1. (10 points) Consider in this problem a marriage market with 4 men and 4 women. The strict preferences of the men and women are as follows:

- \( P(m_1) : w_1, w_2, w_3, w_4 \)  \( P(w_1) : m_4, m_3, m_2, m_1 \)
- \( P(m_2) : w_4, w_2, w_3, w_1 \)  \( P(w_2) : m_2, m_3, m_4, m_1 \)
- \( P(m_3) : w_1, w_2, w_5, w_3 \)  \( P(w_3) : m_2, m_3, m_4, m_1 \)
- \( P(m_4) : w_2, w_4, w_3 \)  \( P(w_4) : m_1, m_4, m_3, m_2 \)

(a) (5 points) Determine the matching that is obtained through the deferred acceptance algorithm in which men make the proposals.

(b) (5 points) Show in this example that the algorithm used in a. is not strategy-proof.

2. (15 points) There are two bidders for a financial asset whose value \( v \) is unknown. Each bidder \( i \) privately observes a signal \( \theta_i \in [0,1] \). Given his signal \( \theta_i \), bidder \( i \) believes that \( v \) is uniformly distributed on the interval \([\theta_i, 1]\). His signal \( \theta_i \) is therefore "good news" in the sense that \( E[v|\theta_i] \) is increasing in \( \theta_i \). Consider a first price auction in which the bids consist of equity shares of the financial asset. If bidder \( i \) wins the auction with the bid \( b_i \in [0,1) \), then the seller receives ex post the payment \( b_i v \). The two bidders therefore compete in terms of the fractional share of the financial asset that the winning bidder will pay to the seller. The first price auction awards the item to the highest bidder who then pays the seller ex post his bid times the realized value of \( v \).

(a) (5 points) What is bidder \( i \)'s expected utility when \( \theta_i \) is his signal, \( b_i \) is his bid, and the other bidder bids \( b_j \)? The uncertainty is with respect to the realization of the value \( v \) of the asset, given \( \theta_i \). You can ignore the possibility of ties among bids.

(b) (10 points) Suppose the two bidders use the same increasing strategy \( b : [0,1] \rightarrow [0,1] \) to determine their bids. Derive a first order condition on \( b(\cdot) \) for it to define a Bayesian-Nash equilibrium. You are not asked to solve the first order condition to obtain a formula for \( b(\cdot) \).

Note: This question is flawed in that it fails to specify each agent’s beliefs about the distribution from which the other agent’s signal is drawn. I had intended in writing the question that \( \theta_1 \) and \( \theta_2 \) be i.i.d. according to the uniform distribution on \([0,1]\). Most students worked part b. using a general distribution \( F \), which was acceptable as an answer.
3. (25 points) A public good can be provided at a cost of \( c > 0 \) to \( n \geq 2 \) agents, where \( n > c \). Each agent \( i \) privately knows the value \( v_i \in [0,1] \) that he places on the public good and regards the values of every other agent as i.i.d. according to the distribution \( F \) on \([0,1]\). \( F \) has density denoted as \( f \).

Agent \( i \)'s utility is
\[
u_i(\delta,v_i,t_i) = v_i \cdot q - t_i,
\]
where \( q \in \{0,1\} \) indicates whether or not the public good is provided and \( t_i \) is a tax paid by agent \( i \).

A \textit{revelation mechanism} is a mapping
\[
(q,t_1,...,t_n) : [0,1]^n \rightarrow \{0,1\} \times \mathbb{R}^n
\]
that specifies by \( q(v_1,...,v_n) \) whether or not the public good is provided and by \( t_i(v_1,...,v_n) \) the tax on each agent \( i \), all as functions of the reported values of the agents. Let \( v = (v_1,...,v_n) \),
\[
Q_i(v_i) = \mathbb{E}[q(v)|v_i], \quad T_i(v_i) = \mathbb{E}[t_i(v)|v_i], \quad U_i(v_i,v_i^*) = v_iQ_i(v_i^*) - T_i(v_i^*) \quad \text{and} \quad U_i(v_i) = U(v_i,v_i).
\]

(a) (2 points) When is it ex post classical efficient to provide the public good?

(b) (5 points) Express mathematically the constraint of Bayesian incentive compatibility in this problem. Using this definition, derive a formula using the "revealed preference" argument that expresses \( T_i(v_i) \) in terms of \( Q_i(\cdot) \).

(c) (3 points) Show how the formula that you derived in b. can also be derived using the envelope theorem (assuming that all functions are differentiable). You don’t need to repeat any steps from your derivation in b.

(d) (5 points) Express mathematically the constraints of ex post budget balance and ex ante budget balance. Express mathematically the constraints of ex ante, interim and ex post individual rationality. Clearly label each constraint.

(e) (5 points) Define the family of VCG mechanisms in this problem.

(f) (5 points) Now consider the special case of \( n = 2, c = 1 \) and \( F \) the uniform distribution on \([0,1]\). Determine whether or not there exists an ex post efficient, incentive compatible, interim individually rational, and ex ante budget balanced mechanism in this problem.