A full answer is expected: show your work and your reasoning. Grading will be based on the intelligence of an answer, not its length.

1. (32 points) Consider the following signaling game in which Nature begins by choosing the type of player 1. Determine all pure strategy pooling and separating sequential equilibria in the game. There are four possibilities for you to consider; in each case, either construct the equilibrium or derive the contradiction to equilibrium. You do not need to prove that an equilibrium is sequential.
Separating equilibria: We first consider the separating equilibrium in which 1 chooses U at the left node and D at the right node:

This is not an equilibrium because player 1 would switch from D to U at his right hand node. We next consider the other candidate for a separating equilibrium:

Player 1 in this case would switch from U to D at his right-hand node. We conclude that no separating equilibrium exists in pure strategies.
**Pooling equilibria:** Consider first the case in which both types of player 1 choose U:

![Game Tree Diagram]

Player 2 must choose R at the bottom information set to keep 1 from choosing D at the right hand node. This requires $q \leq 1/2$; any $q \leq 1/2$ completes the definition of a pooling equilibrium.

Consider next the case in which both types of player 1 choose D:

![Game Tree Diagram]

Player 2 must choose R at his top information set to keep player 1 from choosing U at his left-hand node. We therefore need $p \leq 1/2$ to support 2's choice at this information set. Any $p \leq 1/2$ completes the definition of a pooling equilibrium.
2. (16 points) Consider the signaling model in which there are two types of workers, $\theta_H > \theta_L > 0$. The type of a worker equals his marginal product to a firm. Workers can obtain a level of education $e$ at cost $c(e, \theta)$. We assume that

\[
\begin{align*}
    c(0, \theta) &= 0, \\
    \frac{\partial c}{\partial e}(e, \theta) &> 0, \\
    \frac{\partial^2 c}{\partial e^2}(e, \theta) &> 0, \\
    c(e, \theta_H) &< c(e, \theta_L) \text{ for } e > 0, \\
    \frac{\partial c}{\partial e}(e, \theta_H) &< \frac{\partial c}{\partial e}(e, \theta_L).
\end{align*}
\]

A worker of type $\theta$ has utility function

\[u(w, e, \theta) = w - c(e, \theta),\]

where $w$ denotes any wage that he receives. Each worker has reservation wage equal to 0.

(a) (6) What is the single-crossing property of worker indifference curves? Derive it from the above assumptions.

The single-crossing property asserts that an indifference curve of a type $\theta_H$ worker can intersect an indifference curve of a type $\theta_L$ worker at most once. This is depicted in the figure below.

To prove this property, consider the indifference curves of the two types of workers through a given point $(\bar{e}, \bar{w})$. The indifference curve
of the type $\theta$ determined by the level of utility $k$ is the graph of the function
\[ w = c(e, \theta) + k. \]
The slope of this graph at the point $(e, w)$ is
\[ \frac{\partial c}{\partial e}(e, \theta). \]
The assumption
\[ \frac{\partial c}{\partial e}(e, \theta_H) < \frac{\partial c}{\partial e}(e, \theta_L) \]
implies that the indifference curve of the type $\theta_H$ is not at steeply sloped at any value of $e$ as the indifference curve of the type $\theta_L$ worker, which implies that the indifference curves of the two types of workers cross only at $(\tilde{e}, \tilde{w})$.

(b) (10) Assume now that there are two identical firms that engage in Bertrand competition for workers. Consider the game that proceeds as follows:
I. Nature determines the type of each worker.
II. Knowing his type, a worker chooses a level of education.
III. The firms form beliefs about the productivity of a worker given his level of education. Based on these beliefs, they offer wages to workers of different education levels.
IV. The workers accept or reject the offers.

Draw graphs of
i. the worst pooling equilibrium for the workers;
ii. the best separating equilibrium for the workers.

Be sure to label all points of interest on your graphs. You can consider any wage contract function that is consistent with the equilibrium.
Figure 1:

The worst pooling equilibrium for the workers:
The best separating equilibrium for the workers:

Grading: I took at least one point off for failure to include the wage contract $w(e)$ as part of the equilibrium.
3. (24 points) This problem concerns the selection of an optimal level \( y \) of a public good. There are 2 agents. Agent \( i \)'s utility is

\[ u_i(y, t_i, \theta_i) = \theta_i \sqrt{y} + t_i, \]

where \( \theta_i \in [0, 1] \) is agent \( i \)'s type, \( y \in \mathbb{R}^+ \) is the level of the public good, and \( t_i \in \mathbb{R} \) is a monetary transfer. The public good \( y \) is measured in dollars, i.e., the cost of providing the level \( y \) is \( y \).

(a) (6) Determine for each state \( \theta = (\theta_1, \theta_2) \) the efficient level \( y^*(\theta) \) of the public good. Please provide an explicit formula for the function \( y^*(\theta) \). Hint: Don’t forget the cost of providing the public good.

The efficient level of the public good maximizes

\[ \theta_1 \sqrt{y} + \theta_2 \sqrt{y} - y = (\theta_1 + \theta_2) \sqrt{y} - y. \]

The first order condition is

\[ \frac{(\theta_1 + \theta_2)}{2\sqrt{y}} - 1 = 0 \iff y = \frac{(\theta_1 + \theta_2)^2}{4}. \]

The first derivative test shows that

\[ y^*(\theta) = \frac{(\theta_1 + \theta_2)^2}{4} \]

maximizes the total utility from the public good minus its cost.

(b) (6) Define the family of Vickrey-Clarke-Groves (VCG) mechanisms in this setting. You can assume that there is a "government" that operates the mechanism. Be sure to substitute in your answer to a. The agents report types \((\theta_1^*, \theta_2^*)\). Based upon the reported types, the government provides the level of public good \( y^*(\theta_1^*, \theta_2^*) \) and a transfer to each agent \( i \) of the form

\[ t_i(\theta_1^*, \theta_2^*) = \theta_i^* \sqrt{y^*(\theta_1^*, \theta_2^*)} - y^*(\theta_1^*, \theta_2^*) + \tau_i(\theta_i^* - \theta_i) \]

\[ = \theta_i^* \left( \frac{(\theta_1^* + \theta_2^*)}{2} - \frac{(\theta_1 + \theta_2)^2}{4} \right) + \tau_i(\theta_i^* - \theta_i) \]

Here, \( \tau_i(\theta_i^* - \theta_i) \) can be any function of the reported type of agent \(-i\). Many of the answers omitted the term \(-y^*(\theta_1^*, \theta_2^*)\) from the transfer. The importance of this term is clear if one actually tries to prove that honest reporting is a dominant strategy for each agent.

(c) (6) Provide a formula for the budgetary surplus/deficit of the government in equilibrium of each member of the family of VCG mechanisms. Be sure to reduce your formula using your answers to a. and b.
The VCG mechanisms have the property that honest reporting is a dominant strategy for each agent and so we address here this dominant strategy equilibrium. The deficit/surplus of the government is

\[- (t_1 (\theta) + t_2 (\theta)) - y^* (\theta)\]

where \(- (t_1 (\theta) + t_2 (\theta))\) is the revenue of the government and \(y^* (\theta)\) is the cost of providing the efficient level of the public good. This reduces to

\[- \frac{(\theta_1 + \theta_2)^2}{4} + \frac{(\theta_1 + \theta_2)^2}{2} - \tau_1 (\theta_2) - \tau_2 (\theta_1) - \frac{(\theta_1 + \theta_2)^2}{4}\]

\[= - \frac{(\theta_1 + \theta_2)^2}{4} - \tau_1 (\theta_2) - \tau_2 (\theta_1).\]

Many answers calculated the deficit/surplus as

\[(t_1 (\theta) + t_2 (\theta)) - y^* (\theta),\]

i.e., the sign of the first term is incorrect. Notice that the transfers are paid to the agents in the VCG mechanism, which means they are a cost to the government.

(d) (6) Does there exist a member of the family of VCG mechanisms that runs a deficit/subsidy of zero in every state? In other words, is it possible for the government to collect exactly enough to cover the cost of the efficient level of the public good in every state?

We need functions \(\tau_1 (\theta_2), \tau_2 (\theta_1)\) such that in each state \(\theta,\)

\[- \frac{(\theta_1 + \theta_2)^2}{4} - \tau_1 (\theta_2) - \tau_2 (\theta_1) = 0 \Leftrightarrow\]

\[- \frac{(\theta_1 + \theta_2)^2}{4} - \tau_1 (\theta_2) = \tau_2 (\theta_1).\]

This last equation means that

\[- \frac{(\theta_1 + \theta_2)^2}{4} - \tau_1 (\theta_2)\]

does not depend upon \(\theta_2.\) Taking the derivative w.r.t. \(\theta_2\) implies

\[- \frac{(\theta_1 + \theta_2)^2}{2} = \tau'_1 (\theta_2).\]

The derivative \(\tau'_1 (\theta_2)\) must exist because both

\[- \frac{(\theta_1 + \theta_2)^2}{4} \text{ and } - \frac{(\theta_1 + \theta_2)^2}{4} - \tau_1 (\theta_2)\]
are differentiable w.r.t. $\theta_2$ (where the differentiability of the second formula holds because it doesn't depend upon $\theta_2$). The equation

$$-\frac{3(\theta_1 + \theta_2)}{2} = \tau'_1(\theta_2)$$

provides a contradiction to the existence of a suitable function $\tau'_1(\theta_2)$ (the left side depends on $\theta_1$ and the right side does not).

One answer from a student was notably original:

$$-\left(\frac{0+0}{4}\right)^2 - \tau_1(0) - \tau_2(0) = 0 \Leftrightarrow \tau_1(0) + \tau_2(0) = 0$$

$$-\left(\frac{1+0}{4}\right)^2 - \tau_1(1) - \tau_2(0) = 0 \Leftrightarrow \tau_1(1) + \tau_2(0) = \frac{1}{4}$$

$$-\left(\frac{0+1}{4}\right)^2 - \tau_1(0) - \tau_2(1) = 0 \Leftrightarrow \tau_1(0) + \tau_2(1) = \frac{1}{4}$$

$$-\left(\frac{1+1}{4}\right)^2 - \tau_1(1) - \tau_2(1) = 0 \Leftrightarrow \tau_1(1) + \tau_2(1) = -1$$

We therefore have the linear system

$$\begin{align*}
\tau_1(0) + \tau_2(0) &= 0 \\
\tau_1(1) + \tau_2(0) &= \frac{1}{4} \\
\tau_1(0) + \tau_2(1) &= \frac{1}{4} \\
\tau_1(1) + \tau_2(1) &= -1
\end{align*}$$

\Rightarrow

$$\begin{align*}
\tau_1(1) + -\tau_1(0) &= \frac{1}{4} \\
\tau_1(0) + \tau_2(1) &= \frac{1}{4} \\
\tau_1(1) + \tau_2(1) &= -1
\end{align*}$$

\Rightarrow

$$\begin{align*}
\tau_1(1) + \tau_2(1) &= \frac{1}{2} \\
\tau_1(1) + \tau_2(1) &= -1
\end{align*}$$

which cannot be solved.
4. (28 points) Suppose there are three agents (1, 2, and 3) and let the set of alternatives be \( A = \{a, b, c, d\} \). We consider for each agent the set \( R_i \) consisting of all possible strict orderings of the four alternatives. Let \( R = R_1 \times R_2 \times R_3 \). According to the philosopher Rawls, we evaluate an alternative for the three agents according to how it treats the worst-off member of the group. For \( x \in A \) and \((R_1, R_2, R_3) \in R\) define the Rawls number by

\[
\rho(R_1, R_2, R_3, x) = \max \{y \neq x \mid y \preceq_i x\}.
\]

In words, \( \rho(R_1, R_2, R_3, x) \) counts the maximum across the agents \( i \) of the number of alternatives that \( i \) ranks above \( x \). For example, if the rankings of the 3 agents are

- \( R_1 : a > b > c > d \)
- \( R_2 : b > c > d > a \)
- \( R_3 : c > d > a > b \)

then

\[
\rho(R_1, R_2, R_3, d) = \rho(R_1, R_2, R_3, a) = \rho(R_1, R_2, R_3, b) = 3
\]

because for each of \( a, b, \) and \( d \) there is an agent who ranks the alternative at the bottom, and

\[
\rho(R_1, R_2, R_3, c) = 2
\]

because at most two alternatives are ranked above \( c \) by some agent (in this case, agent 1). The Rawls social choice correspondence \( f^{rw} : R \rightarrow A \) by

\[
f^{rw}(R_1, R_2, R_3) = \{x \in A \mid \rho(R_1, R_2, R_3, x) \leq \rho(R_1, R_2, R_3, y) \text{ for all } y \in A\}.
\]

In words, an alternative \( x \) is in \( f^{rw}(R_1, R_2, R_3) \) if it is not possible to reduce the Rawls number by moving to another alternative \( y \). Notice that an alternative is evaluated according to its effect upon the person who ranks it lowest. In the example above,

\[
f^{rw}(R_1, R_2, R_3) = \{c\}.
\]

(a) (8) Does the Rawls correspondence satisfy no veto power? Explain your answer.

An example clarifies that \( f^{rw} \) does not satisfy no veto power. Suppose the rankings are as follows:

- \( R_1 : a > b > c > d \)
- \( R_2 : a > b > c > d \)
- \( R_3 : b > c > d > a \)

No veto power requires that \( a \in f^{rw}(R_1, R_2, R_3) \). We have, however,

\[
\rho(R_1, R_2, R_3, a) = 3 > 1 = \rho(R_1, R_2, R_3, b),
\]

i.e., it is possible to reduce the Rawls number by changing from \( a \) to \( b \). This implies \( a \notin f^{rw}(R_1, R_2, R_3) \).
(b) (8) Show that the Rawls correspondence does not satisfy monotonicity in the sense of Maskin’s paper.

Suppose the rankings are as follows:

\[ R_1 : a > c > b > d \]
\[ R_2 : c > b > a > d \]
\[ R_3 : b > d > c > a \]

We have \( f^{rw} (R_1, R_2, R_3) = \{ b, c \} \). We now move \( c \) upwards in agent 3’s ranking:

\[ R'_1 : a > c > b > d \]
\[ R'_2 : c > b > a > d \]
\[ R'_3 : b > c > d > a \]

We have \( f^{rw} (R'_1, R'_2, R'_3) = \{ c \} \). The alternative \( b \) does not fall in any agent’s ranking in moving from \( R_i \) to \( R'_i \), but we do not have \( b \in f^{rw} (R'_1, R'_2, R'_3) \), which contradicts monotonicity.

(c) (4) Given problem 2. b., what is the main conclusion of Maskin’s paper concerning the social choice correspondence \( f^{rw} \)? You can answer this question even if you can not answer b.

The paper proves that monotonicity is necessary for Nash implementability. Therefore, there cannot exist a game that implements \( f^{rw} \) in Nash equilibrium.

(d) (8) Is the Rawlsian correspondence dictatorial in the sense in which this term is used in the Gibbard-Satterthwaite Theorem? Explain your answer.

"Dictatorial" in this sense means that there is a selected person whose top choice is always chosen by \( f^{rw} \). The following example demonstrates that \( f^{rw} \) is not dictatorial:

\[ R_1 : a > b > c > d \]
\[ R_2 : b > c > d > a \]
\[ R_3 : d > c > a > b \]

We have

\[ \rho (R_1, R_2, R_3, a) = \rho (R_1, R_2, R_3, b) = \rho (R_1, R_2, R_3, d) = 3 \]
\[ > 2 = \rho (R_1, R_2, R_3, c) , \]

and so

\[ f^{rw} (R_1, R_2, R_3) = \{ c \} . \]

The only alternative in \( f^{rw} (R_1, R_2, R_3) \) is therefore the only alternative that is not the top choice of any agent.