right)L_{m,n}^B\left(b|B,S,\mu\right)$ is the probability that the focal buyer i jumps over buyer k's bid $b_k=B\left(\sigma_k\right)$ by increasing his bid b by Δb in the event that $b_k=s_{(m)}$ in the vector of bids/asks the focal buyer faces. As a result the focal buyer goes from not trading to trading. Derivation of this event's probability exactly parallels the derivation of the probability that buyer i jumps over seller k with two differences: the probability buyer k's bid b_k is in the interval $(b,b+\Delta b)$ is $\dot{\sigma}_B\left(b\right)g_{\varepsilon+\delta}\left(\sigma_B(b)-\mu\right)\Delta b$ and there are $m-1$ buyers who could be the marginal buyer k.

3. As observed above, $M_{m,n}^B\left(b|B,S,\mu\right)$ is the probability that the focal buyer's bid lies between $s_{(m)}$ and $s_{(m+1)}$ in a sample of $m-1$ buyers each using B and n sellers each using S. Thus, for the full sample of m bids (including the focal buyer's bid of b) and n asks, $M_{m,n}^B\left(b|B,S,\mu\right)$ is the probability that the focal buyer's bid b equals $s_{(m+1)}$ within the full sample of asks/bids

and therefore sets the price. Consequently, increasing b by Δb increases his expected cost of successfully trading by $M^B_{m,n}\left(b|B,S,\mu\right)\Delta b$.

Finally, the state μ must be integrated out of each of the three terms using the conditional density $f_{\mu|\sigma}\left(\mu|\sigma_B\right) = g_{\varepsilon+\delta}(\sigma_B - \mu)$.

A.2 Seller's FOC

Select a focal seller with signal σ_S and cost c, and let x, y be the m^{th} and $(m+1)^{\text{st}}$ order statistics of the other traders' bids/asks. The seller's ex post utility when he asks a is $(x-c)\mathbb{I}\{a \le x < y\}$. His interim expected utility is therefore

$$\begin{aligned}
\pi^S(\sigma_S, a; B, S) &= \int\int\int (x-c)\mathbb{I}\{a \le x < y\} f^S_{cxy|\sigma}(c,x,y|\sigma_S)dcdxdy \\
&= \int \left\{ \int_a^\infty \left(x - \mathbb{E}[c|\mu,\sigma_S]\right) f^S_{x|\mu}(x|\mu)dx \right\} f_{\mu|\sigma}(\mu|\sigma_S)d\mu
\end{aligned}$$

where the second line follows the same logic as that for the focal buyer in going from (19) to (21). Taking the derivative with respect to a gives the seller's first order condition:

$$\pi^S_a\left(\sigma_S, a; B, S\right) = \int \left(\mathbb{E}[c|\mu,\sigma_S] - a\right) f^S_{x|\mu}(a|\mu) g_{\varepsilon+\delta}(\sigma_S - \mu)d\mu = 0 \tag{29}$$

where we substituted $f_{\mu|\sigma}(\mu|\sigma_S) = g_{\varepsilon+\delta}(\sigma_S - \mu)$ from (3). Similar to (28), we have

$$\begin{aligned}
f^S_{x|\mu}(a|\mu) = \dot{\sigma}_S(a) & \overbrace{(n-1)g_{\varepsilon+\delta}\left(\sigma_S(a) - \mu\right) K^S_{m,n}\left(a|B,S,\mu\right)}^{A^S_1(a|B,S,\mu)} \\
+ \dot{\sigma}_B\left(a\right) & \overbrace{m g_{\varepsilon+\delta}\left(\sigma_B(a) - \mu\right) L^S_{m,n}\left(a|B,S,\mu\right)}^{A^S_2(a|B,S,\mu)}.
\end{aligned} \tag{30}$$

$K^S_{m,n}\left(a|B,S,\mu\right)$ is the probability that the focal seller's ask a lies between $s_{(m-1)}$ and $s_{(m)}$ in a sample of m buyers' bids and $n-2$ sellers' asks:

$$K^S_{m,n} = \sum_{\substack{i+k=m-1\\0\le i\le m\\0\le k\le n-2}} \binom{m}{i}\binom{n-2}{k} \left\{ \begin{array}{c} G_{\varepsilon+\delta}\left(\sigma_B\left(a\right) - \mu\right)^i G_{\varepsilon+\delta}\left(\sigma_S(a) - \mu\right)^k \\ \times \overline{G}_{\varepsilon+\delta}\left(\sigma_B\left(a\right) - \mu\right)^{m-i} \overline{G}_{\varepsilon+\delta}\left(\sigma_S(a) - \mu\right)^{n-2-k} \end{array} \right\}. \tag{31}$$

$L^S_{m,n}\left(a|B,S,\mu\right)$ is the probability that the focal seller's bid a lies between $s_{(m-1)}$ and $s_{(m)}$ in a sample of $m-1$ buyers' bids and $n-1$ sellers' asks:

$$L^S_{m,n} = \sum_{\substack{i+k=m-1\\0\le i\le m-1\\0\le k\le n-1}} \binom{m-1}{i}\binom{n-1}{k} \left\{ \begin{array}{c} G_{\varepsilon+\delta}\left(\sigma_B\left(a\right) - \mu\right)^i G_{\varepsilon+\delta}\left(\sigma_S\left(a\right) - \mu\right)^k \\ \times \overline{G}_{\varepsilon+\delta}\left(\sigma_B\left(a\right) - \mu\right)^{m-1-i} \overline{G}_{\varepsilon+\delta}\left(\sigma_S(a) - \mu\right)^{n-1-k} \end{array} \right\}. \tag{32}$$

To understand formula (30) for $f^S_{x|\mu}(a|\mu)$, suppose the focal seller with signal σ_S increases his bid from a to $a+\Delta a$. Conditional on μ, consider the first term $\dot{\sigma}_S(a)\, A_1^S(a|B,S,\mu)$ on its right side. Pick a seller k with signal σ_k and ask $a_k = S(\sigma_k)$. The term $\dot{\sigma}_S(a)\, g_{\varepsilon+\delta}(\sigma_S(a)-\mu)\, K^S_{m,n}(a|B,S,\mu)\times \Delta a$ is the probability that seller i's increase Δa in his ask causes him to jump over a_k and go from trading at a to not trading at $a+\Delta a$. This switch to not trading requires that $a_k = s_{(m)}$ in the sample of m bids and $n-1$ asks the focal seller faces. Because the focal seller j jumps over seller k, $\dot{\sigma}_S(a) g_{\varepsilon+\delta}(\sigma_S(a)-\mu)\,\Delta a$ is the probability that $a_k \in (a, a+\Delta a)$.[14] Since Δa is small, $a_k \approx a$ whenever j jumps over a_k. $K^S_{m,n}(a|B,S,\mu)$ is the probability that, excluding both seller k and the focal seller j, the bids/asks of the other m buyers and $n-2$ sellers bracket a_k, making it the marginal bid/ask (the m^{th} order statistic) that the focal seller must jump over to no longer trade. Finally, the probability that k is both marginal and jumped over is multiplied by $n-1$ because there are $n-1$ sellers who could be the marginal seller k. Derivation of the second term is similar.

B Invariance of the Vector Field

We begin by rewriting the formulas for $K^B_{m,n}$, $L^B_{m,n}$, $M^B_{m,n}$, $K^S_{m,n}$, and $L^S_{m,n}$ as given by (25), (26), (24), (31), and (32), respectively, in terms of a buyer's signal σ_B, a seller's signal σ_S, and a bid/ask β for which $B(\sigma_B) = \beta = S(\sigma_S)$. In terms of the inverse functions, we have $\sigma_B(\beta) = \sigma_B$ and $\sigma_S(\beta) = \sigma_S$. This is purely a matter of substitution that we now illustrate in the case of $K^B_{m,n}(b|B,S,\mu)$ as defined in (25). Each term in the formula for $K^B_{m,n}(\beta|B,S,\mu)$ can be expressed as a function of (σ_B, σ_S) and the state μ:

$$\begin{aligned}
&G_{\varepsilon+\delta}(\sigma_B(\beta)-\mu)^k\, G_{\varepsilon+\delta}(\sigma_S(\beta)-\mu)^j\, \overline{G}_{\varepsilon+\delta}(\sigma_B(\beta)-\mu)^{m-1-k}\, \overline{G}_{\varepsilon+\delta}(\sigma_S(\beta)-\mu)^{n-1-j} \\
&= G_{\varepsilon+\delta}(\sigma_B-\mu)^k\, G_{\varepsilon+\delta}(\sigma_S-\mu)^j\, \overline{G}_{\varepsilon+\delta}(\sigma_B-\mu)^{m-1-k}\, \overline{G}_{\varepsilon+\delta}(\sigma_S-\mu)^{n-1-j}\,. \qquad (33)
\end{aligned}$$

In this way, $K^B_{m,n}$ becomes

$$\widehat{K}^B_{m,n}(\sigma_B,\sigma_S|\mu) \equiv \sum_{\substack{k+j=m \\ 0\le k\le m-1 \\ 0\le j\le n-1}} \binom{m-1}{k}\binom{n-1}{j} \left\{ \begin{array}{l} G_{\varepsilon+\delta}(\sigma_B-\mu)^k\, G_{\varepsilon+\delta}(\sigma_S-\mu)^j \times \\ \overline{G}_{\varepsilon+\delta}(\sigma_B-\mu)^{m-1-k}\, \overline{G}_{\varepsilon+\delta}(\sigma_S-\mu)^{n-1-j} \end{array} \right\} \qquad (34)$$

when expressed as a function of $(\sigma_B,\sigma_S)\in\mathbb{R}^2$. Similarly, define $\widehat{L}^B_{m,n}(\sigma_B,\sigma_S|\mu)$, $\widehat{M}^B_{m,n}(\sigma_B,\sigma_S|\mu)$ for buyers and $\widehat{K}^S_{m,n}(\sigma_B,\sigma_S|\mu), \widehat{L}^S_{m,n}(\sigma_B,\sigma_S|\mu)$ for sellers. In the second line of (27), the density $g_{\varepsilon+\delta}(\sigma_B(\beta)-\mu)$ becomes $g_{\varepsilon+\delta}(\sigma_B-\mu)$. These simplifications convert $A_1^B, A_2^B, M^B_{m,n}$ in (27) and A_1^S, A_2^S in (30) into the functions $\widehat{A}_1^B, \widehat{A}_2^B, \widehat{M}^B_{m,n}$ and $\widehat{A}_1^S, \widehat{A}_2^S$ each with arguments σ_B, σ_S and μ.

The density $f^B_{x|\mu}(\beta|\mu)$ can now be written as

$$f^B_{x|\mu}(\beta|\mu) = \dot{\sigma}_S(\beta)\, \widehat{A}_1^B(\sigma_B,\sigma_S|\mu) + \dot{\sigma}_B(\beta)\, \widehat{A}_2^B(\sigma_B,\sigma_S|\mu)\,.$$

[14] Similar to (23) we have $\Pr[a_j \le S(\sigma_k) \le a_j + \Delta a|\mu] = g_{\varepsilon+\delta}(\sigma_S(a_j)-\mu)\,\dot{\sigma}_S(a_j)\,\Delta a$.

Similarly, the density $f^S_{x|\mu}(\beta|\mu)$ can be expressed as

$$f^S_{x|\mu}(\beta|\mu) = \dot{\sigma}_S(\beta)\,\widehat{A}^S_1(\sigma_B,\sigma_S|\mu) + \dot{\sigma}_B(\beta)\,\widehat{A}^S_2(\sigma_B,\sigma_S|\mu)$$

and

$$\int_{-\infty}^{b}\int_{b}^{\infty} f^B_{xy|\mu}(x,y|\mu)dydx = \widehat{M}^B_{m,n}(\sigma_B,\sigma_S|\mu).$$

The FOCs (22) and (29) can therefore be written at any point $\omega = (\sigma_B,\beta,\sigma_S) \in \mathbb{R}^3$ as

$$\int \Big\{(\mathbb{E}[v|\mu,\sigma_B]-\beta)\Big[\dot{\sigma}_S\widehat{A}^B_1(\sigma_B,\sigma_S|\mu) + \dot{\sigma}_B\widehat{A}^B_2(\sigma_B,\sigma_S|\mu)\Big] - \widehat{M}^B_{m,n}(\sigma_B,\sigma_S|\mu)\Big\} f_{\mu|\sigma}(\mu|\sigma_B)d\mu = 0, \tag{35}$$

$$\int \Big\{(\mathbb{E}[c|\mu,\sigma_B]-\beta)\Big[\dot{\sigma}_S\widehat{A}^S_1(\sigma_B,\sigma_S|\mu) + \dot{\sigma}_B\widehat{A}^S_2(\sigma_B,\sigma_S|\mu)\Big]\Big\} f_{\mu|\sigma}(\mu|\sigma_S)d\mu = 0. \tag{36}$$

This is a linear system in $\dot{\sigma}_B$ and $\dot{\sigma}_S$ that is representable in the matrix form

$$\begin{bmatrix} B_{11}(\omega) & B_{12}(\omega) \\ B_{21}(\omega) & B_{22}(\omega) \end{bmatrix} \cdot \begin{bmatrix} \dot{\sigma}_S \\ \dot{\sigma}_B \end{bmatrix} = \begin{bmatrix} C(\sigma_B,\sigma_S) \\ 0 \end{bmatrix} \tag{37}$$

where

$$\begin{aligned}
B_{11} &\equiv \int (\mathbb{E}[v|\mu,\sigma_B]-\beta)\,\widehat{A}^B_1(\sigma_B,\sigma_S|\mu)\,f_{\mu|\sigma}(\mu|\sigma_B)d\mu, \\
B_{12} &\equiv \int (\mathbb{E}[v|\mu,\sigma_B]-\beta)\,\widehat{A}^B_2(\sigma_B,\sigma_S|\mu)\,f_{\mu|\sigma}(\mu|\sigma_B)d\mu, \\
B_{21} &\equiv \int (\mathbb{E}[c|\mu,\sigma_S]-\beta)\,\widehat{A}^S_1(\sigma_B,\sigma_S|\mu)\,f_{\mu|\sigma}(\mu|\sigma_S)d\mu, \\
B_{22} &\equiv \int (\mathbb{E}[c|\mu,\sigma_S]-\beta)\,\widehat{A}^S_2(\sigma_B,\sigma_S|\mu)\,f_{\mu|\sigma}(\,\mu|\sigma_S)d\mu, \\
C &\equiv \int \widehat{M}^B_{m,n}(\sigma_B,\sigma_S|\mu) f_{\mu|\sigma}(\mu|\sigma_B)d\mu.
\end{aligned} \tag{38}$$

Line of Singularity. We now derive the line of singularity $\dot{\sigma}_B = 0$ that appears in Figure 2. Consider a buyer's FOC (35) in the case in which sellers play an offset strategy. We therefore set $\dot{\sigma}_S = 1$ in (35), which reduces it to

$$\int (\mathbb{E}[v|\mu,\sigma_B]-\beta)\,\widehat{A}^B_2(\sigma_B,\sigma_S|\mu)\,f_{\mu|\sigma}(\mu|\sigma_B)d\mu = 0,$$

or equivalently,

$$\frac{\int (\mathbb{E}[v|\mu,\sigma_B]-\beta)\,\widehat{A}^B_2(\sigma_B,\sigma_S|\mu)\,f_{\mu|\sigma}(\mu|\sigma_B)d\mu}{\int \widehat{A}^B_2(\sigma_B,\sigma_S|\mu)\,f_{\mu|\sigma}(\mu|\sigma_B)d\mu} = 0. \tag{39}$$

$\widehat{A}_2^B$ is a probability density.[15] Interpreting $\widehat{A}_2^B \cdot \Delta\mu$ as a probability, it is the likelihood in the state μ that the focal buyer's bid β equals the bid of some other buyer when this other bid is the m^{th} smallest among the $m+n-1$ bids/asks of the non-focal traders. The denominator is therefore the probability that this event occurs. Let $\mathcal{B}$ denote the event in which the m^{th} smallest bid/ask x of the other traders is the bid of a buyer. The left side of (39) therefore equals $\mathbb{E}\left[v-\beta|\sigma_B, x=\beta, \mathcal{B}\right]$, the expected gain to the focal buyer of acquiring an item at the price β given that $\beta = x$ in event $\mathcal{B}$.

Proof of Lemma 1. The lemma is proven by demonstrating that the five coefficients B_{11}, B_{12}, B_{21}, B_{22}, C are invariant to translations along the diagonal. Since these coefficients share a common structure, we show invariance here only in the case of B_{11}. We consider in turn the three terms in the integrand of (38) that defines this coefficient.

For arbitrary $\rho \in \mathbb{R}$, consider the translation $\sigma_B^* = \sigma_B + \rho$, $\beta^* = \beta + \rho, \sigma_S^* = \sigma_S + \rho$, and $\mu^* = \mu + \rho$. Using equation (20), which gives the formula for the first term, we check that the translation has no effect:

$$\begin{aligned}
\mathbb{E}[v|\mu^*, \sigma_B^*] - \beta^* &= \frac{\int v g_\varepsilon(v_B - \mu^*) g_\delta(\sigma_B^* - v)dv}{f_{\mu|\sigma}\left(\mu^*|\sigma_B^*\right)} - \beta^* \\
&= \frac{\int v g_\varepsilon(v - \mu - \rho) g_\delta(\sigma_B + \rho - v)dv}{\int g_\varepsilon(v - \mu - \rho) g_\delta(\sigma_B + \rho - v)dv} - \beta - \rho \\
&= \frac{\int (v+\rho) g_\varepsilon(v + \rho - \mu - \rho) g_\delta(\sigma_B + \rho - v - \rho)dv}{\int g_\varepsilon(v + \rho - \mu - \rho) g_\delta(\sigma_B + \rho - v - \rho)dv} - \beta - \rho \\
&= \frac{\int v g_\varepsilon(v - \mu) g_\delta(\sigma_B - v)dv}{\int g_\varepsilon(v - \mu) g_\delta(\sigma_B - v)dv} + \rho \frac{\int g_\varepsilon(v - \mu^*) g_\delta(\sigma_B - v)dv}{\int g_\varepsilon(v - \mu) g_\delta(\sigma_B - v)dv} - \beta - \rho \\
&= \mathbb{E}[v|\mu, \sigma_B] - \beta
\end{aligned} \tag{40}$$

where the third equality is the change of variable $v + \rho \to v$.

The second term is

$$\widehat{A}_1^B\left(\sigma_B, \sigma_S|\mu\right) = n g_{\varepsilon+\delta}\left(\sigma_S - \mu\right) \widehat{K}_{m,n}^B\left(\sigma_B, \sigma_S|\mu\right).$$

The translation leaves the density unchanged: $g_{\varepsilon+\delta}\left(\sigma_S^* - \mu^*\right) = g_{\varepsilon+\delta}\left(\sigma_S - \mu\right)$. Applying the translation to the (j,k) term in the expansion (33) of $\widehat{K}_{m,n}^B\left(\sigma_B, \sigma_S|\mu\right)$ shows that it is invariant:

$$\begin{aligned}
&G_{\varepsilon+\delta}\left(\sigma_B^* - \mu^*\right)^k G_{\varepsilon+\delta}\left(\sigma_S^* - \mu^*\right)^j \overline{G}_{\varepsilon+\delta}\left(\sigma_B^* - \mu^*\right)^{m-1-k} \overline{G}_{\varepsilon+\delta}\left(\sigma_S^* - \mu^*\right)^{n-1-j} \\
&= G_{\varepsilon+\delta}\left(\sigma_B - \mu\right)^k G_{\varepsilon+\delta}\left(\sigma_S - \mu\right)^j \overline{G}_{\varepsilon+\delta}\left(\sigma_B - \mu\right)^{m-1-k} \overline{G}_{\varepsilon+\delta}\left(\sigma_S - \mu\right)^{n-1-j}.
\end{aligned}$$

Finally, recall from equation (3) that $f_{\mu|\sigma}(\mu|\sigma_B) = g_{\varepsilon+\delta}\left(\sigma_B - \mu\right)$ and so

$$f_{\mu|\sigma}(\mu + \rho|\sigma_B + \rho) = g_{\varepsilon+\delta}\left((\sigma_B + \rho) - (\mu + \rho)\right) = g_{\varepsilon+\delta}\left(\sigma_B - \mu\right) = f_{\mu|\sigma}(\mu|\sigma_B). \quad \blacksquare$$

[15] A_2^B is defined in (27), and its reduction to $\widehat{A}_2^B$ is explained after equation (34).

C Existence of an Offset Solution to the Buyer's FOC in the CPV Case

Proof of Theorem 1. Define $\pi(\lambda, \lambda_B)$ as a focal buyer's expected profit when v^* is his value, he bids $v^* + \lambda$, the other buyers use the strategy $B(v) = v + \lambda_B$, and sellers are honest in their asks. The fact that all other buyers use an offset strategy and sellers report honestly implies that $\pi(\lambda, \lambda_B)$ has the same value for all possible values of v^*. This permits us to omit v^* as a variable and for convenience set it equal to zero. We seek a value of λ_B^* that solves the first order condition,

$$\left.\frac{\partial \pi(\lambda, \lambda_B^*)}{\partial \lambda}\right|_{\lambda=\lambda_B^*} = 0.$$

The formula for marginal utility in the FOC (5) implies that at $\lambda = \lambda_B = 0$ the focal buyer has an incentive to underbid his value $v^* = 0$:

$$\left.\frac{\partial \pi(\lambda, \lambda_B)}{\partial \lambda}\right|_{\lambda=\lambda_B=0} < 0. \tag{41}$$

Since A1 implies that $\pi(\lambda, \lambda_B)$ is continuously differentiable, if a $\lambda_B < 0$ exists such that

$$\left.\frac{\partial \pi(\lambda, \lambda_B)}{\partial \lambda}\right|_{\lambda=\lambda_B} > 0, \tag{42}$$

then the Intermediate Value Theorem implies that λ_B^* exists.

The argument is now made in four steps. First, note that

$$\pi(0,0) > 0$$

because if all traders report their values/costs honestly, then gains from trade exist and are realized in expectation. We prove below that

$$\lim_{\lambda_B \to -\infty} \pi(\lambda_B, \lambda_B) = 0 \tag{43}$$

using A1's assumption that G_ε has a first moment. Therefore $\pi(0,0) > \lim_{\lambda_B \to -\infty} \pi(\lambda_B, \lambda_B)$, which implies that $\lambda_C \in \arg\max_{\lambda_B < 0} \pi(\lambda_B, \lambda_B)$ exists. Second, observe that the focal buyer's expected profit declines as the other buyers increase their bids by increasing λ_B because the focal buyer's probability of trade goes down and the expected price if he should trade goes up. Therefore, for any λ, $\lambda_B \in \mathbb{R}$,

$$\frac{\partial \pi(\lambda, \lambda_B)}{\partial \lambda_B} < 0. \tag{44}$$

Third, λ_C satisfies its first order condition for a maximum,

$$0 = \left.\frac{d\pi(\lambda_B, \lambda_B)}{d\lambda_B}\right|_{\lambda_B=\lambda_C} = \left.\frac{\partial \pi(\lambda, \lambda_B)}{\partial \lambda}\right|_{\lambda=\lambda_B=\lambda_C} + \left.\frac{\partial \pi(\lambda, \lambda_B)}{\partial \lambda_B}\right|_{\lambda=\lambda_B=\lambda_C}; \tag{45}$$

the equation's right-most derivative is negative by inequality (44), and therefore

$$\left.\frac{\partial \pi\left(\lambda, \lambda_B\right)}{\partial \lambda}\right|_{\lambda=\lambda_B=\lambda_C}>0$$

as needed to imply existence of λ_B^*.

Returning to (43), we bound $\pi\left(\lambda_B, \lambda_B\right)$ by

$$\pi\left(\lambda_B, \lambda_B\right) \leq \int\left[\int_{-\infty}^{\lambda_B}(-c)\, n g_{\varepsilon}(c-\mu)\, \overline{G}_{\varepsilon}(c-\mu)^{n-1}\, d c\right] g_{\varepsilon}(-\mu)\, d \mu \tag{46}$$

where the logic of this bound is as follows. The focal buyer trades if and only if at least m asks and bids from among the other $m+n-1$ traders are below his bid λ_B Consequently, the focal buyer trades only if the smallest seller's cost $c_{(1)}$ is below λ_B. Given his value $v^*=0$, his gain from trading is bounded above by $0-c_{(1)}=-c_{(1)}$ (the price he pays may in fact be larger than $c_{(1)}$). Conditional on μ, the interior integral on the right-side is the expected value of the gain from trade computed with respect to the density of the smallest seller's cost where the subscript on $c_{(1)}$ is suppressed. This interior integral in (46) can be further bounded:[16]

$$\begin{aligned}
\int_{-\infty}^{\lambda_B}(-c) n g_{\varepsilon}(c-\mu) \overline{G}_{\varepsilon}(c-\mu)^{n-1} d c & \leq -n \int_{-\infty}^{\lambda_B}(c-\mu+\mu)\, g_{\varepsilon}(c-\mu) d c \\
& =-n\left[\int_{-\infty}^{\lambda_B}(c-\mu)\, g_{\varepsilon}(c-\mu) d c+\mu \int_{-\infty}^{\lambda_B} g_{\varepsilon}(c-\mu) d c\right] \\
& =-n\left[\int_{-\infty}^{\lambda_B}(c-\mu)\, g_{\varepsilon}(c-\mu) d c+\mu G_{\varepsilon}\left(\lambda_B-\mu\right)\right] .
\end{aligned}$$

Therefore

$$\begin{aligned}
\pi\left(\lambda_B, \lambda_B\right) & \leq-n \int\left[\int_{-\infty}^{\lambda_B}(c-\mu)\, g_{\varepsilon}(c-\mu) d c+\mu G_{\varepsilon}\left(\lambda_B-\mu\right)\right] g_{\varepsilon}(-\mu)\, d \mu \\
& =-n \int\left[\int_{-\infty}^{\lambda_B}(c-\mu)\, g_{\varepsilon}(c-\mu) d c\right] g_{\varepsilon}(-\mu)\, d \mu-n \int \mu G_{\varepsilon}\left(\lambda_B-\mu\right) g_{\varepsilon}(-\mu)\, d \mu .
\end{aligned}$$

The limit (43) then follows from the two limits

$$\lim_{\lambda_B \rightarrow-\infty} \int_{-\infty}^{\lambda_B}(c-\mu)\, g_{\varepsilon}(c-\mu) d c=0 \quad \text { and } \quad \lim_{\lambda_B \rightarrow-\infty} G_{\varepsilon}\left(\lambda_B-\mu\right)=0$$

that are true for all μ, with the first holding because G_{ε} has a finite first moment. ■

[16] This proof breaks down at this point if G_{ε} does not have a finite first moment, as is the case with the Cauchy distribution. We discuss this further in online Appendix L.

D Convergence to Price-Taking Behavior in the CPV Case

Theorems 3 and 4 state rates of convergence for sequences of markets with ηm buyers and ηn sellers for $\eta \in \mathbb{N}$ and fixed m and n. We are able here to establish rates for more general sequences. Consider m,n that satisfy the bound

$$\frac{1}{\varphi} \leq \frac{m}{n}, \frac{n}{m} \leq \varphi. \tag{47}$$

for some constant $\varphi \geq 1$. We also use in this section the notation

$$m \wedge n \equiv \min(m, n).$$

We prove first that a buyer's strategic term is $O(1/m \wedge n)$, where the constant that establishes the rate depends on G_ε and φ. This reduces to Theorem 3 in the case of $\varphi = 1$, i.e., fixed m/n. In online Appendix I, we then apply this result to show that relative inefficiency is $O\left(1/(m \wedge n)^2\right)$, where again, the constant in the rate depends on G_ε and φ. This reduces to Theorem 4 in the case of $\varphi = 1$.

We begin by reducing several formulas from our analysis of the first order approach to the specific case of the CPV case. In this case there is only one FOC, that of a buyer, since sellers simply report their costs. A buyer's FOC (35) reduces in this case to

$$\dot{v}(v,b) = \frac{\int \widehat{M}^B_{m,n} g_\varepsilon(v-\mu)d\mu - (v-b)\int n g_\varepsilon(b-\mu)\widehat{K}^B_{m,n} g_\varepsilon(v-\mu)\,d\mu}{(v-b)\int (m-1) g_\varepsilon(v-\mu) L^*_{m,n} g_\varepsilon(v-\mu)d\mu} \tag{48}$$

where $\dot{v}$ is the reduced form of $\dot{\sigma}_B$ from Section 4 to the CPV case. The functions $\widehat{M}^B_{m,n}$, $\widehat{K}^B_{m,n}$, and $\widehat{L}^B_{m,n}$ from Appendix B reduce as follows in the CPV case:

$$\widehat{K}^B_{m,n}(v,b|\mu) = \sum_{\substack{i+j=m-1\\0\leq i\leq m-1\\0\leq j\leq n-1}} \begin{pmatrix} m-1 \\ i \end{pmatrix} \begin{pmatrix} n-1 \\ j \end{pmatrix} G_\varepsilon(v-\mu)^i\, G_\varepsilon(b-\mu)^j\, \overline{G}_\varepsilon(v-\mu)^{m-1-i}\, \overline{G}_\varepsilon(b-\mu)^{n-1-j}, \tag{49}$$

$$\widehat{L}^B_{m,n}(v,b|\mu) = \sum_{\substack{i+j=m-1\\0\leq i\leq m-2\\0\leq j\leq n}} \begin{pmatrix} m-2 \\ i \end{pmatrix} \begin{pmatrix} n \\ j \end{pmatrix} G_\varepsilon(v-\mu)^i\, G_\varepsilon(b-\mu)^j\, \overline{G}_\varepsilon(v-\mu)^{m-2-i}\, \overline{G}_\varepsilon(b-\mu)^{n-j}, \tag{50}$$

$$\widehat{M}^B_{m,n}(v,b|\mu) = \sum_{\substack{i+j=m\\0\leq i\leq m-1\\0\leq j\leq n}} \begin{pmatrix} m-1 \\ i \end{pmatrix} \begin{pmatrix} n \\ j \end{pmatrix} G_\varepsilon(v-\mu)^i\, G_\varepsilon(b-\mu)^j\, \overline{G}_\varepsilon(v-\mu)^{m-1-i}\, \overline{G}_\varepsilon(b-\mu)^{n-j}. \tag{51}$$

We next prove a lemma that implies that an equilibrium offset cannot be arbitrarily large in magnitude, regardless of the size of the market.

Lemma 2 *For given values of m and n, a constant $L^*(G_\varepsilon) > 0$ exists such that*

$$v - b > L^*(G_\varepsilon) \Rightarrow \dot{v}(v, b) < 1. \tag{52}$$

Any equilibrium offset λ in the market with m buyers and n sellers therefore satisfies $|\lambda| \leq L^*(G_\varepsilon)$.

Proof. The invariance of $\dot{v}(v, b)$ allows us to set $v = 0$ with no loss of generality. We also restrict attention to $b \leq v = 0$ because setting $b > v$ is a dominated strategy for buyers. The functions $\widehat{K}^B_{m,n}(0, b|\mu)$, $\widehat{L}^B_{m,n}(0, b|\mu)$ and $\widehat{M}^B_{m,n}(0, b|\mu)$ that appear in formula (48) and are given by (49)-(51) satisfy the bounds

$$\widehat{K}^B_{m,n}(0, b|\mu) \geq G_\varepsilon(-\mu)^{m-1} \overline{G}_\varepsilon(b-\mu)^{n-1},$$

$$\widehat{L}^B_{m,n}(0, b|\mu) \geq nG_\varepsilon(-\mu)^{m-2} G_\varepsilon(b-\mu) \overline{G}_\varepsilon(b-\mu)^{n-1},$$

and

$$\widehat{M}^B_{m,n}(0, b|\mu) \leq k_{m,n} G_\varepsilon(-\mu)^{m-1} G_\varepsilon(b-\mu) \overline{G}_\varepsilon(b-\mu)^{n-1}$$

where $k_{m,n}$ in the last formula denotes

$$k_{m,n} = \sum_{\substack{i+j=m \\ 0 \leq i \leq m-1 \\ 1 \leq j \leq n}} \begin{pmatrix} m-1 \\ i \end{pmatrix} \begin{pmatrix} n \\ j \end{pmatrix}.$$

The bound on $\widehat{K}^B_{m,n}(0, b|\mu)$ is obtained by focusing on its $i = m-1$, $j = 0$ term and the bound on $\widehat{L}^B_{m,n}(0, b|\mu)$ is obtained from its $i = m-2$, $j = 1$ term.

The bound on $\widehat{M}^B_{m,n}(0, b|\mu)$ is obtained as follows. $\overline{G}_\varepsilon$ is strictly decreasing because G_ε is strictly increasing. Therefore, given that $b \leq v = 0$, $G_\varepsilon(b-\mu) \leq G_\varepsilon(-\mu)$ and $\overline{G}_\varepsilon(b-\mu) \geq \overline{G}_\varepsilon(-\mu)$. This implies that the monomial in its i^{th} term satisfies

$$\begin{aligned}
& G_\varepsilon(-\mu)^i G_\varepsilon(b-\mu)^j \overline{G}_\varepsilon(-\mu)^{m-1-i} \overline{G}_\varepsilon(b-\mu)^{n-j} \\
= \; & \frac{G_\varepsilon(-\mu)^i G_\varepsilon(b-\mu)^j \overline{G}_\varepsilon(-\mu)^{m-1-i} \overline{G}_\varepsilon(b-\mu)^{n-j}}{G_\varepsilon(-\mu)^{1-j} G_\varepsilon(b-\mu)^{j-1} \overline{G}_\varepsilon(-\mu)^{m-1-i} \overline{G}_\varepsilon(b-\mu)^{i+1-m}} \\
\times \; & G_\varepsilon(-\mu)^{1-j} G_\varepsilon(b-\mu)^{j-1} \overline{G}_\varepsilon(-\mu)^{m-1-i} \overline{G}_\varepsilon(b-\mu)^{i+1-m} \\
= \; & G_\varepsilon(-\mu)^{i+j-1} G_\varepsilon(b-\mu) \overline{G}_\varepsilon(b-\mu)^{m-1-i+n-j} \left(\frac{G_\varepsilon(b-\mu)}{G_\varepsilon(-\mu)}\right)^{j-1} \left(\frac{\overline{G}_\varepsilon(-\mu)}{\overline{G}_\varepsilon(b-\mu)}\right)^{m-1-i} \\
\leq \; & G_\varepsilon(-\mu)^{i+j-1} G_\varepsilon(b-\mu) \overline{G}_\varepsilon(b-\mu)^{m-1-i+n-j} \\
= \; & G_\varepsilon(-\mu)^{m-1} G_\varepsilon(b-\mu) \overline{G}_\varepsilon(b-\mu)^{n-1}
\end{aligned}$$

because (i) both fractions on line three are at most unity and (ii) $i \leq m-1$ and $i + j = m$ imply that $j \geq 1$. Substitution into (48) and factoring the integrand in the numerator implies that $\dot{v}(0, b)$

is bounded above by

$$\frac{\int \left[k_{m,n} G_{\varepsilon}\left(b-\mu\right)+nbg_{\varepsilon}(b-\mu)\right] G_{\varepsilon}\left(-\mu\right)^{m-1} \overline{G}_{\varepsilon}\left(b-\mu\right)^{n-1} g_{\varepsilon}(-\mu) d\mu}{-(m-1)nb \int G_{\varepsilon}\left(-\mu\right)^{m-2} G_{\varepsilon}\left(b-\mu\right) \overline{G}_{\varepsilon}\left(b-\mu\right)^{n-1} g_{\varepsilon}(-\mu)^{2} d\mu}. \tag{53}$$

The integral in the numerator of (53) is now bounded separately over the intervals $(-\infty, b]$ and $[b, \infty)$. The proof is completed by showing that: (i) the integral over (-∞, b] divided by the denominator of (53) converges to zero as $b \to -\infty$; (ii) the integral over $[b, \infty)$ is negative for b sufficiently negative. These two steps are considered separately below.

Integral over $(-\infty, b]$. This integral is bounded above by

$$\begin{aligned}
&\int_{-\infty}^{b} k_{m,n} G_{\varepsilon}\left(b-\mu\right) G_{\varepsilon}\left(-\mu\right)^{m-1} \overline{G}_{\varepsilon}\left(b-\mu\right)^{n-1} g_{\varepsilon}(-\mu) d\mu \\
&\leq k_{m,n} \int_{-\infty}^{b} g_{\varepsilon}(-\mu) d\mu \\
&= k_{m,n} \overline{G}_{\varepsilon}(-b) \\
&= k_{m,n} G_{\varepsilon}(b).
\end{aligned}$$

The first line follows from $b \leq 0$, the second from

$$G_{\varepsilon}\left(b-\mu\right) G_{\varepsilon}\left(-\mu\right)^{m-1} \overline{G}_{\varepsilon}\left(b-\mu\right)^{n-1} \leq 1,$$

and the last equality follows from the symmetry of G_{ε}. The integral in the denominator of (53) satisfies

$$\begin{aligned}
&\int G_{\varepsilon}\left(-\mu\right)^{m-2} G_{\varepsilon}\left(b-\mu\right) \overline{G}_{\varepsilon}\left(b-\mu\right)^{n-1} g_{\varepsilon}(-\mu)^{2} d\mu \\
&\geq \int_{-\infty}^{0} G_{\varepsilon}\left(-\mu\right)^{m-2} G_{\varepsilon}\left(b-\mu\right) \overline{G}_{\varepsilon}\left(b-\mu\right)^{n-1} g_{\varepsilon}(-\mu)^{2} d\mu \\
&\geq G_{\varepsilon}\left(b\right) \int_{-\infty}^{0} G_{\varepsilon}\left(-\mu\right)^{m-2} \overline{G}_{\varepsilon}\left(b-\mu\right)^{n-1} g_{\varepsilon}(-\mu)^{2} d\mu
\end{aligned}$$

where the last inequality reflects $G_{\varepsilon}\left(b-\mu\right) \geq G_{\varepsilon}(b)$ for $\mu \leq 0$. Combining these two bounds implies

$$\begin{aligned}
&\frac{\int_{-\infty}^{b} \left[k_{m,n} G_{\varepsilon}\left(b-\mu\right)+nbg_{\varepsilon}(b-\mu)\right] G_{\varepsilon}\left(-\mu\right)^{m-1} \overline{G}_{\varepsilon}\left(b-\mu\right)^{n-1} g_{\varepsilon}(-\mu) d\mu}{-(m-1)nb \int \int G_{\varepsilon}\left(-\mu\right)^{m-2} G_{\varepsilon}\left(b-\mu\right) \overline{G}_{\varepsilon}\left(b-\mu\right)^{n-1} g_{\varepsilon}(-\mu)^{2} d\mu} \\
&\leq \frac{k_{m,n} G_{\varepsilon}(b)}{-(m-1)nbG_{\varepsilon}\left(b\right) \int_{-\infty}^{0} G_{\varepsilon}\left(-\mu\right)^{m-2} \overline{G}_{\varepsilon}\left(b-\mu\right)^{n-1} g_{\varepsilon}(-\mu)^{2} d\mu} \\
&= \frac{k_{m,n}}{-(m-1)nb \int_{-\infty}^{0} G_{\varepsilon}\left(-\mu\right)^{m-2} \overline{G}_{\varepsilon}\left(b-\mu\right)^{n-1} g_{\varepsilon}(-\mu)^{2} d\mu}.
\end{aligned}$$

The integrand in the denominator increases as b decreases and the ratio therefore goes to 0 as $b \to -\infty$.

Integral over $[b,\infty)$. This integral equals

$$\int_b^\infty \left[k_{m,n} + nb\frac{g_\varepsilon(b-\mu)}{G_\varepsilon(b-\mu)}\right] G_\varepsilon(b-\mu)\, G_\varepsilon(-\mu)^{m-1}\, \overline{G}_\varepsilon(b-\mu)^{n-1}\, g_\varepsilon(-\mu)d\mu. \tag{54}$$

We have $b-\mu \le 0$ because $\mu \ge b$. For G_ε that satisfies (12) there exists a constant $K>0$ such that

$$\frac{g_\varepsilon(x)}{G_\varepsilon(x)} \ge K \text{ for } x \le 0.$$

The integral (54) is therefore bounded above by

$$\int_b^\infty [k_{m,n} + nbK]\, G_\varepsilon(b-\mu)\, G_\varepsilon(-\mu)^{m-1}\, \overline{G}_\varepsilon(b-\mu)^{n-1}\, g_\varepsilon(-\mu)d\mu.$$

The integrand and hence the integral itself are negative for

$$k_{m,n} + nbK < 0 \Leftrightarrow b < -\frac{k_{m,n}}{nK}. \quad \blacksquare$$

Proof of Theorem 3. As discussed at the beginning of this appendix, we prove that the strategic term of buyers is $O(1/m \wedge n)$ for any m,n satisfying (47). Our focus on offset equilibria implies that $\dot{v}(v,b)=1$, which means that attention can be restricted to $(v,b) \in \mathbb{R}^2$ such that

$$v - b \le L^*(G_\varepsilon) \tag{55}$$

where $L^*(G_\varepsilon)$ is the constant whose value is provided by Lemma 2. Select $L > L^*$ such that

$$\frac{G_\varepsilon(-L)}{\left(G_\varepsilon(-2L^*)\overline{G}_\varepsilon(2L^*)\right)^\varphi}, \frac{\overline{G}_\varepsilon(L)}{\left(G_\varepsilon(-2L^*)\overline{G}_\varepsilon(2L^*)\right)^\varphi} < \frac{1}{2}. \tag{56}$$

The denominator of each ratio on the left side of (56) is constant and so a sufficiently large L satisfies the inequalities. The number L is selected with foresight for its use later in the proof.

Formula (48) implies that $\dot{v}(v,b) \ge \delta > 0$ only if

$$0 \le \int \left[\widehat{M}^B_{m,n} - ng_\varepsilon(b-\mu)(v-b)\widehat{K}^B_{m,n}\right] g_\varepsilon(v-\mu)d\mu. \tag{57}$$

This integral is now computed over three intervals:

$$\begin{aligned} 0 \quad \le \quad & \int_{-\infty}^{b-L} \widehat{M}^B_{m,n} g_\varepsilon(v-\mu)d\mu \\ & + \int_{b-L}^{v+L} \left[\widehat{M}^B_{m,n} - ng_\varepsilon(b-\mu)(v-b)\widehat{K}^B_{m,n}\right] g_\varepsilon(v-\mu)d\mu \\ & + \int_{v+L}^{\infty} \widehat{M}^B_{m,n} g_\varepsilon(v-\mu)d\mu. \end{aligned}$$

The $ng_\varepsilon(b-\mu)(v-b)\widehat{K}^B_{m,n}$ term has been dropped from the integrands of the first and third integrals, which can only increase their values. This implies that $v-b$ is at most

$$\frac{\int_{b-L}^{v+L}\widehat{M}_{m,n}^{B}g_{\varepsilon}(v-\mu)d\mu}{\int_{b-L}^{v+L}ng_{\varepsilon}(b-\mu)\widehat{K}_{m,n}^{B}g_{\varepsilon}(v-\mu)d\mu} \tag{58}$$

$$+\frac{\int_{-\infty}^{b-L}\widehat{M}_{m,n}^{B}g_{\varepsilon}(v-\mu)d\mu+\int_{v+L}^{\infty}\widehat{M}_{m,n}^{B}g_{\varepsilon}(v-\mu)d\mu}{\int_{b-L}^{v+L}ng_{\varepsilon}(b-\mu)\widehat{K}_{m,n}^{B}g_{\varepsilon}(v-\mu)d\mu}. \tag{59}$$

The proof is completed below by showing that that (58) is $O\left(1/(m\wedge n)\right)$ and (59) is $O(2^{-m\wedge n})$. The bound we seek is thus determined over the states μ near v and b in $[b-L,v+L]$, with states μ in the tails $[-\infty,b-L]$ and $[v+L,\infty]$ shown to be relatively inconsequential.

Expression (58) is $O\left(1/(m\wedge n)\right)$. For $\mu\in[b-L,v+L]$, the bounds

$$\mu\geq b-L\Leftrightarrow b-\mu\leq L \tag{60}$$

and

$$\mu\leq v+L\Leftrightarrow -L\leq v-\mu$$

hold. The bound (55) then implies

$$v-\mu=v-b+b-\mu\leq L^{*}+L$$

and

$$b-\mu=b-v+v-\mu\geq -L^{*}-L.$$

So

$$v-\mu,b-\mu\in\left[-L^{*}-L,L+L^{*}\right]. \tag{61}$$

Turning to the numerator of (58), we have

$$\begin{aligned}\int_{b-L}^{v+L}\widehat{M}_{m,n}^{B}g_{\varepsilon}(v-\mu)d\mu &= \int_{b-L}^{v+L}\frac{\widehat{M}_{m,n}^{B}}{n\widehat{K}_{m,n}^{B}}n\widehat{K}_{m,n}^{B}g_{\varepsilon}(v-\mu)d\mu \\ &\leq \int_{b-L}^{v+L}\frac{M_{m\wedge n,m\wedge n}^{*}}{(n\wedge m)\,K_{m\wedge n,m\wedge n}^{*}}n\widehat{K}_{m,n}^{B}g_{\varepsilon}(v-\mu)d\mu \\ &\leq \frac{2}{m\wedge n}\int_{b-L}^{v+L}\frac{G_{\varepsilon}(v-\mu)\overline{G}_{\varepsilon}\left(b-\mu\right)}{\overline{G}_{\varepsilon}\left(v-\mu\right)}n\widehat{K}_{m,n}^{B}g_{\varepsilon}(v-\mu)d\mu.\end{aligned} \tag{62}$$

The step from the first to the second line follows from the fact $\widehat{M}_{m,n}^{B}/n\widehat{K}_{m,n}^{B}$ is non-increasing in n and in m when $m\geq 2$ (Williams (1991, Lem. 4.2)). The inequality

$$\frac{M_{m\wedge n,m\wedge n}^{*}}{K_{m\wedge n,m\wedge n}^{*}}<2\frac{G_{\varepsilon}(v-\mu)\overline{G}_{\varepsilon}\left(b-\mu\right)}{\overline{G}_{\varepsilon}\left(v-\mu\right)}$$

follows from substitution into formulas (5.3) and (5.4) for $z(v_2,b)$ of Satterthwaite and Williams

(1989b, p. 490) with $v-\mu$ replacing v_2, $b-\mu$ replacing b, and G_ε replacing F_1 and F_2.

The fact that $v-\mu$ and $b-\mu$ lie in the compact interval (61) implies that

$$\frac{G_\varepsilon(v-\mu)\overline{G}_\varepsilon\left(b-\mu\right)}{\overline{G}_\varepsilon\left(v-\mu\right)}$$

is bounded above and $g_\varepsilon(b-\mu)$ is bounded below for $\mu\in\left[b-L,v+L\right]$. Applying these bounds and (62), there exists a constant F^* such that (58) is at most

$$\frac{F^*}{m\wedge n}\cdot\frac{\int_{b-L}^{v+L}n\widehat{K}_{m,n}^{B}g_\varepsilon(v-\mu)d\mu}{\int_{b-L}^{v+L}n\widehat{K}_{m,n}^{B}g_\varepsilon(v-\mu)d\mu}=\frac{F^*}{m\wedge n},$$

which completes the proof that (58) is $O\left(1/\left(n\wedge m\right)\right)$.

Expression (59) is $O(2^{-m\wedge n})$**.** Consider first the denominator of (59). Reduce the support of this integral from $[b-L,v+L]$ to $[b-L^*,v+L^*]$. Replacing L with L^* in the analysis (60)-(61) implies

$$v-\mu,b-\mu\in\left[-2L^*,2L^*\right]$$

for $\mu\in[b-L^*,v+L^*]$. The fact that $v\geq b$ together with formula (49) for $\widehat{K}_{m,n}^{B}$ imply

$$\begin{aligned}n\widehat{K}_{m,n}^{B}(v,b,\mu) &\geq \sum_{\substack{i+j=m-1\\0\leq i\leq m-1\\0\leq j\leq n-1}} n\begin{pmatrix}m-1\\i\end{pmatrix}\begin{pmatrix}n-1\\j\end{pmatrix}\cdot G_\varepsilon\left(b-\mu\right)^{m-1}\overline{G}_\varepsilon\left(v-\mu\right)^{n-1}\\ &\geq \sum_{\substack{i+j=m-1\\0\leq i\leq m-1\\0\leq j\leq n-1}} n\begin{pmatrix}m-1\\i\end{pmatrix}\begin{pmatrix}n\\j\end{pmatrix}\cdot G_\varepsilon\left(-2L^*\right)^{m-1}\overline{G}_\varepsilon\left(2L^*\right)^{n-1}.\end{aligned}$$

Letting

$$g_\varepsilon^*=\inf_{x\in[-2L^*,2L^*]}g_\varepsilon(x),$$

the remaining term in the denominator satisfies

$$\begin{aligned}\int_{b-L^*}^{v+L^*}g_\varepsilon(b-\mu)g_\varepsilon(v-\mu)d\mu &\geq \left(g_\varepsilon^*\right)^2\left(v-b+2L^*\right)\\ &\geq \left(g_\varepsilon^*\right)^2\left(2L^*\right).\end{aligned}$$

The denominator of (59) is thus at least

$$\sum_{\substack{i+j=m-1\\0\leq i,j\leq m-1\\0\leq j\leq n-1}}\begin{pmatrix}m-1\\i\end{pmatrix}\begin{pmatrix}n\\j\end{pmatrix}\cdot G_\varepsilon\left(-2L^*\right)^{m-1}\overline{G}_\varepsilon\left(2L^*\right)^{n-1}\left(g_\varepsilon^*\right)^2\left(2L^*\right). \tag{63}$$

Turning to the numerator of (59), notice that

$$\mu \leq b - L \Leftrightarrow L \leq b - \mu$$

and

$$\mu \geq v + L \Leftrightarrow v - \mu \leq -L.$$

Applying $b \leq v$, formula (51) implies

$$\widehat{M}^{B}_{m,n} \leq \sum_{\substack{i+j=m \\ 0\leq i\leq m-1 \\ 1\leq j\leq n}} \binom{m-1}{i} \binom{n}{j} G_{\varepsilon}\left(v-\mu\right)^{m} \overline{G}_{\varepsilon}\left(b-\mu\right)^{n-1}.$$

The change of index $j+1 \to j$ implies

$$\begin{aligned}
\sum_{\substack{i+j=m \\ 0\leq i\leq m-1 \\ 1\leq j\leq n}} \binom{m-1}{i} \binom{n}{j} &= \sum_{\substack{i+j=m-1 \\ 0\leq i\leq m-1 \\ 0\leq j\leq n-1}} \binom{m-1}{i} \binom{n}{j+1} \\
&= \sum_{\substack{i+j=m-1 \\ 0\leq i\leq m-1 \\ 0\leq j\leq n-1}} \binom{m-1}{i} \binom{n-1}{j} \frac{n}{j+1} \\
&\leq n \sum_{\substack{i+j=m-1 \\ 0\leq i\leq m-1 \\ 0\leq j\leq n-1}} \binom{m-1}{i} \binom{n-1}{j}.
\end{aligned}$$

Therefore, for $\mu \in [-\infty, b-L]$,

$$\widehat{M}^{B}_{m,n} \leq \sum_{\substack{i+j=m-1 \\ 0\leq i\leq m-1 \\ 0\leq j\leq n-1}} n \binom{m-1}{i} \binom{n-1}{j} \overline{G}_{\varepsilon}\left(L\right)^{n-1} \tag{64}$$

and, for $\mu \in [v+L, \infty]$,

$$\widehat{M}^{B}_{m,n} \leq \sum_{\substack{i+j=m-1 \\ 0\leq i\leq m-1 \\ 0\leq j\leq n-1}} n \binom{m-1}{i} \binom{n-1}{j} G_{\varepsilon}\left(-L\right)^{m}. \tag{65}$$

We now combine the bounds (63), (64) and (65). Notice that

$$\sum_{\substack{i+j=m-1 \\ 0\leq i\leq m-1 \\ 0\leq j\leq n-1}} n \binom{m-1}{i} \binom{n-1}{j}$$

appears in each of the bounds (63), (64) and (65) and therefore cancels. The ratio (59) is therefore bounded above with

$$
\begin{aligned}
& \frac{\overline{G}_{\varepsilon}(L)^{n-1} \int_{-\infty}^{b-L} g_{\varepsilon}(v-\mu) d\mu + G_{\varepsilon}(-L)^{m} \int_{v+L}^{\infty} g_{\varepsilon}(v-\mu) d\mu}{G_{\varepsilon}(-2L^{*})^{m-1} \overline{G}_{\varepsilon}(2L^{*})^{n-1} (g_{\varepsilon}^{*})^{2}(2L^{*})} \\
\leq \; & \frac{1}{(g_{\varepsilon}^{*})^{2}(2L^{*})} \cdot \frac{\overline{G}_{\varepsilon}(L)^{n} + G_{\varepsilon}(-L)^{m}}{G_{\varepsilon}(-2L^{*})^{m} \overline{G}_{\varepsilon}(2L^{*})^{n}} \\
\leq \; & \frac{1}{(g_{\varepsilon}^{*})^{2}(2L^{*})} \cdot \left[\frac{\overline{G}_{\varepsilon}(L)^{n}}{\overline{G}_{\varepsilon}(2L^{*})^{\varphi n} \overline{G}_{\varepsilon}(2L^{*})^{\varphi n}} + \frac{G_{\varepsilon}(-L)^{m}}{\overline{G}_{\varepsilon}(2L^{*})^{\varphi m} \overline{G}_{\varepsilon}(2L^{*})^{\varphi m}} \right] \\
\leq \; & \frac{1}{(g_{\varepsilon}^{*})^{2}(2L^{*})} \cdot \left[\left(\frac{1}{2}\right)^{n} + \left(\frac{1}{2}\right)^{m} \right] \\
\leq \; & \frac{1}{(g_{\varepsilon}^{*})^{2}(L^{*})} \cdot \left(\frac{1}{2}\right)^{m \wedge n}
\end{aligned}
$$

where the bound (56) is applied in the fourth line of this calculation. ■

E Sampling Error

Proof of Theorem 6. We set $\mu = 0$ with no loss in generality. In this case $p^{\mathrm{REE}} = \xi_q^{\varepsilon}$. We first note that $z_{(\eta m+1)} - \xi_q^{\varepsilon}$ is asymptotically normal after a suitable rescaling, i.e., for any $r \in \mathbb{R}$,

$$\lim_{\eta \to \infty} \Pr\left(\left(z_{(\eta m+1)} - \xi_q^{\varepsilon}\right) \frac{g_{\varepsilon}\left(\xi_q^{\varepsilon}\right) \sqrt{\eta (m+n)}}{\sqrt{q(1-q)}} \leq r \right) = \Phi(r).$$

See, for example, Arnold, Balakrishnan, and Nagaraja (1992, Thm. 8.5.1). The random variable

$$\left(z_{(\eta m+1)} - \xi_q^{\varepsilon}\right) \frac{g_{\varepsilon}\left(\xi_q^{\varepsilon}\right) \sqrt{\eta (m+n)}}{\sqrt{q(1-q)}}$$

thus converges in distribution to the standard normal. Theorem 6 rests upon the convergence of the first absolute moment of this random variable to the corresponding first absolute moment of the limiting standard normal distribution. Condition (18) ensures that[17]

$$\lim_{\eta \to \infty} \mathbb{E}\left[\left| z_{(\eta m+1)} - \xi_q^{\varepsilon} \right| \middle| \mu = 0 \right] \cdot \frac{g_{\varepsilon}\left(\xi_q^{\varepsilon}\right) \sqrt{\eta (m+n)}}{\sqrt{q(1-q)}} = \mathbb{E}_{\Phi}\left[|r|\right] \tag{66}$$

where $r \sim \mathcal{N}(0,1)$. From (66) it is clear that for sufficiently large η,

$$\frac{1}{2\sqrt{\eta}} \cdot K \cdot \mathbb{E}_{\Phi}\left[|r|\right] < \mathbb{E}\left[\left| z_{(\eta m+1)} - \xi_q^{\varepsilon} \right| \middle| \mu = 0 \right] < \frac{2}{\sqrt{\eta}} \cdot K \cdot \mathbb{E}_{\Phi}\left[|r|\right]$$

[17] This result originates in Anderson (1982). We apply here Shorack and Wellner (1986, Thm. 4, p. 475) in the case of $h(x) = |x|$. Condition (18) is assumption (12) of this theorem and its assumption (14) holds for $h(x) = |x|$ with $M = 1$ and $x^{*} = 3$.

where K denotes the constant

$$K \equiv \frac{\sqrt{q\left(1-q\right)}}{g_{\varepsilon}\left(\xi_{q}^{\varepsilon}\right)\sqrt{\left(m+n\right)}}.$$

This completes the proof. ■

Online Appendices for "Price Discovery Using a Double Auction"

F An Example of the Failure of Affiliation in the CIV case

This section presents an example in which, conditional on the state μ, affiliation holds (i) among traders' signals, (ii) among the focal buyer's signal and the ordered signals of the other traders, (iii) among the focal buyer's signal and the bids/asks of other traders, but (iv) not among the focal buyer's signal and the *ordered* bids/asks of the other traders. It is case (iv) that is relevant to the focal buyer's decision problem. There are two buyers and two sellers ($m = n = 2$) and buyer one is the focal buyer. We assume that $\varepsilon \sim \mathcal{N}(0, 1/\tau_\varepsilon)$ and $\delta \sim \mathcal{N}(0, 1/\tau_\delta)$. It follows that $\varepsilon + \delta \sim \mathcal{N}(0, 1/\tau)$ for $\tau \equiv (1/\tau_\varepsilon + 1/\tau_\delta)^{-1}$.

Conditional on μ the traders' values/costs (v_1, v_2, c_1, c_2) are independent and identically distributed. Additionally, conditional on μ their signals $(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$ are also independent and identically distributed (here the buyers are indexed by signals 1 and 2, while the two sellers by signals 3 and 4). Hence the density of $(\mu, \sigma_1, \sigma_2, \sigma_3, \sigma_4)$ is

$$f_{\mu,\sigma_1,\sigma_2,\sigma_3,\sigma_4}(x_0, x_1, x_2, x_3, x_4) \quad = \quad g_\mu(x_0)\Pi_{i=1}^4 g_{\varepsilon+\delta}(x_i - x_0).$$

The density g_μ of μ can be either proper or improper for this section. Affiliation requires that

$$\frac{\partial^2 \ln f_{\mu,\sigma_1,\sigma_2,\sigma_3,\sigma_4}(x_0, x_1, x_2, x_3, x_4)}{\partial x_i \partial x_j} \quad \geq \quad 0$$

for all $i \neq j$ and $i, j \in \{0, 1, 2, 3, 4\}$ (Milgrom and Weber (1982, Thm. 1(i))). For $g_{\varepsilon+\delta}$ that is log-concave, so that the conditional $f_{\sigma|\mu}(\sigma|\mu) = g_{\varepsilon+\delta}(\sigma - \mu)$ (from (3)) satisfies the *monotone likelihood ratio property* (MLRP), the above inequality holds (Milgrom and Weber (1982, p. 1099)). The normal is log-concave so the vector $(\mu, \sigma_1, \sigma_2, \sigma_3, \sigma_4)$ is affiliated, which establishes (i).[18]

We now show that the ordered signals of the nonfocal traders are affiliated with the signal of the focal buyer. Let $\tilde{s}_{(i)}$ be the i^{th} order statistic of $(\sigma_2, \sigma_3, \sigma_4)$. The density of $(\mu, \sigma_1, \tilde{s}_{(1)}, \tilde{s}_{(2)}, \tilde{s}_{(3)})$ is

$$\begin{aligned} f_{\mu,\sigma_1,\tilde{s}_{(1)},\tilde{s}_{(2)},\tilde{s}_{(3)}}(x_0, x_1, x_2, x_3, x_4) &= g_\mu(x_0)3!\Pi_{i=1}^4 g_{\varepsilon+\delta}(x_i - x_0) \\ &= g_\mu(x_0)g_{\varepsilon+\delta}(x_1 - x_0)3!\Pi_{i=2}^4 g_{\varepsilon+\delta}(x_i - x_0) \end{aligned}$$

for $x_2 \leq x_3 \leq x_4$. Since the initial density $f_{\mu,\sigma_1,\sigma_2,\sigma_3,\sigma_4}$ is symmetric in its last $m + n - 1 = 2$ arguments, for $g_{\varepsilon+\delta}$ normal affiliation holds (Milgrom and Weber (1982, Thm. 2)). This establishes (ii).

We now show that the bid/asks of the nonfocal traders are affiliated with the focal buyer's signal. Consider the sample of bid/asks where the nonfocal buyer uses offset λ_B and the sellers

[18] Note that affiliation of a vector of random variables implies affiliation of any subset of the random variables in that vector; in particular affiliation of $(\mu, \sigma_1, \sigma_2, \sigma_3, \sigma_4)$ implies affiliation of $(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$ (Milgrom and Weber (1982, Thm. 4)).

use offset λ_S. Note that $(\lambda_B, \lambda_S) \in \mathbb{R}^2$ are arbitrary constants, not necessarily the solutions to the buyers' and sellers' FOCs. Then conditional on μ the elements of the vector $(\sigma_1, b_2, a_1, a_2)$, where $b_2 = \sigma_2 + \lambda_B$, $a_1 = \sigma_3 + \lambda_S$, $a_2 = \sigma_4 + \lambda_S$, are independent but not identically distributed. In particular,

$$\begin{aligned} f_{\mu,\sigma_1,b_2,a_1,a_2}(x_0,x_1,x_2,x_3,x_4) &= g_\mu(x_0)g_{\varepsilon+\delta}(x_1-x_0)g_{\varepsilon+\delta}(x_2-\lambda_B-x_0)g_{\varepsilon+\delta}(x_3-\lambda_S-x_0) \\ &\times \; g_{\varepsilon+\delta}(x_4-\lambda_S-x_0). \end{aligned}$$

However, since the bid and the asks are increasing functions affiliation holds for the vector $(\mu, \sigma_1, b_2, a_1, a_2)$ (Milgrom and Weber (1982, Thm. 3)). This establishes (iii).

We now show that the ordered bid/asks of the nonfocal traders are not affiliated with the focal buyer's signal. Let $s_{(i)}$ be the i^{th} order statistic of the vector of bid/asks (b_2, a_1, a_2). From David and Nagaraja (2003, p. 98) the density of $(\mu, \sigma_1, s_{(1)}, s_{(2)}, s_{(3)})$ is

$$\begin{aligned} f_{\mu,\sigma_1,s_{(1)},s_{(2)},s_{(3)}}(x_0,x_1,x_2,x_3,x_4) &= 2g_\mu(x_0)g_{\varepsilon+\delta}(x_1-x_0) \\ &\times \Big[\underbrace{g_{\varepsilon+\delta}(x_2-\lambda_B-x_0)g_{\varepsilon+\delta}(x_3-\lambda_S-x_0)g_{\varepsilon+\delta}(x_4-\lambda_S-x_0)}_{Z_1} \\ &+ \underbrace{g_{\varepsilon+\delta}(x_2-\lambda_S-x_0)g_{\varepsilon+\delta}(x_3-\lambda_B-x_0)g_{\varepsilon+\delta}(x_4-\lambda_S-x_0)}_{Z_2} \\ &+ \underbrace{g_{\varepsilon+\delta}(x_2-\lambda_S-x_0)g_{\varepsilon+\delta}(x_3-\lambda_S-x_0)g_{\varepsilon+\delta}(x_4-\lambda_B-x_0)}_{Z_3}\Big], \end{aligned} \tag{67}$$

for $x_2 \le x_3 \le x_4$. Focus on $\partial^2 \ln f_{\mu,\sigma_1,s_{(1)},s_{(2)},s_{(3)}}(x_0,x_1,x_2,x_3,x_4)/\partial x_2 \partial x_3$. If this is greater than or equal to zero, then the focal buyer's signal may be affiliated with the order statistics of other traders' bid/asks. If it is negative, then affiliation fails. Since the term $2g_\mu(x_0)g_{\varepsilon+\delta}(x_1-x_0)$ does not depend on either x_2 or x_3 we have

$$\frac{\partial^2 \ln f_{\mu,\sigma_1,s_{(1)},s_{(2)},s_{(3)}}(x_0,x_1,x_2,x_3,x_4)}{\partial x_2 \partial x_3} = \frac{\partial^2 \ln Z}{\partial x_2 \partial x_3},$$

where $Z \equiv Z_1 + Z_2 + Z_3$ and Z_1, Z_2, Z_3 are defined in (67) above.[19]

Since $\varepsilon + \delta \sim \mathcal{N}(0, 1/\tau)$ we have $g_{\varepsilon+\delta}(x) = \sqrt{\tau}\exp(-\tau x^2/2)/\sqrt{2\pi}$ and so

$$\begin{aligned} Z_1 &= g_{\varepsilon+\delta}(x_2-\lambda_B-x_0)g_{\varepsilon+\delta}(x_3-\lambda_B-x_0)g_{\varepsilon+\delta}(x_4-\lambda_S-x_0) \\ &= \left(\sqrt{\frac{\tau}{2\pi}}\right)^3 \exp\Big\{-\frac{\tau}{2}\big[(x_2-x_0)^2+(x_3-x_0)^2+(x_4-x_0)^2+\lambda_B^2+2\lambda_S^2 \\ &\quad -2\lambda_B(x_2-x_0)-2\lambda_S(x_3-x_0)-2\lambda_S(x_4-x_0)\big]\Big\}, \end{aligned}$$

[19]The terms Z_1, Z_2, Z_3 depend on $(x_0, x_2, x_3, x_4, \lambda_B, \lambda_S)$, but we omit these arguments for simplicity.

and similarly for Z_2 and Z_3. Therefore

$$\begin{aligned} Z &= \left(\sqrt{\frac{\tau}{2\pi}}\right)^3 \exp\left\{-\frac{\tau}{2}\left[(x_2-x_0)^2+(x_3-x_0)^2+(x_4-x_0)^2+\lambda_B^2+2\lambda_S^2\right]\right\} \\ &\times \Big[\underbrace{\exp\left[\tau\lambda_B(x_2-x_0)+\tau\lambda_S(x_3-x_0)+\tau\lambda_S(x_4-x_0)\right]}_{\tilde{Z}_1} \\ &+ \underbrace{\exp\left[\tau\lambda_S(x_2-x_0)+\tau\lambda_B(x_3-x_0)+\tau\lambda_S(x_4-x_0)\right]}_{\tilde{Z}_2} \\ &+ \underbrace{\exp\left[\tau\lambda_S(x_2-x_0)+\tau\lambda_S(x_3-x_0)+\tau\lambda_B(x_4-x_0)\right]}_{\tilde{Z}_3}\Big]. \end{aligned} \tag{68}$$

Let $\tilde{Z} \equiv \tilde{Z}_1 + \tilde{Z}_2 + \tilde{Z}_3$ where $\tilde{Z}_1, \tilde{Z}_2, \tilde{Z}_3$ are defined in (68) above. Taking the natural logarithm of Z in (68) separates the first line from the rest. The term produced by the first line is

$$\ln\left(\sqrt{\frac{\tau}{2\pi}}\right)^3 - \frac{\tau}{2}\left[(x_2-x_0)^2+(x_3-x_0)^2+(x_4-x_0)^2+\lambda_B^2+2\lambda_S^2\right].$$

This has no x_2 and x_3 cross-terms and hence does not affect the cross-partial with respect to these variables. Therefore

$$\frac{\partial^2 \ln f_{\mu,\sigma_1,s_{(1)},s_{(2)},s_{(3)}}(x_0,x_1,x_2,x_3,x_4)}{\partial x_2 \partial x_3} = \frac{\partial^2 \ln \tilde{Z}}{\partial x_2 \partial x_3}.$$

We now have:

$$\begin{aligned} \frac{\partial^2 \ln \tilde{Z}}{\partial x_2 \partial x_3} &= \partial\left(\frac{\partial \ln \tilde{Z}}{\partial x_2}\right)/\partial x_3 = \partial\left(\frac{1}{\tilde{Z}}\frac{\partial \tilde{Z}}{\partial x_2}\right)/\partial x_3 = -\frac{1}{\tilde{Z}^2}\frac{\partial \tilde{Z}}{\partial x_3}\frac{\partial \tilde{Z}}{\partial x_2} + \frac{1}{\tilde{Z}}\frac{\partial^2 \tilde{Z}}{\partial x_2 \partial x_3} \\ &= \frac{1}{\tilde{Z}}\left[\frac{\partial^2 \tilde{Z}}{\partial x_2 \partial x_3} - \frac{1}{\tilde{Z}}\frac{\partial \tilde{Z}}{\partial x_3}\frac{\partial \tilde{Z}}{\partial x_2}\right]. \end{aligned} \tag{69}$$

The sign of the partial is determined by the sign of the terms inside the square brackets of (69). In turn:

$$\begin{aligned} \frac{\partial \tilde{Z}}{\partial x_2} &= \tau\left(\lambda_B \tilde{Z}_1 + \lambda_S \tilde{Z}_2 + \lambda_S \tilde{Z}_3\right), \\ \frac{\partial \tilde{Z}}{\partial x_3} &= \tau\left(\lambda_S \tilde{Z}_1 + \lambda_B \tilde{Z}_2 + \lambda_S \tilde{Z}_3\right), \\ \frac{\partial^2 \tilde{Z}}{\partial x_2 \partial x_3} &= \tau\left(\lambda_B \lambda_S \tilde{Z}_1 + \lambda_S \lambda_B \tilde{Z}_2 + \lambda_S^2 \tilde{Z}_3\right), \end{aligned}$$

so that

$$\begin{aligned}
\frac{\partial^2 \tilde{Z}}{\partial x_2 \partial x_3} - \frac{1}{\tilde{Z}} \frac{\partial \tilde{Z}}{\partial x_3} \frac{\partial \tilde{Z}}{\partial x_2} &= \tau \left(\lambda_B \lambda_S \tilde{Z}_1 + \lambda_S \lambda_B \tilde{Z}_2 + \lambda_S^2 \tilde{Z}_3 \right) \\
&- \frac{\tau^2 \left(\lambda_B \tilde{Z}_1 + \lambda_S \tilde{Z}_2 + \lambda_S \tilde{Z}_3 \right) \left(\lambda_B \tilde{Z}_1 + \lambda_S \tilde{Z}_2 + \lambda_S \tilde{Z}_3 \right)}{\tilde{Z}_1 + \tilde{Z}_2 + \tilde{Z}_3} \\
&= -\frac{\tau^2 \tilde{Z}_1 \tilde{Z}_2}{\tilde{Z}} (\lambda_B - \lambda_S)^2.
\end{aligned}$$

Putting it all together, the partial of interest is

$$\frac{\partial^2 \ln f_{\mu,\sigma_1,s_{(1)},s_{(2)},s_{(3)}}(x_0, x_1, x_2, x_3, x_4)}{\partial x_2 \partial x_3} = -\frac{\tau^2 \tilde{Z}_1 \tilde{Z}_2}{\tilde{Z}^2} (\lambda_B - \lambda_S)^2.$$

Since $\tilde{Z}_1$ and $\tilde{Z}_2$ are strictly positive, it is clear that for any $\lambda_B \neq \lambda_S$ the partial is strictly negative and hence the focal buyer's signal is not affiliated with the ordered bid/asks of the other traders. This establishes (iv).

We note two points. First, the proof can be replicated for a focal seller by simply changing the interpretation of λ_B and λ_S. Second, the same arguments hold trivially for the CPV case in which $\tau_\delta \to \infty$. We then have that $g_{\varepsilon+\delta} \to g_\varepsilon$ and $(\sigma_1, \sigma_2) \to (v_1, v_2)$, $(\sigma_3, \sigma_4) \to (c_1, c_2)$. In the CPV case sellers always ask their cost, so $\lambda_S \to 0$ and affiliation fails for $\lambda_B \neq 0$.

G Equivalence of the Two Forms of the FOCs

In Appendix A, we derived the alternative forms (22) and (29) of the FOCs (5) and (7), or equivalently, (4) and (6). The following lemma establishes the equivalence of the two derivations.

Lemma 3 *The two forms (4) and (22) of the buyer's first order condition,*

$$\left(\mathbb{E}[v|\sigma_B, x = b] - b \right) f^B_{x|\sigma} \left(b|\sigma_B \right) - \Pr\left[x < b < y | \sigma_B \right] = 0$$

and

$$\int \left\{ \left(\mathbb{E}[v|\mu, \sigma_B] - b \right) f^B_{x|\mu}(b|\mu) - \int_{-\infty}^{b} \int_{b}^{\infty} f^B_{xy|\mu}(x, y|\mu) dy dx \right\} f_{\mu|\sigma}(\mu|\sigma_B) d\mu = 0,$$

are equivalent. Similarly, the two forms (6) and (29) of the seller's first order condition are equivalent.

Proof. We focus on the more complicated case of the buyer's FOC. We need to prove three equalities. The first is

$$\begin{aligned}\int b f^B_{x|\mu}(b|\mu) f_{\mu|\sigma}(\mu|\sigma_B) d\mu &= b \int f^B_{x|\mu,\sigma}(b|\mu,\sigma_B) f_{\mu|\sigma}(\mu|\sigma_B) d\mu \\ &= b \int f^B_{x,\mu|\sigma}(b,\mu|\sigma_B) d\mu \\ &= b f^B_{x|\sigma}(b|\sigma_B).\end{aligned}$$

The second line above follows from Bayes' Rule together with the fact that, conditional on μ, the buyer's signal σ_B offers no new information regarding other traders' signals and resulting bids/asks. The second equality follows from integrating with respect to μ:

$$\int \Pr\left[x < b < y|\mu\right] f_{\mu|\sigma}(\mu|\sigma_B) d\mu = \Pr\left[x < b < y|\sigma_B\right].$$

The last equality is

$$\mathbb{E}[v|\sigma_B, x = b] f^B_{x|\sigma}(b|\sigma_B) = \int \mathbb{E}[v|\mu,\sigma_B] f^B_{x|\mu}(b|\mu) f_{\mu|\sigma}(\mu|\sigma_B) d\mu.$$

Conditional on μ, x and σ_B are independent. Therefore

$$\begin{aligned}\int \mathbb{E}[v|\mu,\sigma_B] f^B_{x|\mu}(b|\mu) f_{\mu|\sigma}(\mu|\sigma_B) d\mu &= \int\int v f_{v|\mu,\sigma}(v|\mu,\sigma_B) f^B_{x|\mu}(b|\mu) f_{\mu|\sigma}(\mu|\sigma_B) dv d\mu \\ &= \int\int v f_{v|\mu,\sigma}(v|\mu,\sigma_B) f^B_{x|\mu,\sigma}(b|\mu,\sigma_B) f_{\mu|\sigma}(\mu|\sigma_B) dv d\mu.\end{aligned}$$

Observe that

$$f_{v|\mu,\sigma}(v|\mu,\sigma_B) f^B_{x|\mu,\sigma}(b|\mu,\sigma_B) = f^B_{v,x|\mu,\sigma}(v,b|\mu,\sigma_B) = \frac{f^B_{v,x,\mu|\sigma}(v,b,\mu|\sigma_B)}{f_{\mu|\sigma}(\mu|\sigma_B)}.$$

Substitution produces

$$\begin{aligned}& \int \mathbb{E}[v|\mu,\sigma_B] f^B_{x|\mu}(b|\mu) f_{\mu|\sigma}(\mu|\sigma_B) d\mu \\ = & \int\int v f^B_{v,x,\mu|\sigma}(v,b,\mu|\sigma_B) dv d\mu = \int v \left\{\int f^B_{v,x,\mu|\sigma}(v,b,\mu|\sigma_B) d\mu\right\} dv \\ = & \int v f^B_{v,x|\sigma}(v,b|\sigma_B) dv = \int v f^B_{v|x,\sigma}(v|x=b,\sigma_B) f^B_{x|\sigma}(b|\sigma_B) dv \\ = & \left\{\int v f^B_{v|x,\sigma}(v|x=b,\sigma_B) dv\right\} f^B_{x|\sigma}(b|\sigma_B) = \mathbb{E}[v|\sigma_B, x=b] f^B_{x|\sigma}(b|\sigma_B).\end{aligned}$$

This completes the proof of equivalence for buyers. The proof for sellers is similar except that the second of the three equalities is not present. ■

H Sufficiency

Proof of Theorem 2. Consider a focal buyer with signal σ_B and assume that all other buyers and sellers play strategies B and S that satisfy the FOCs (4) and (6) for all $(\sigma_B, \sigma_S) \in \mathbb{R}^2$. For all (σ_B, b) such that $B(\sigma_B) = b$ the buyer's FOC (4) is

$$\pi_b^B(\sigma_B, b|S, B) = (\mathbb{E}[v|\sigma_B, x = b] - b)\, f_{x|\sigma}^B(b|\sigma_B) - \Pr[x < b < y|\sigma_B] = 0. \tag{70}$$

A sufficient condition for B to be a best response is that the following statements hold for all $(\sigma_B, b) \in \mathbb{R}^2$:

$$\left\{\begin{matrix} \pi_b^B(\sigma_B, b|S, B) \geq 0 \text{ if } b < B(\sigma_B) \\ \pi_b^B(\sigma_B, b|S, B) = 0 \text{ if } b = B(\sigma_B) \\ \pi_b^B(\sigma_B, b|S, B) \leq 0 \text{ if } B(\sigma_B) < b \end{matrix}\right\} \Leftrightarrow \left\{\begin{matrix} \pi_b^B(\sigma_B, b|S, B) \geq 0 \text{ if } \sigma_B(b) < \sigma_B \\ \pi_b^B(\sigma_B, b|S, B) = 0 \text{ if } \sigma_B(b) = \sigma_B \\ \pi_b^B(\sigma_B, b|S, B) \leq 0 \text{ if } \sigma_B < \sigma_B(b) \end{matrix}\right\}. \tag{71}$$

The left-hand column of conditions is the first derivative test on the variable b for a fixed value of σ_B. The equivalent conditions in the right-hand column are obtained by applying the inverse mapping $\sigma_B(\cdot) = B^{-1}(\cdot)$ to the variables. They are useful below when we fix the bid b and interpret σ_B as the variable.

The right column is satisfied if, for a fixed $b \in \mathbb{R}$, $\pi_b(\sigma_B, b|B, S)$ is increasing in σ_B and equals zero at $\sigma_B = \sigma_B(b)$. Rewrite the focal buyer's marginal utility (70) as

$$\begin{aligned} \pi_b(\sigma_B, b|B, S) &= (\mathbb{E}[v|\sigma_B, x = b] - b)\, f_{x|\sigma}^B(b|\sigma_B) - \Pr[x < b < y|\sigma_B] \\ &= f_{x|\sigma}^B(b|\sigma_B) \left\{ (\mathbb{E}[v|\sigma_B, x = b] - b) - \frac{\Pr[x \leq b \leq y|\sigma_B]}{f_{x|\sigma}^B(b|\sigma_B)} \right\}. \end{aligned} \tag{72}$$

Given that $f_{x|\sigma}^B(b|\sigma_B) > 0$ for all $(\sigma_B, b) \in \mathbb{R}^2$, the term in brackets determines the sign of $\pi_b^B(\sigma_B, b|B, S)$. The entire term equals 0 at $\sigma_B = \sigma_B(b) = B^{-1}(b)$ because B is a solution to the FOC. Hence, if the term in brackets is increasing in σ_B for all $b \in \mathbb{R}$ then (71) is satisfied.

Consider now a focal seller j and continue to assume that B and S are solutions to the buyers' and sellers' FOCs. For all (σ_S, a) such that $a = S(\sigma_S)$, seller j's FOC (6) is

$$\pi_a^S(a|\sigma_S, B, S) = (\mathbb{E}[c|\sigma_S, x = a] - a)\, f_{x|\sigma}^S(a|\sigma_S) = 0. \tag{73}$$

Similar to the logic of (71), S is a best response if

$$\left\{\begin{matrix} \pi_a^S(a|\sigma_S, B, S) \geq 0 \text{ if } \sigma_S(a) < \sigma_S \\ \pi_a^S(a|\sigma_S, B, S) = 0 \text{ if } \sigma_S(a) = \sigma_S \\ \pi_a^S(a|\sigma_S, B, S) \leq 0 \text{ if } \sigma_S < \sigma_S(a) \end{matrix}\right\}.$$

Since S is a solution to the FOC, $\pi_a^S(\sigma_S, a|B) = 0$ at $\sigma_S = \sigma_S(a) = S^{-1}(a)$, and the above is satisfied if $\mathbb{E}[c|\sigma_S, x = a]$ is increasing in σ_S for all $a \in \mathbb{R}$. ■

I Convergence to Efficiency in the CPV Case

Proof of Theorem 4. As discussed at the beginning of Appendix D, we consider a market with m buyers and n sellers such that m, n satisfy (47) for some constant φ. The state μ is fixed. The denominator of relative inefficiency is the potential gains from trade. In each state μ, the gains from trade are at least $m \wedge n$ times the expected gains from trade $\Gamma\left(G_{\varepsilon}\right)$ between a single buyer and a single seller. Notice that the number $\Gamma\left(G_{\varepsilon}\right)$ is the same in each state μ. The expected gains from trade in state μ are therefore at least $(m \wedge n)\,\Gamma\left(G_{\varepsilon}\right)$.

The numerator of relative inefficiency in (16) is the expected value of the efficient trades in state μ that fail to be made because of buyer strategic behavior. Given the lower bound $(m \wedge n)\,\Gamma\left(G_{\varepsilon}\right)$ on the denominator, it remains to be shown that the numerator is $O\left(1/\left(m \wedge n\right)\right)$. We first show that the value of a trade that inefficiently fails to occur is $O(1/\left(m \wedge n\right))$. The second step is then to show that the expected number of trades that inefficiently fail to occur is bounded above by a constant that does not depend on m or n.

Given a sample of m values and n costs, we first consider the value of a trade between a buyer with value v and a seller with cost c that occurs in the efficient allocation but does not occur in the equilibrium $\langle B, S\rangle$, where S is the truth-telling dominant strategy of sellers. Let p denote the market price in the BBDA determined by the sample and by $\langle B, S\rangle$. This buyer and this seller each fail to trade in the BBDA only if

$$B(v) \leq p \leq c. \tag{74}$$

Efficiency requires that both should trade only if v is among the largest n values/costs and c is among the m smallest values/costs. Consequently, trade between this buyer and this seller occurs in the efficient allocation only if

$$c \leq v. \tag{75}$$

Combining (74) and (75) implies

$$B(v) \leq c \leq v.$$

Theorem 3 then implies that the value $v - c$ of a trade that inefficiently fails to occur satisfies

$$v - c \leq v - B(v) \leq \frac{K}{m \wedge n}.$$

We suppress the dependence of K upon the distribution G_{ε} and the bound φ on the relative size of m and n for notational simplicity in this proof.

The proof is completed by bounding above the expected number of trades that inefficiently fail to occur in the state μ by a constant determined by F that is independent of m, n and μ. Here, $z_{(m+1)}$ denotes the $(m+1)^{\text{st}}$ smallest value/cost in a sample of m buyers' values and n sellers' costs, $F_{z_{(m+1)}}\left(z_{(m+1)}\,|\mu\right)$ denotes the distribution of $z_{(m+1)}$ conditional on μ, and $L(z_{(m+1)}, \mu)$ is the expected number of trades that inefficiently fail to occur given the realization of $z_{(m+1)}$ in the state μ. The proof is completed by bounding the expected number of missed trades in the state μ,

$$\int L(z_{(m+1)},\mu)dF_{z_{(m+1)}}\left(z_{(m+1)}\,|\mu\right), \tag{76}$$

by a constant that does not depend on m, n or μ. The integral (76) is represented as the sum

$$\int_{-\infty}^{\mu+\gamma} L(z_{(m+1)}\,|\mu)dF_{z_{(m+1)}}\left(z_{(m+1)}\,|\mu\right) \tag{77}$$

$$+\int_{\mu+\gamma}^{\infty} L(z_{(m+1)}\,|\mu)dF_{z_{(m+1)}}\left(z_{(m+1)}\,|\mu\right) \tag{78}$$

where γ is a constant chosen with foresight so that

$$\overline{G}_{\varepsilon}(\gamma)\leq\frac{1}{2^{3\varphi}}. \tag{79}$$

The two integrals (77) and (78) are bounded above in separate arguments.

Bounding (77). Because buyers underbid by at most $K/(m\wedge n)$, the equilibrium market price (i.e., the $(m+1)^{\text{st}}$ bid/ask) necessarily lies in the interval

$$\left[z_{(m+1)}-\frac{K}{m\wedge n},z_{(m+1)}\right].$$

The value of a buyer who inefficiently fails to trade must lie (i) at or below $z_{(m+1)}+K/\left(m\wedge n\right)$ so that his bid can fall below the market price, and (ii) at or above $z_{(m+1)}$ so that he should in fact trade. Given μ and $z_{(m+1)}$, the expected number of trades that inefficiently fail to occur is therefore at most equal to the expected number of buyers' values in the interval

$$\left[z_{(m+1)},z_{(m+1)}+\frac{K}{m\wedge n}\right]. \tag{80}$$

This expected value is bounded using an upper bound on the density of trader's value/cost in this interval given μ.

We now compute the bound on this density. Let z denote a trader's value/cost above $z_{(m+1)}$. Conditional on μ and $z_{(m+1)}$, there are $n-1$ traders' values above $z_{(m+1)}$ that are independently distributed with density

$$\frac{g_{\varepsilon}(z-\mu)}{\overline{G}_{\varepsilon}(z_{(m+1)}-\mu)}.$$

This satisfies the bound

$$\frac{g_{\varepsilon}(z-\mu)}{\overline{G}_{\varepsilon}(z_{(m+1)}-\mu)}\leq\frac{g_{\varepsilon}(z-\mu)}{\overline{G}_{\varepsilon}(z-\mu)}$$

because $z \geq z_{(m+1)}$. In particular, for

$$z \in \left[z_{(m+1)}, z_{(m+1)} + \frac{K}{m \wedge n}\right] \tag{81}$$

with

$$z_{(m+1)} \in (-\infty, \mu + \gamma], \tag{82}$$

it is true that

$$z - \mu \leq z_{(m+1)} + \frac{K}{m \wedge n} - \mu \leq \gamma + \frac{K}{m \wedge n} \leq \gamma + K.$$

Within this proof define the function ζ to be

$$\zeta\left(G_\varepsilon\right) = \sup_{y \leq \gamma + K} \frac{g_\varepsilon(y)}{\overline{G}_\varepsilon(y)},$$

so that

$$\frac{g_\varepsilon(z-\mu)}{\overline{G}_\varepsilon(z_{(m+1)} - \mu)} \leq \zeta\left(G_\varepsilon\right)$$

for z and $z_{(m+1)}$ satisfying (81) and (82).

The expected number of buyers' values in the interval (80) is therefore at most

$$1 + ((n-1) \wedge m)\, \zeta\left(G_\varepsilon\right) \cdot \frac{K}{m \wedge n} \leq 1 + \zeta\left(G_\varepsilon\right) \cdot K,$$

where the "1" counts the trader's value/cost equal to $z_{(m+1)}$ and $\zeta\left(G_\varepsilon\right) \cdot K / \left(m \wedge n\right)$ bounds above the probability that the value/cost of any of the other $n-1$ traders above $z_{(m+1)}$ lies in the interval (80). Because there are m buyers, the term $(n-1) \wedge m$ bounds above the number of buyers' values that are above $z_{(m+1)}$. The desired bound on (77) is now computed as follows:

$$\begin{aligned}
\int_{-\infty}^{\mu+\gamma} L(z_{(m+1)}, \mu) dF_{z_{(m+1)}}\left(z_{(m+1)} \,|\mu\right) &\leq \int_{-\infty}^{\mu+\gamma} \left(1 + \zeta\left(G_\varepsilon\right) \cdot K\right) dF_{z_{(m+1)}}\left(z_{(m+1)} \,|\mu\right) \\
&\leq 1 + \zeta\left(G_\varepsilon\right) \cdot K.
\end{aligned}$$

Bounding (78). We focus on the low probability that $z_{(m+1)} \geq \mu + \gamma$. Specifically,

$$\begin{aligned}
\int_{\mu+\gamma}^{\infty} L(z_{(m+1)}, \mu) dF_{z_{(m+1)}}\left(z_{(m+1)} \,|\mu\right) &\leq \int_{\mu+\gamma}^{\infty} \left(m \wedge n\right) dF_{z_{(m+1)}}\left(t_{(m+1)} \,|\mu\right) \qquad (83) \\
&= m \wedge n \cdot \left(1 - F_{z_{(m+1)}}\left(\mu + \gamma \,|\mu\right)\right).
\end{aligned}$$

The event "$z_{(m+1)} \geq \mu + \gamma$" requires that n values/costs be above $\mu + \gamma$. The term $1 - F_{z_{(m+1)}}\left(\mu + \gamma \,|\mu\right)$ is therefore the probability given μ that at least n in a sample of $m+n$ values/costs are greater than or equal to $\mu + \gamma$. This is calculated by summing over the number i of values/costs below $\mu + \gamma$ and then reducing as follows:

$$\begin{aligned} 1 - F_{z_{(m+1)}}(\mu + \gamma \,|\mu) &= \sum_{i=0}^{m} \binom{n+m}{i} G_\varepsilon(\gamma)^i \overline{G}_\varepsilon(\gamma)^{n+m-i} \\ &= \overline{G}_\varepsilon(\gamma)^n \sum_{i=0}^{m} \binom{n+m}{i} G_\varepsilon(\gamma)^i \overline{G}_\varepsilon(\gamma)^{m-i} \\ &\leq \overline{G}_\varepsilon(\gamma)^n \sum_{i=0}^{n+m} \binom{n+m}{i} 1^i 1^{n+m-i} \\ &\leq \frac{1}{2^{3\varphi n}} \cdot (1+1)^{n+m} \\ &= \frac{1}{2^{2\varphi n - m}} \\ &\leq \frac{1}{2^{\varphi n}}. \end{aligned}$$

The first inequality drops $G_\varepsilon(\gamma)^i \overline{G}_\varepsilon(\gamma)^{m-i} < 1$ from the i^{th} term, the second inequality increases the sum by adding terms from $i = m+1$ to $i = n+m$, and the final inequality applies the binomial formula and the bound (79).

Substitution into (83) implies

$$\int_{\mu+\gamma}^{\infty} L(z_{(m+1)}, \mu) dF_{z_{(m+1)}}\left(z_{(m+1)} \,|\mu\right) \leq \frac{m \wedge n}{2^{\varphi n}} < 1,$$

which completes the proof. ■

J The Limit Market

Proof of Theorem 5. Start with the property that for every μ there exists a price p^{REE} with which the traders can infer μ. After all trades are completed at p^{REE}, only traders for whom $\mathbb{E}[z|\mu, \sigma] < p^{\text{REE}}$ are without a unit of the good. The fraction q is the proportion of buyers in the market. By definition, they are the only traders who start with an initial endowment of zero units of the traded good. Given that each trader always has either zero or one unit, market clearing requires that the proportion of traders with zero units of the good remain constant at q both before and after trade. Therefore, conditional on μ, p^{REE} is market clearing if and only if

$$\begin{aligned} q &= \Pr\left[\mathbb{E}[z|\mu, \sigma] \leq p^{\text{REE}}\right] = \Pr\left[\mu + \mathbb{E}[z|0, \sigma - \mu] \leq p^{\text{REE}}\right] \\ &= \Pr\left[V(\sigma - \mu) \leq p^{\text{REE}} - \mu\right] = \Pr\left[\sigma - \mu \leq V^{-1}(p^{\text{REE}} - \mu)\right] \\ &= G_{\varepsilon+\delta}(V^{-1}(p^{\text{REE}} - \mu)). \end{aligned}$$

The second equality follows from the equation

$$\mathbb{E}[z|\mu,\sigma] = \gamma + \mathbb{E}[z|\mu-\gamma,\sigma-\gamma], \tag{84}$$

which is shown in (40) in Appendix B to hold for any $\gamma \in \mathbb{R}$ as a consequence of our model's invariance. The last equality follows from the fact that $\sigma - \mu = \varepsilon + \delta$ and consequently has distribution $G_{\varepsilon+\delta}$. Solving for p^{REE} we have

$$\begin{aligned} \xi_q^{\varepsilon+\delta} = G_{\varepsilon+\delta}^{-1}(q) = V^{-1}\left(p^{\text{REE}} - \mu\right) \quad &\Longleftrightarrow \quad V\left(\xi_q^{\varepsilon+\delta}\right) = p^{\text{REE}} - \mu \\ &\Longleftrightarrow \quad p^{\text{REE}} = \mu + V\left(\xi_q^{\varepsilon+\delta}\right). \end{aligned} \tag{85}$$

We next characterize equilibrium strategies in the BBDA in the limit market and show that the equilibrium price is the REE price. Suppose every trader with signal σ follows the offset strategy of bidding/asking

$$s = \sigma + V\left(\xi_q^{\varepsilon+\delta}\right) - \xi_q^{\varepsilon+\delta}. \tag{86}$$

We show that this is every trader's equilibrium strategy in the limit market. As a function of the state μ, this strategy results in the deterministic market-clearing price $\mu + V\left(\xi_q^{\varepsilon+\delta}\right) = p^{\text{REE}}$. This is shown in two steps. First, if traders were to truthfully report their signals σ, which conditional on μ are distributed according to $G_{\varepsilon+\delta}(\sigma-\mu)$, then the limit market price would be $\mu + \xi_q^{\varepsilon+\delta}$. Second, strategy (86) translates traders' bids/asks by the constant $V\left(\xi_q^{\varepsilon+\delta}\right) - \xi_q^{\varepsilon+\delta}$, which therefore translates the market-clearing price the same amount.

Pick a focal trader with signal σ and value/cost z who anticipates all others using strategy (86) and considers his optimal response. In the limit market, no individual trader can affect the price and so regardless of the focal trader's bid/ask the price remains equal to $\mu + V\left(\xi_q^{\varepsilon+\delta}\right)$. If the focal trader submits $\sigma + V\left(\xi_q^{\varepsilon+\delta}\right) - \xi_q^{\varepsilon+\delta}$ then he leaves the market with an item if

$$\begin{aligned} \sigma + V\left(\xi_q^{\varepsilon+\delta}\right) - \xi_q^{\varepsilon+\delta} \geq \mu + V\left(\xi_q^{\varepsilon+\delta}\right) \quad &\Longleftrightarrow \quad \sigma - \mu \geq \xi_q^{\varepsilon+\delta} \\ &\Longleftrightarrow \quad V(\sigma-\mu) \geq V\left(\xi_q^{\varepsilon+\delta}\right) \\ &\Longleftrightarrow \quad \mu + V(\sigma-\mu) \geq \mu + V\left(\xi_q^{\varepsilon+\delta}\right) \\ &\Longleftrightarrow \quad \mathbb{E}[z|\sigma,\mu] \geq \mu + V\left(\xi_q^{\varepsilon+\delta}\right) \end{aligned}$$

where the last line follows from (84). Hence, if the focal trader plays strategy (86) when all others use this strategy, then he leaves the market with an item if and only if his expected value/cost conditional on μ and σ is above the market price for *any* value of the state μ. He can certainly not improve on that by deviating and hence the offset strategy $\sigma + V\left(\xi_q^{\varepsilon+\delta}\right) - \xi_q^{\varepsilon+\delta}$ is the equilibrium strategy of all traders. ■

K Robustness: A Proper Prior

Our assumption that the state μ is drawn from the uniform improper prior simplifies our theoretical and numerical analyses through the invariance properties it implies on traders' decision problems. In this section we investigate a modified CIV model in which the common prior on μ is $\mathcal{N}(0, 1/\tau)$. We demonstrate for small values of the precision τ that there exist equilibria that approximate the offset equilibrium in which the prior on μ is uniform improper. This exercise shows that equilibria for the uniform improper prior are representative of equilibria for an important, empirically plausible class of proper priors.

We now assume that the state μ is drawn from $\mathcal{N}(0, 1/\tau)$ for $\tau > 0$, with the case of $\tau = 0$ interpreted as the limiting uniform improper prior of our model. Each trader's preference and noise terms are drawn from $\mathcal{N}(0, 1)$. Consider a trader with signal σ and let $f_{\mu|\sigma}(\mu|\sigma; \tau)$ be his posterior density of μ:[20]

$$\mu|\sigma \sim \mathcal{N}\left(\frac{\sigma}{1+2\tau}, \frac{2}{1+2\tau}\right). \tag{87}$$

As in our development of the uniform improper prior case observe that if $\tau = 0$, then $\mu|\sigma \sim \mathcal{N}(\sigma, 2)$. Denote with $\langle B^\tau(\sigma_B), S^\tau(\sigma_S)\rangle$ the equilibrium strategies of buyers and sellers and, as in section 4, let $\langle \sigma_B^\tau(\beta), \sigma_S^\tau(\beta)\rangle$ be the corresponding inverse strategies for $\beta \in \mathbb{R}$. As in system (37) within Appendix B, inverse strategies here satisfy the system of differential equations

$$\begin{bmatrix} B_{11}^\tau(\omega) & B_{12}^\tau(\omega) \\ B_{21}^\tau(\omega) & B_{22}^\tau(\omega) \end{bmatrix} \cdot \begin{bmatrix} \dot{\sigma}_S^\tau \\ \dot{\sigma}_B^\tau \end{bmatrix} = \begin{bmatrix} C^\tau(\omega) \\ 0 \end{bmatrix}, \tag{88}$$

where the formulas for terms $B_{11}^\tau, B_{12}^\tau, B_{21}^\tau, B_{22}^\tau$ and C^τ are straightforward adaptations of the corresponding formulas (38) in the $\tau = 0$ case. Making the prior distribution proper changes only how each trader integrates his uncertainty over μ; all other terms in formulas (38) remain unaffected. Linear system (88) defines the vector field $\overrightarrow{\mathcal{V}}^\tau(\omega) = (\dot{\sigma}_B^\tau, 1, \dot{\sigma}_S^\tau)$ at every point $\omega \in \mathbb{R}^3$.

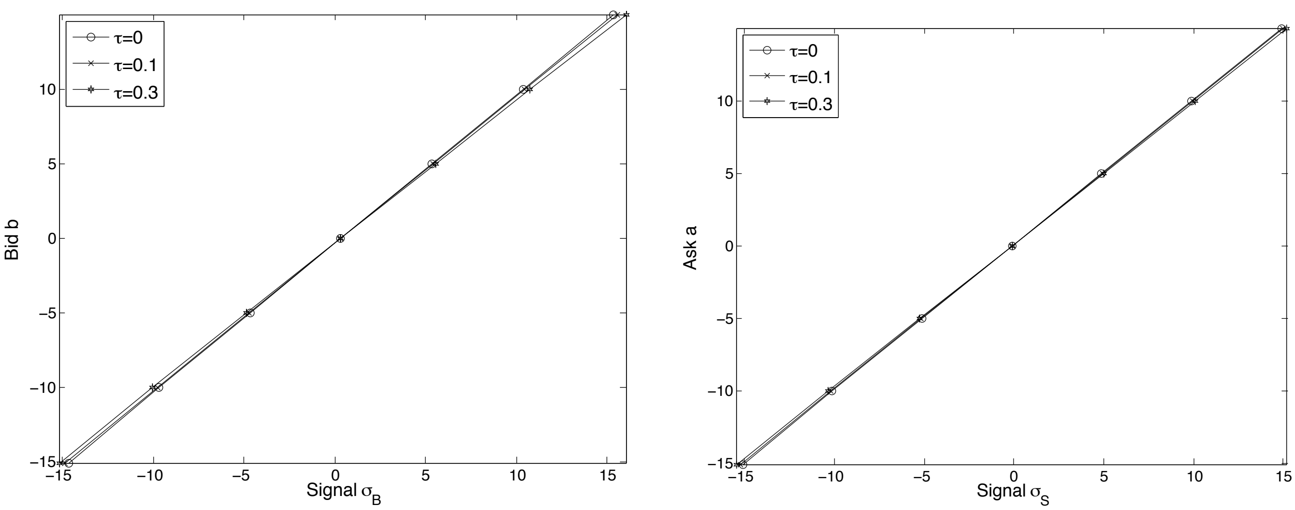
We compute solutions to (88) for $\tau \in \{0.1, 0.3\}$ and then verify that they represent approximate equilibria by examining marginal expected utility at a sample of points. Section K.1 below outlines the algorithm we use to compute these equilibria and Figure 5 presents their graphs in comparison to the equilibrium in the improper case ($\tau = 0$) when $m = n = 8$.[21]. In the left panel we plot the strategies B^τ for buyers and in the right panel we plot the strategies S^τ for sellers. As shown in the figure, these strategies are defined only for signals $\sigma \in (-15, 15)$. These strategies are computed over this interval under the assumption that traders with signals outside this range do not participate. In this sense, we compute approximate equilibria. Nevertheless these strategies ex ante are good approximations to equilibrium strategies because this nonparticipation almost never happens.[22]

[20] See Tong (1990, Thm. 3.3.4).

[21] Results for $m = n = 12$ are available upon request.

[22] Consider the case of $\tau = 0.1$. Ex ante, traders' signals are distributed $\sigma|\mu \sim \mathcal{N}(0, 12)$. The variance 12 corresponds to a standard deviation of 3.46. The domain of signals is $15/3.46 = 4.34$ standard deviations on each side of the origin. The probability that a trader's signal σ falls outside $(-15, 15)$ is therefore 1.49×10^{-5}, i.e., on

Figure 5 shows graphically the main message of this section: for the three values of τ the strategies are essentially indiscernible from each other. As the normal proper priors approach the uniform improper prior the offset equilibrium strategies for the improper prior more closely approximate the equilibrium strategies for the proper priors.



(a) Buyer (b) Seller

Figure 5: A trader's equilibrium bid/ask versus his signal when $\mu \sim \mathcal{N}(0, 1/\tau)$ in the cases of $\tau = 0, 0.1, 0.3$ ($m = n = 8$ and $\varepsilon, \delta \sim \mathcal{N}(0, 1)$).

Figure 6 illustrates more clearly how equilibrium changes as one moves from the uniform improper prior to a proper prior. Notably, invariance no longer holds and this has predictable effects on equilibrium strategies. The figure graphs the equilibrium offsets $\lambda_B^\tau(\sigma_B) = B^\tau(\sigma_B) - \sigma_B$ and $\lambda_S^\tau(\sigma_S) = S^\tau(\sigma_S) - \sigma_S$ for buyers and sellers as nonconstant functions of their signals. We make two observations. First, a trader's offset becomes larger as his signal increases, reflecting the fact that a trader in the proper prior case has a sense of whether or not his signal is relatively high or low and adjusts his bid/ask accordingly. Second, the offset functions become more steeply sloped as τ increases. This reflects the fact that a trader has a stronger sense of the relative size of his signal in the case of a larger τ. Specifically, the focal buyer's posterior over trader j's signal is

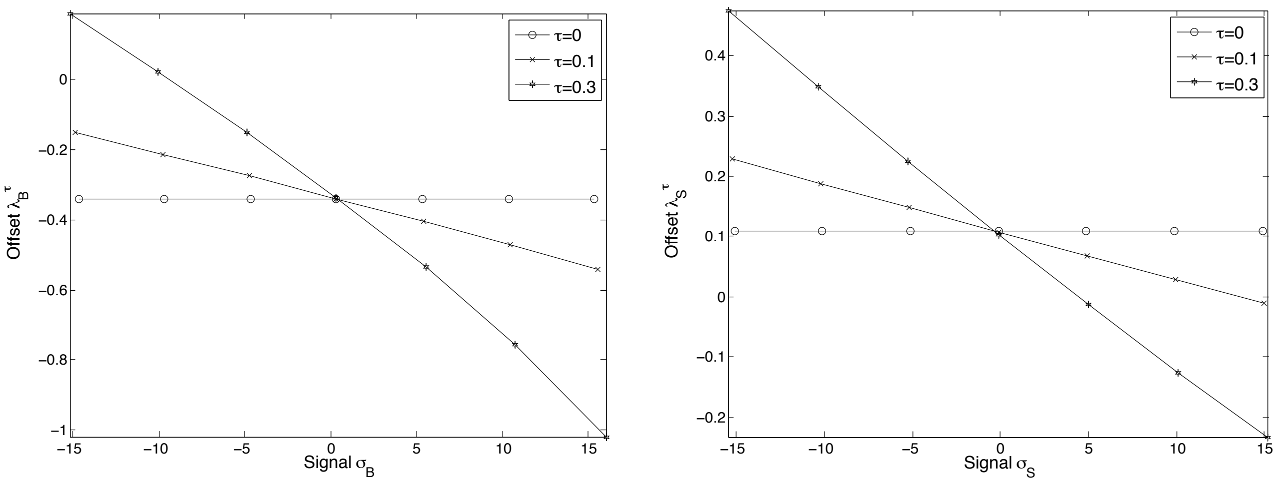
$$\sigma_j | \sigma \sim \mathcal{N}\left(\frac{\sigma}{1+2\tau}, \frac{4+4\tau}{1+2\tau}\right). \tag{89}$$

Hence, as a function of his signal σ, the probability that his signal is larger than j's signal σ_j is

$$\Pr[\sigma > \sigma_j] = \Phi\left(\frac{1 - \frac{1}{1+2\tau}}{\sqrt{\frac{4+4\tau}{1+2\tau}}}\,\sigma\right) = \Phi\left(\frac{\tau}{1+\tau}\sqrt{\frac{1+\tau}{1+2\tau}}\,\sigma\right).$$

average approximately 3 out of 200,000 signals draws fall outside the interval $(-15, 15)$. The same odds for $\tau = 0.3$ are approximately 8 out of 100 billion.

For the uniform improper prior ($\tau = 0$) invariance implies that this probability equals $\Phi(0) = 0.5$ no matter what his signal is. For a proper prior the focal buyer knows he has a higher than average signal whenever his signal is greater than zero. An illustration shows that this is economically significant increase in each trader's information. Let $\tau = 0.1$, if $\sigma = 3.46$ (one standard deviation above its ex ante mean) then $\Pr[\sigma > \sigma_j] = 0.618$.



(a) Buyer

(b) Seller

Figure 6: The difference of a trader's equilibrium bid/ask from his signal versus his signal when $\mu \sim \mathcal{N}(0, 1/\tau)$ in the cases of $\tau = 0, 0.1, 0.3$ ($m = n = 8$ and $\varepsilon, \delta \sim \mathcal{N}(0,1)$).

This discussion of how a positive precision prior causes buyers and sellers to deviate from the offset strategy as the precision increases from zero should not obscure the message of Figure 5: the offset equilibrium is a good approximation for the equilibria for proper priors with moderate-size precisions.

K.1 Numerical Calculation of Approximate Equilibria

Define $Q \equiv \left\{(\sigma_B, 0, \sigma_S) | (\sigma_B, \sigma_S) \in \mathbb{R}^2\right\}$ to be a plane through the origin of $\mathbb{R}^3$. Pick an initial condition $\omega \in Q$ and follow the vector field $\overrightarrow{\mathcal{V}}^\tau$ defined in (88) in both directions from ω as long as both $\dot{\sigma}_B^\tau, \dot{\sigma}_S^\tau \in (0, \infty)$. Each $\omega \in Q$ indexes a solution to the first order conditions. We are interested in solutions that define differentiable strictly increasing strategies over as large domain as numerically feasible. Define an approximate solution s to be the differentiable function of β, parameterized by $\omega \in Q$,

$$s\left(\beta|\omega\right) = \langle s_B\left(\beta|\omega\right), s_S\left(\beta|\omega\right)\rangle,$$

that has three properties:

1. Its domain is the open interval $(\beta_L, \beta_U) \subset \mathbb{R}$ where $\beta_L < 0 < \beta_U$.
2. For all $\beta \in (\beta_L, \beta_U), \langle \dot{s}_B\left(\beta|\omega\right), \beta, \dot{s}_S\left(\beta|\omega\right)\rangle = \overrightarrow{\mathcal{V}}^\tau\left(s_B\left(\beta|\omega\right), \beta, s_S\left(\beta|\omega\right)\right)$ where $\dot{s}_B\left(\beta|\omega\right)$ and $\dot{s}_S\left(\beta|\omega\right)$ are the derivatives of s_B and s_S with respect to β. Thus s follows the vector field.

3. For all $\beta \in (\beta_L, \beta_U)$, $\dot{s}_B(\beta|\omega)$, $\dot{s}_B(\beta|\omega) \in (0,\infty)$ and no interval (β'_L, β'_U) exists that both strictly contains (β_L, β_U) and satisfies $\dot{s}_B(\beta|\omega)$, $\dot{s}_S(\beta|\omega) \in (0,\infty)$ for all $\beta \in (\beta'_L, \beta'_U)$.

Observe that the last property uniquely determines β_L and β_U as functions of s because, given the parameter ω, the interval (β_L, β_U) is the largest interval over which $s(\beta|\omega)$ is increasing and differentiable.

Given $s(\beta|\omega)$ and the interval (β_L, β_U) it implies, define the function

$$\chi(\omega) \equiv \min(-\beta_L, \beta_U).$$

An equilibrium is a ω^* and $s(\beta|\omega^*)$ such that (i) $\chi(\omega^*) = \infty$, (ii) $s_B(\beta|\omega^*)$ is a buyer's best response to all other traders playing s_B and s_S, and (iii) $s_S(\beta|\omega^*)$ is a seller's best response to all other traders playing s_B and s_S. A $\omega \in Q$ identifies an approximate equilibrium $s(\beta|\omega)$ if $\chi(\omega)$ is large, the ex ante probability a trader's signal σ is an element of (β_L, β_S) is essentially one, and, for all $\beta \in (\beta_L, \beta_U)$, $s_B(\beta|\omega)$ and $s_S(\beta|\omega)$ are best responses for buyers and sellers, respectively. To find the approximate equilibria in Figures 5–6 we first used a MATLAB optimization routine to maximize $\chi(\omega)$ within the limits of MATLAB's numerical precision and then checked at a sample of points $\beta \in (\beta_L, \beta_U)$ that both $s_B(\beta|\omega)$ and $s_S(\beta|\omega)$ are in fact best responses. The reason that we must settle for an approximate equilibrium is that $\chi(\omega)$ is badly behaved on some open neighborhood of ω^* since $\chi(\omega^*) = \infty$.

This algorithm seems appropriate to the problem because the support of each trader's signal σ is $\mathbb{R}$. Consequently this system of differential equations has no natural initial or boundary value conditions. Our interest here, as in the rest of the paper, is in equilibria in which strategies are everywhere strictly increasing and therefore invertible. Requiring strictly increasing strategies is a powerful criterion for selecting among the double continuum of solutions to the system (88). For the uniform improper prior case it selects the offset solution as the unique equilibrium.[23] In the proper prior case that we consider here the monotonicity requirement appears to have the same effect.

A reasonable concern is that the monotonicity requirement is too effective and may cause non-existence. For example suppose the densities g_ε and g_δ are not symmetric about the origin. Then the $\omega \in Q$ for which $-\beta_L = \infty$ may not equal the $\omega' \in Q$ for which $\beta_U = \infty$ as an equilibrium in increasing and differentiable strategies requires. But even in this case a $\omega'' \in Q$ may exist for which $\chi(\omega'')$ is large enough that it is one of many approximate equilibria.

L Robustness: The Cauchy Distribution

Theorem 1's proof of the existence of an offset solution to a buyer's FOC in the CPV model uses finiteness of the first moment of the distribution G_ε that determines the traders' preferences. The-

[23] Its power as a selection criterion is not limited to this paper's model in which values/costs and signals can take on values over the entire real line. It operates almost as powerfully in an independent private values model where values/costs are drawn from a compact interval. Figures 3 and 4 in Rustichini, Satterthwaite, and Williams (1994) demonstrates this power graphically.

orem 3's proof of convergence to price-taking behavior in the CPV model uses regularity condition (12) that restricts the left tail of this distribution. The proof of convergence to efficiency in Theorem 4 applies Theorem 3 and so it too uses (12). We explore in this section the dependence of our results on these conditions by considering the standard Cauchy distribution, which satisfies neither of them. Figure 7 depicts the density of the standard Cauchy distribution in relation to the density of the standard normal distribution.

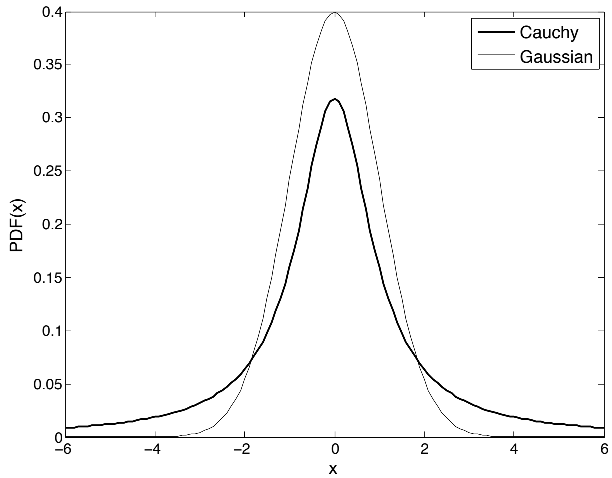


Figure 7: Probability density functions of standard Cauchy and standard normal.

We begin by proving in Theorem 7 that the Cauchy distribution causes nonexistence of equilibrium in the bilateral CPV case. Recall that a buyer in the BBDA equates the marginal benefit from lowering his bid and thereby decreasing his expected price when he trades to the marginal cost of losing a profitable trade with the lower bid. Because of the fat tail of the Cauchy distribution, the marginal benefit from lowering the bid always exceeds the marginal cost. As a consequence, the buyer always has the incentive to lower his bid further and further, and so an optimal bid does not exist for him.

We then investigate through a series of numerical experiments both the CPV and the CIV cases with G_ε, G_δ standard Cauchy for $m = n = 1$ and $\eta = 2$, 4, 8, and 16. What is most notable about the results of these numerical experiments is how closely they resemble the results from the standard normal case that are presented in sections 4.2, 4.3, 5.4.1 and 5.4.2. The results indicate that the nonexistence of equilibrium in the bilateral case is resolved immediately as one moves to the multilateral case: Numerical Result I concerning existence and uniqueness of equilibrium holds for all four market sizes considered.[24] Intuitively, the BBDA works well in the case of the multilateral Cauchy distribution because this mechanism focuses on the "inner" order statistics $s_{(m)}$ and $s_{(m+1)}$ as opposed to the highest and second highest order statistics that are focal in auction theory. As discussed in Section 4.3, the focus on $s_{(m)}$ and $s_{(m+1)}$ complicates the issue of sufficiency of the

[24] Additional numerical experiments in fact demonstrate that the nonexistence of equilibrium in the bilateral Cauchy case is resolved with the introduction of a single additional trader on either side of the market.

first order approach in the BBDA. Because the pathology of the Cauchy distribution is its fat tails, however, the BBDA's focus on $s_{(m)}$ and $s_{(m+1)}$ is advantageous in this case because the fat tails quickly recede in significance as the market increases in size and the probability that $s_{(m)}$ or $s_{(m+1)}$ lies in the tails goes to zero. We then demonstrate that: (i) the computed equilibria demonstrate the rate of convergence to price-taking behavior stated in Numerical Result II; the price that is determined by these equilibria have the properties as an estimate of p^{REE} that are stated in Numerical Result IV. Numerical Result III concerning convergence to efficiency is omitted from this discussion because the fat tails of the Cauchy distribution imply that both the ex ante equilibrium and the ex ante potential gains from trade in any state are infinite.

L.1 The CPV case

L.1.1 Nonexistence of Equilibrium in the Bilateral Case

Theorem 7 *Consider the BBDA in the bilateral CPV case with G_ε the standard Cauchy distribution. Given honest reporting by the seller, a buyer's marginal expected utility is negative for all bids below the buyer's value. An equilibrium bidding strategy for the buyer therefore does not exist.*

Proof. The theorem concerns the Cauchy distribution with location parameter 0 and scale parameter 1. Hence for a buyer conditional on his value v, the seller's cost c is Cauchy distributed with location parameter v and scale parameter 2. Using the formula for the cumulative of the Cauchy distribution, we have

$$\pi(v,b) = (v-b)\left(\frac{1}{\pi}\arctan\left(\frac{b-v}{2}\right)+\frac{1}{2}\right),$$

and his marginal expected utility is

$$\pi_b(v,b) = -\left(\frac{1}{\pi}\arctan\left(\frac{b-v}{2}\right)+\frac{1}{2}\right)+(v-b)\frac{1}{\pi}\left(\frac{2}{4+(b-v)^2}\right).$$

It is sufficient to consider the case of $v=0$ and $b\le 0$, which simplifies the notation to

$$\begin{aligned}\pi_b(0,b) &= -\left(\frac{1}{\pi}\arctan\left(\frac{b}{2}\right)+\frac{1}{2}\right)-\frac{b}{\pi}\left(\frac{2}{4+b^2}\right)\\ &= -\frac{1}{\pi}\left(\arctan\left(\frac{b}{2}\right)+\frac{\pi}{2}+\frac{2b}{4+b^2}\right).\end{aligned} \tag{90}$$

At $b=0$ we have

$$\pi_b(0,0) = -\frac{1}{2} < 0,$$

and

$$\lim_{b\to-\infty}\pi_b(0,b) = 0.$$

The remainder of the argument shows that

$$\frac{\partial \pi_b}{\partial b}(0, b) < 0$$

for $b < 0$, which implies that $\pi_b(0, b)$ decreases from its limiting value of 0 at $b = -\infty$ to its value of $-1/2$ at $b = 0$. This implies

$$\pi_b(0, b) < 0$$

for all $b < 0$, and so the buyer's expected utility strictly increases as b becomes more and more negative.

Working from (90), the second derivative is

$$\begin{aligned}
\frac{\partial \pi_b}{\partial b}(0, b) &= -\frac{1}{\pi}\left(\frac{2}{4+b^2} + \frac{8-2b^2}{\left(4+b^2\right)^2}\right) \\
&= -\frac{1}{\pi\left(4+b^2\right)^2}\left(8 + 2b^2 + 8 - 2b^2\right) \\
&= -\frac{16}{\pi\left(4+b^2\right)^2} < 0,
\end{aligned}$$

which completes the proof. ■

L.1.2 Numerical Example: G_ε Standard Cauchy in the CPV Case

Column 2 of Table 5 presents the calculated values of the equilibrium offset λ_{Cauchy} in the CPV case for markets with $\eta = 2$, 4, 8, and 16 traders on each side ($m = n = 1$). The value of the offset roughly falls by half as η doubles, reflecting a $O(1/\eta)$ rate of convergence. Column 3 of this table displays the ratio between λ_{Cauchy} and $\lambda_{Gaussian}$, the equilibrium offset in the case in which G_ε is the standard normal distribution from Table 3, Panel A, column 1. While the ratio is infinite in the bilateral case, the equilibrium offsets become comparable in size across the two distributions as soon as the number of traders increases to 2 on a side. Given that the standard Cauchy has a higher dispersion than the standard normal it is not surprising that the offset is larger in magnitude for the standard Cauchy.

η	λ_{Cauchy}	$\lambda_{Cauchy}/\lambda_{Gaussian}$
2	-1.1656	1.6903
4	-0.4936	1.4630
8	-0.2235	1.4364
16	-0.1053	1.3819

Table 5: Comparison of the equilibrium offsets in the CPV case when traders' idiosyncratic noise is standard Cauchy and standard Gaussian. Column 1 is the market size η ($m = n = 1$), column 2 is the equilibrium offset in the Cauchy case, and column 3 is the ratio of the equilibrium offsets of the Cauchy and the Gaussian cases.

It is worth noting that these equilibrium offsets do not uniquely maximize a focal buyer's expected utility in the case of the Cauchy distribution. Because of the fat downward tail, the expected price faced by a focal buyer conditional upon his value in this case is negatively infinite when sellers report honestly and the other buyers bid less than their values.[25] As a consequence, all bids $b < v$ provide the focal buyer with an infinite expected utility. In what sense does the computed offset represent utility maximization and equilibrium? The answer lies in the use of a distribution with an infinite support as an approximation of a finite world. Truncating the Cauchy distribution to the support $[-c, c]$ for $c > 0$ and scaling its density so that its integral over this interval equals one produces a distribution for which a buyer's equilibrium offset and corresponding expected utility are well-defined. This equilibrium offset in this case converges to the value presented in Table 5 as c increases to infinity. Though the limit of expected utility is infinite, the marginal analysis in the Cauchy case derives a meaningful limit of the solution to a buyer's decision problem in the finite case.

L.2 Numerical Example: The CIV Case with G_ε, G_δ Standard Cauchy

We next consider the CIV case in which both G_ε and G_δ are standard Cauchy. In Table 6 column 2 we report the computed offsets (λ_B, λ_S) for buyers and sellers in markets with $m = n = 1$ and market size η. The offsets behave qualitatively as those in Table 1 for the Gaussian case and in particular exhibit the rate of convergence stated in Numerical Result II. Computing the vector

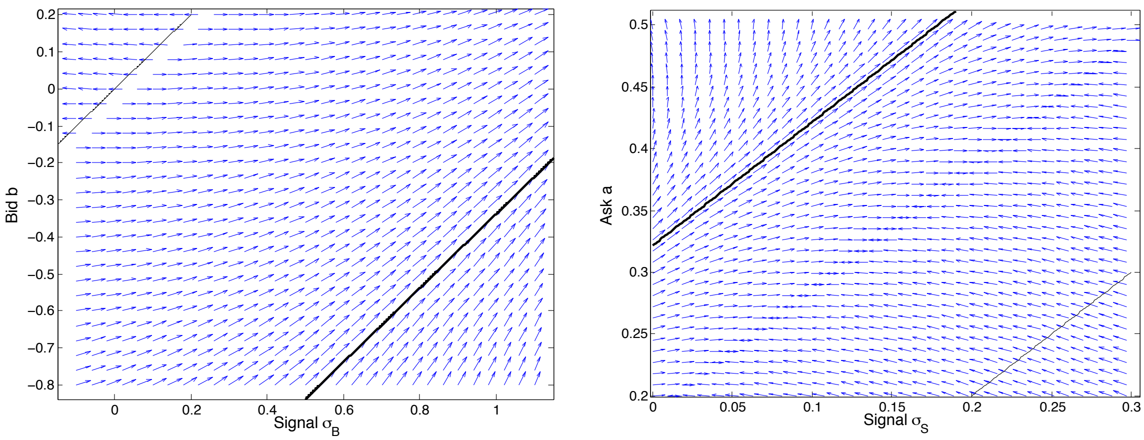
η	λ_B, λ_S
2	$-3.3480, 0.7446$
4	$-1.3381, 0.3217$
8	$-0.6150, 0.1650$
16	$-0.2983, 0.0872$

Table 6: Column 2 are the equilibrium offset solutions for the different sizes of market η reported in column 1 ($m = n = 1$, G_ε, G_δ standard Cauchy).

field for the Cauchy case produces Figure 8 that is qualitatively indistinguishable to the one for the Gaussian case in Figure 2.

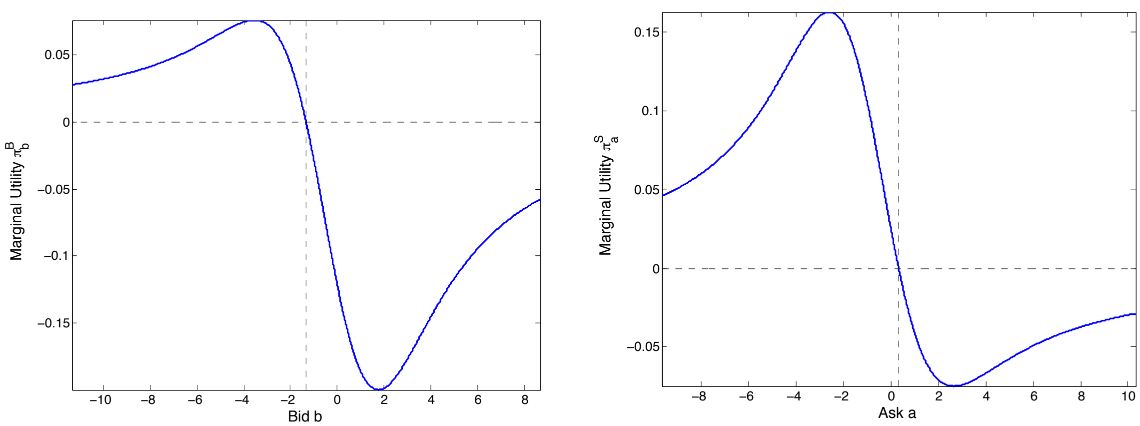
Consistent with Numerical Result I, this suggests that the offset solutions (λ_B, λ_S) in Table 6 to the buyers' and sellers' FOCs are in fact the unique solutions that define equilibrium satisfying A2. We verify that the offset solutions define equilibrium by plotting in Figure 9 a focal trader's marginal utility for each class (corresponding to Figure 3 in the Gaussian case). We see how both

[25] The argument for large $m = n$ in the CPV case is as follows. The distribution of the state μ conditional upon the focal buyer's value v is Cauchy with location parameter v and scale parameter 1. The symmetry of the Cauchy distribution implies that state μ is less than v with probability 1/2. Because of the fat downward tail of the Cauchy distribution, the expected value of μ conditional on being below v is $-\infty$. Regardless of how the focal buyer bids, the price in the BBDA is bounded above by the $(m+1)^{\text{st}}$ smallest value/cost among the $2m-1$ values/costs of the other traders. This is true because the other buyers bid less than their values. This order statistic converges in distribution to μ as m increases (Rothenberg, Franklin, and Tilanus (1964)).



(a) Buyer (b) Seller

Figure 8: The normalized vector field for traders ($m = n = 4$, G_ε, G_δ standard Cauchy). Figure (a) assumes that sellers use the offset strategy $S(\sigma_S) = \sigma_S + 0.3217$ and Figure (b) assumes that buyers use the offset strategy $B(\sigma_B) = \sigma_B - 1.3381$. The thick line is the offset strategy for this indicated trader. The thin line is the 45° diagonal.



(a) Buyer (b) Seller

Figure 9: Marginal utility for focal traders ($m = n = 4$, G_ε, G_δ standard Cauchy). The vertical dashed line indicates the offset solution to the focal trader's FOC.

marginal utilities change sign from positive to negative at the computed offsets.[26]

Table 7 replicates the results of Table 4 for the Cauchy distribution in the CIV case with two exceptions: (i) computations of expected gains from trade are missing from this table because they are infinite; (ii) the variance of the sample median is only finite for $\eta \geq 4$ (Rider (1960)) and so $\eta = 2$ is omitted from Panel B. The strategic term of buyers in Panel A, the variances in Panel B and the expectations of absolute errors in Panel C, all exhibit the same behavior as for the normal distribution in Table 4: the strategic term of buyers is $O(1/\eta)$, the strategic error is $O(1/\eta)$, the

[26]Figure 10 in online Appendix M illustrates that the sufficiency conditions of Theorem 2 are also satisfied for the standard Cauchy.

Table 7: Convergence in the CIV case ($m = n = 1$, G_ε, G_δ standard Cauchy) In all panels, the size of market η is reported in column 1. **Panel A:** Column 2 is the buyer's strategic term computed at the equilibrium offsets. **Panel B:** Columns 2 and 3 are the variances of the errors in the price-taking price and the equilibrium price as estimates of the REE price. **Panel C:** Columns 2 and 3 are the expected absolute errors of the price-taking price and the equilibrium price as estimates of the REE price; column 4 is the expected absolute difference between these two prices.

Panel A

η	$\frac{\Pr[x<\lambda_B<y\|\sigma_B]}{f_x^B(\lambda_B\|\sigma_B)}$
2	2.6034
4	1.0164
8	0.4501
16	0.2110

Panel B

η	$\mathrm{VAR}\left(t_{(\eta m+1)} - p^{\mathrm{REE}}\|\mu\right)$	$\mathrm{VAR}\left(s_{(\eta m+1)} - p^{\mathrm{REE}}\|\mu\right)$
4	2.2186	2.4239
8	0.7740	0.8019
16	0.3423	0.3464

Panel C

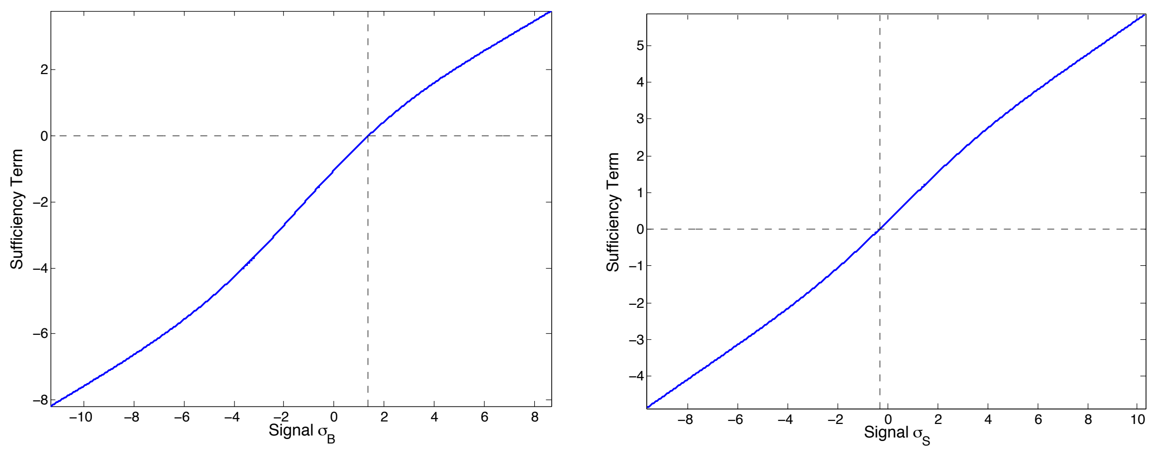
η	Exp. Sampling Error $\mathbb{E}\left[\left\|t_{(\eta m+1)} - p^{\mathrm{REE}}\right\| \|\mu\right]$	Exp. Total Error $\mathbb{E}\left[\left\|s_{(\eta m+1)} - p^{\mathrm{REE}}\right\| \|\mu\right]$	Exp. Strategic Error $\mathbb{E}\left[\left\|s_{(\eta m+1)} - t_{(\eta m+1)}\right\| \|\mu\right]$
2	2.1288	2.3108	1.4296
4	1.0942	1.1160	0.6459
8	0.6895	0.6854	0.3054
16	0.4624	0.4609	0.1500

sampling error is $\Theta(1/\sqrt{\eta})$, the total error is $O(1/\sqrt{\eta})$, and the equilibrium price $s_{(\eta m+1)}$ is virtually indistinguishable from the price-taking price $t_{(\eta m+1)}$ as an estimator of p^{REE} for η as small as 4. Numerical Result IV thus holds in this case.

M Numerical Example: Additional Figures and Tables in the CIV Case for G_ε, G_δ Standard Cauchy and Standard Laplace

$m\backslash n$	2	4	8	16
2	$-3.3480, 0.7446$	$-2.3149, 0.9569$	$-2.1775, 1.6295$	$-2.6344, 2.9595$
4	$-2.7363, 0.2300$	$-1.3381, 0.3217$	$-0.7111, 0.7110$	$-0.3860, 1.3947$
8	$-3.5205, -0.0597$	$-1.4871, -0.1215$	$-0.6150, 0.1650$	$-0.0586, 0.6335$
16	$-5.9899, -0.1730$	$-2.2888, -0.5237$	$-1.0016, -0.3194$	$-0.2983, 0.0872$

Table 8: Results for buyers' and sellers' offsets λ_B, λ_S for different values of m and n and G_ε, G_δ standard Cauchy.



(a) Buyer

(b) Seller

Figure 10: Sufficiency terms for focal traders ($m = n = 4$, G_ε, G_δ standard Cauchy). The vertical dashed line indicates the offset solution to the focal trader's FOC.

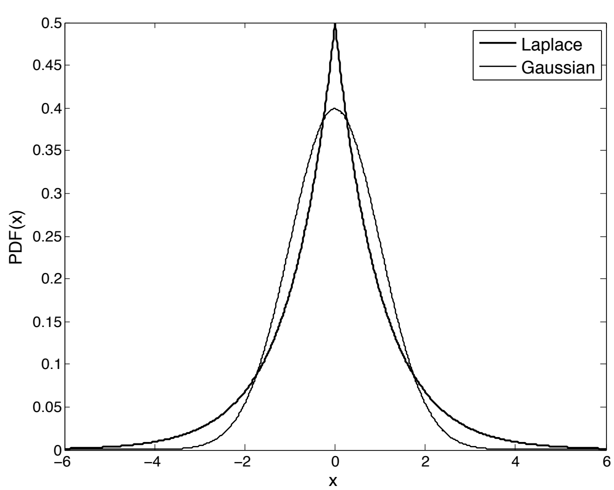
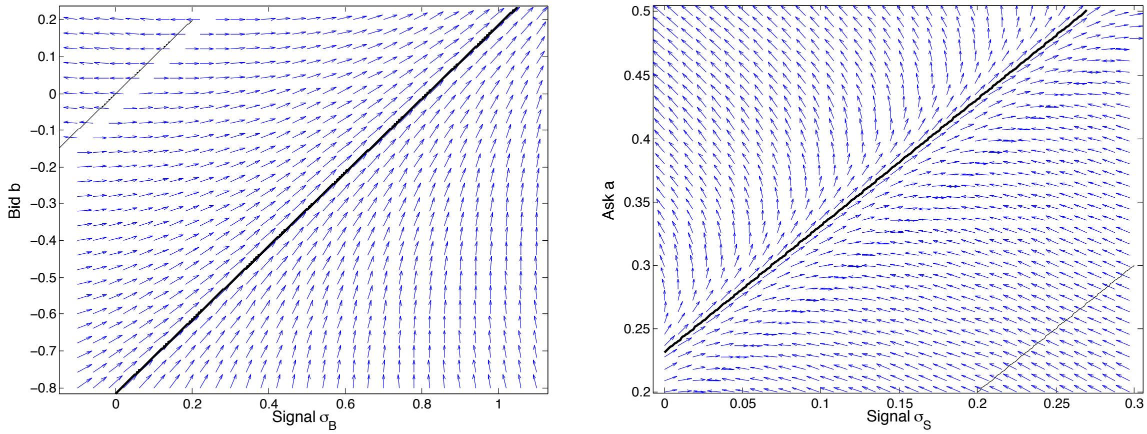


Figure 11: Probability density functions of standard Laplace and standard normal.

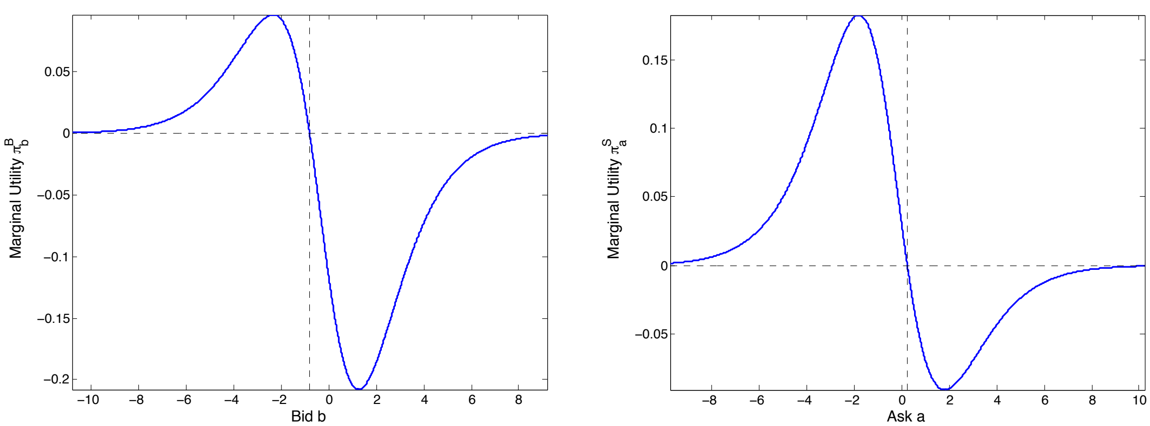


(a) Buyer (b) Seller

Figure 12: The normalized vector field for traders ($m = n = 4$, G_ε, G_δ standard Laplace). The thick line signifies the solution to the system (37) for each trader class. The thin line is the 45° diagonal.

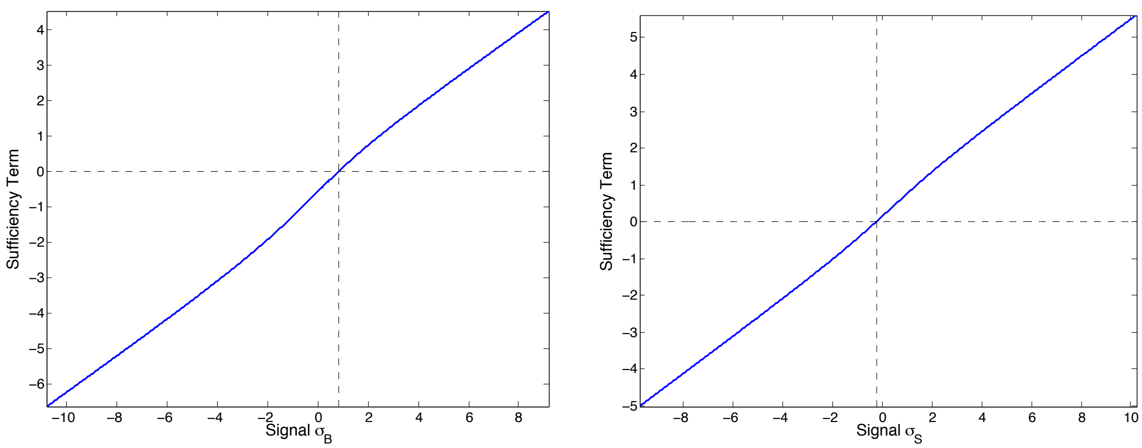
$m \backslash n$	2	4	8	16
2	$-1.6701, 0.4747$	$-1.0556, 0.5785$	$-0.5939, 0.8110$	$-0.1968, 1.1174$
4	$-1.4900, 0.1398$	$-0.8153, 0.2312$	$-0.3417, 0.4548$	$0.0523, 0.7597$
8	$-1.5629, -0.1662$	$-0.8830, -0.0895$	$-0.3888, 0.1162$	$0.0061, 0.4020$
16	$-1.7933, -0.4838$	$-1.1228, -0.4176$	$-0.6077, -0.2125$	$-0.1903, 0.0593$

Table 9: Results for buyers' and sellers' offsets λ_B, λ_S for different values of m and n and G_ε, G_δ standard Laplace.



(a) Buyer (b) Seller

Figure 13: Marginal expected utility for focal traders ($m = n = 4$, G_ε, G_δ standard Laplace). The vertical dashed line indicates the offset solution to the focal trader's FOC.



(a) Buyer

(b) Seller

Figure 14: Sufficiency terms for focal traders ($m = n = 4$, G_ε, G_δ standard Laplace). The vertical dashed line indicates the offset solution to the focal trader's FOC.

Table 10: Convergence in the CIV case ($m = n = 1$, G_ε, G_δ standard Laplace). In all panels, the size of market η is reported in column 1. **Panel A:** Column 2 is the buyer's strategic term computed at the equilibrium offsets; column 3 is the expected gains from trade $\overline{GFT}^{\mathrm{eq}}$ in equilibrium and column 4 is the expected gains from trade $\overline{GFT}^{\mathrm{pt}}$ when traders submit their price-taking bids/asks; column 5 is the relative inefficiency. **Panel B:** Columns 2 and 3 are the variances of the errors in the price-taking price and the equilibrium price as estimates of the REE price. **Panel C:** Columns 2 and 3 are the expected absolute errors of the price-taking price and the equilibrium price as estimates of the REE price; column 4 is the expected absolute difference between these two prices.

Panel A

η	$\frac{\Pr[x<\lambda_B<y\|\sigma_B]}{f_x^B(\lambda_B\|\sigma_B)}$	$\overline{GFT}^{\mathrm{pt}}$	$\overline{GFT}^{\mathrm{eq}}$	$(\overline{GFT}^{\mathrm{pt}} - \overline{GFT}^{\mathrm{eq}})/\overline{GFT}^{\mathrm{pt}}$
2	1.1954	1.2740	1.0116	0.2059
4	0.5841	2.7584	2.6302	0.0465
8	0.2726	5.7535	5.6914	0.0108
16	0.1310	11.7612	11.7303	0.0026

Panel B

η	$\mathrm{VAR}\left(t_{(\eta m+1)} - p^{\mathrm{REE}}\|\mu\right)$	$\mathrm{VAR}\left(s_{(\eta m+1)} - p^{\mathrm{REE}}\|\mu\right)$
2	1.2035	1.4516
4	0.5571	0.5976
8	0.2679	0.2736
16	0.1304	0.1311

Panel C

η	Exp. Sampling Error $\mathbb{E}\left[\left\|t_{(\eta m+1)} - p^{\mathrm{REE}}\right\| \|\mu\right]$	Exp. Total Error $\mathbb{E}\left[\left\|s_{(\eta m+1)} - p^{\mathrm{REE}}\right\| \|\mu\right]$	Exp. Strategic Error $\mathbb{E}\left[\left\|s_{(\eta m+1)} - t_{(\eta m+1)}\right\| \|\mu\right]$
2	0.9364	0.9457	0.7310
4	0.6162	0.6075	0.3901
8	0.4194	0.4124	0.1925
16	0.2904	0.2868	0.0951