1. (14 points) The cost to a monopolist of producing a quantity $q$ is $C(q, \theta)$, where $\theta \in [\theta_0, \theta_1]$. The monopolist privately observes $\theta$, while the regulatory agency regards $\theta$ as a draw from the interval $[\theta_0, \theta_1]$ according to the cumulative distribution $F$. Let $f$ denote the density function of $F$. The cost function $C(q, \theta)$ is strictly increasing in both $q$ and $\theta$. The inverse demand function for the monopolist’s output is $P(q)$, which is a positive, decreasing function. It is known by both the monopolist and the regulatory agency. The regulatory agency asks the monopolist to make a report $\theta^*$ concerning its cost parameter $\theta$. Based on this reported value, it then tells the monopolist the output $q(\theta^*)$ that it must produce. This output is priced at $P(q(\theta^*))$, the price determined by market demand. The choice of the function $q(\theta^*)$ by the regulatory agency defines a revelation mechanism. All functions in this problem are assumed to be differentiable.

(a) (3 points) Provide a formula for the monopolist’s profit $\pi(\theta, \theta^*)$ as a function of his true cost parameter $\theta$ and his reported cost parameter $\theta^*$.

(b) (3 points) What is the constraint of incentive compatibility in this problem?

(c) (4 points) Assuming incentive compatibility, apply the Envelope Theorem to express the monopolist’s equilibrium profit $\pi(\theta, \theta)$ in terms of $C(q, \theta)$ and his equilibrium profit $\pi(\theta_0, \theta_0)$ for the lowest possible value of his cost parameter.

(d) (4 points) The monopolist can choose to go out of business at the interim stage and bear a loss equal to $C(0, \theta)$. What are the constraints of ex ante and interim individual rationality for the monopolist in this problem?

2. (10 points) There are 5 men $\{m_1, m_2, m_3, m_4, m_5\}$ and 4 women $\{w_1, w_2, w_3, w_4\}$. The preferences of the men and women over the opposite sex are as follows:

$m_1 : w_1, w_2, w_3$ \quad $w_1 : m_4, m_3, m_5$
$m_2 : w_1, w_3, w_4, w_2$ \quad $w_2 : m_2, m_4, m_5, m_1, m_3$
$m_3 : w_3, w_2, w_4$ \quad $w_3 : m_1, m_3, m_2, m_4, m_5$
$m_4 : w_2, w_3, w_1$ \quad $w_4 : m_5, m_3, m_2, m_1$
$m_5 : w_1, w_3, w_2, w_4$

Determine the woman-optimal stable matching and the man-optimal stable matching.
3. (26 points) There are two sellers, each of whom owns one unit of an indivisible, homogeneous good, and one buyer, who may want to buy at most one of the units. Each seller \( i \) privately knows the cost \( c_i \) at which he can provide the item, while the buyer privately knows the value \( v \) that he places on acquiring one unit. The utility of each trader is quasilinear. We assume that \( v, c_1, c_2 \) are i.i.d. according to the uniform distribution on \([0, 1]\).

(a) (2 points) Let \( c_{(1)} \) denote the smaller of the two costs. What is the ex post efficient outcome of trading?

1. (4 points) What is the cumulative distribution and the density function for the smaller of the two costs \( c_{(1)} \)?
2. (4 points) What is the ex ante expected potential gains from trade?

(b) (4 points) Define the basic VCG mechanism in this problem.

1. (2 points) Using your answer to b., what is the ex ante expected monetary subsidy that is required by the basic VCG mechanism to operate?
2. (2 points) At what value \( v \) is the buyer’s interim expected payoff in the basic VCG mechanism minimized? What is his interim expected payoff at this value?
3. (4 points) The seller’s interim expected utility is minimized at \( c_i = 1 \). What is his interim expected utility at \( c_i = 1 \)?

(c) (4 points) Can there exist an ex post efficient, Bayesian incentive compatible, interim individually rational and ex ante budget balanced mechanism in this problem? Explain your answer.