1. a. The subgame perfect Nash equilibrium is as follows:
player 1:
stage 1: $x_0 = 1 - \delta_2^2$
stage 2: $x_1 = 1 - \delta_2$
stage 3: accept any $x_2$
player 2:
stage 1: accept if and only if $1 - x_0 \geq \delta_2^2$
stage 2: accept if and only if $1 - x_1 \geq \delta_2$
stage 3: $x_2 = 0$
The strategies should be specified for each player in each stage.

b. player 1:
stage 1: $x_0 = 0$
stage 2: $x_1 = \frac{1}{3}$
stage 3: accept if and only if $x_2 = 1$
player 2:
stage 1: accept if and only if $1 - x_0 = 1$
stage 2: accept if and only if $1 - x_1 \geq \frac{2}{3}$
stage 3: $x_2 = 0$

The strategies should be specified for each player in each stage. There are other correct answers.

2. Consider the following game of imperfect information:

- a. For what values of $p_1$ and $p_2$ does I choose $F$ over $A$?
  
  Expected payoff with $F$: $2p_1 + (-2)p_2 + 3(1 - p_1 - p_2) = 3 - p_1 - 5p_2$
  
  Expected payoff with $A$: $p_1 + 0(p_2) + 2(1 - p_1 - p_2) = 2 - p_1 - 2p_2$
  
  $F \geq A$:
  
  $$3 - p_1 - 5p_2 \geq 2 - p_1 - 2p_2$$
  
  $$1 \geq 3p_2$$
  
  $$\frac{1}{3} \geq p_2$$

- b. Is there a weak perfect Bayesian equilibrium in which E1 chooses OUT1 to start the game? If so, then fully describe the equilibrium.

  If E1 is to choose OUT1 to start the game, then it must be that E2 would decline E1’s offer if he proposed a joint venture (both of E1’s payoffs are positive if E2 accepts his proposal). We must have $p_2 \leq 1/3$ so that I chooses $F$ over $A$; if I instead chooses $A$ over $F$, then E1 would choose Enter over OUT1. The proposed equilibrium is thus as follows:

E1 chooses OUT2 at his right hand node, which completes the definition of the
equilibrium.

c. Fully describe a weak perfect Bayesian equilibrium in E1 proposes the joint venture and E2 accepts.

d. Is your answer to c a sequential equilibrium? Explain. Yes – all information sets are reached in equilibrium, and the beliefs at each information set is obtained from the strategies by Bayes Rule.
3. Consider the following 3 person game:

\[
\begin{array}{ccc}
3: & l & r \\
1/2 & L & R \\
T & 2, 1, 0 & -3, 0, -1 \\
B & 1, -2, 1 & 0, 1, 0 \\
\end{array}
\]

Here, player 3 chooses either the left "l" game or the right "r" game.

a. Determine the Nash equilibria of the game.
   - T,L,l
   - B,R,l

b. Determine whether or not each Nash equilibrium is trembling hand perfect. Explain your answer.
   - This was an unintentionally long question, and so almost no one answered it fully and correctly. I’ve given full credit if the answer shows that the correct calculations for checking trembling hand perfection are begun for one of the two equilibria. The answer below (for instance) is sufficient for full credit.
   - Consider T,L,l:
     - Consider the totally mixed strategies
       1 : (1 - ε, ε)
       2 : (1 - γ, γ)
       3 : (1 - δ, δ)
     - 1’s payoff is
       \[ T: (1 - γ)(1 - δ)(2) + (γ)(1 - δ)(-3) + (1 - γ)(δ)(1) + γδ(0) \]
       \[ B: (1 - γ)(1 - δ)(1) + (γ)(1 - δ)(0) + (1 - γ)(δ)(-1) + γδ(1) \]
     - Subtracting, we get
       \[ T - B : (1 - γ)(1 - δ)(1) - 3(γ)(1 - δ) + 2(1 - γ)(δ) - γδ \]
     - And so on...

4. Consider an economy in which there are two consumers, a public good \( x \) and a private good \( y \). The cost function for making public good out of private good is \( C(x) = x \) (i.e., one unit of the private good can be converted into one unit of the public good).
   - Consider the voluntary contribution game in which each player \( i \) contributes an amount \( z_i \) of his private good whereupon an amount \( x = z_1 + z_2 \) of the public good is produced. Consumer \( i \) has large initial endowment \( \hat{y}_i \) of \( y \) and a utility function
     \[ u_i(x, y_i) = a_i x - \frac{x^2}{2} + y_i, \]
   - where \( y_i = \hat{y}_i - z_i \). Consumer \( i \)'s preference parameter \( a_i \) is either \( a_i = 4 \) or \( a_i = 6 \), but only consumer \( i \) knows which. The other consumer believes that \( a_i = 4 \) with probability 1/2 and \( a_i = 6 \) with probability 1/2.
   - Compute a Bayesian Nash equilibrium of this game in which each player uses the same strategy (i.e., a symmetric Bayesian Nash equilibrium).
   - Let \( z_L \) denote the contribution of a player with type 4 and \( z_H \) denote the contribution of a player
with type 6. \( z = z_L \) must maximize
\[
\frac{1}{2} \left[ 4(z + z_L) - \frac{(z + z_L)^2}{2} + \bar{y} - z \right] + \frac{1}{2} \left[ 4(z + z_H) - \frac{(z + z_H)^2}{2} + \bar{y} - z \right]
\]
which implies the first order condition holding at \( z = z_L \):
\[
0 = \frac{1}{2} [4 - (z_L + z)] + \frac{1}{2} [4 - (z_L + z_H)] - 1
\]
or
\[
0 = 6 - 3z_L - z_H.
\]
Similarly, \( z = z_H \) must maximize
\[
\frac{1}{2} \left[ 6(z_L + z) - \frac{(z_L + z)^2}{2} + \bar{y} - z \right] + \frac{1}{2} \left[ 6(z_H + z) - \frac{(z_H + z)^2}{2} + \bar{y} - z \right]
\]
which implies the first order condition holding at \( z = z_H \):
\[
0 = \frac{1}{2} [6 - (z_L + z_H)] + \frac{1}{2} [6 - 2z_H] - 1
\]
or
\[
0 = 10 - z_L - 3z_H.
\]
The second derivative test shows that a solution to these first order conditions will be a Bayesian-Nash equilibrium. Solving,
\[
0 = 6 - 3z_L - z_H = 3z_L + z_H = 6
\]
\[
0 = 10 - z_L - 3z_H = z_L + 3z_H = 10
\]
\[
1 = z_L, 3 = z_H
\]
5. Each of \( n \) players announces a number in the set \( \{1, 2, \ldots, K\} \). A prize of $1 is split equally among all of the people whose number is closest to \( 2/3 \) of the average number.

a. Show that the game has a unique mixed strategy Nash equilibrium (it is in fact a pure strategy Nash equilibrium). (From Osborne and Rubenstein).

Let \( k^* \) be the largest number used with positive probability by some player, and let \( i \) be a player who uses \( k^* \).
- player \( i \)'s expected payoff when he plays \( k^* \) must equal his expected payoff in his mixed strategy.
- player \( i \) must have a positive expected payoff, else he could change his strategy to insure that he sometimes wins.
- By the last two points, \( k^* \) is with some positive probability the closest number to \( 2/3 \) of the mean \( \mu \) of all reported numbers. Whenever \( i \) wins with \( k^* \), it must therefore be the case that at least one other player reports \( k^* \).
- player \( i \) can thus increase his payoff by choosing \( k^* - 1 \).

This argument works except when \( k = 1 \). As the maximum reported number, we therefore have each player reporting 1 as the unique mixed strategy Nash equilibrium.

b. Show that each player’s only rationalizable action is his unique Nash equilibrium action.
Since the game is symmetric, the set of rationalizable actions is the same, say \( Z \), for all players. Let \( k^* \) be the largest number in \( Z \). The answer to a. shows that \( k^* \) is a best reponse to a subset of \( Z \) only if \( k^* = 1 \).