Section 8 of Matthews: Abstract Auctions

We delve more deeply into the equivalence of different auction forms in this section. We consider an enormous variety of auction rules, even ones that may seem crazy (i.e., the lowest bidder wins, or all bidders pay their bids, or auctions such as \( EA(r, c) \) or \( DA(r, c) \) that take place over time, etc.).

One question that we’d like to answer is, "What is the best auction (i.e., the revenue-maximizing auction) for the seller?" If we want to determine the optimal auction, then we need to find a way to model all possible auctions. This seems impossibly complicated (for example, consider the difficulty of formalizing a bidder’s strategy in an English auction if "jump" bids are allowed or if a bidder can exit and then re-enter the bidding process). The action sets of bidders can be very complicated, especially in dynamic settings, and who knows what we might be able to invent? To be more precise, we need a way to address all possible outcomes of equilibria of auctions, however complicated their action sets may be. By outcome, I mean who gets the item (possibly randomized) and how much does the seller receive.

The key insight that makes it possible to address all possible equilibrium outcomes of auctions is the revelation principle. Suppose we have a Bayesian Nash equilibrium of some auction. The revelation principle states that there is an auction in which (i) bidders are asked to report their values, (ii) honest reporting is a Bayesian Nash equilibrium, and (iii) this honest equilibrium produces exactly the same outcome as the equilibrium of the auction that we started with. The outcomes of auctions in which (I) bidders report their values and (II) honest reporting is a Bayesian Nash equilibrium are thus a kind of "canonical form" in which all equilibrium outcomes of all other auctions can be represented. By studying auctions with properties (I) and (II), we can in effect study all possible (Bayesian Nash) equilibrium outcomes of all possible auctions.

The revelation principle is subtle. We’ll actually prove it and take some time in explaining it.

To start, define a revelation auction as one in which each bidder’s action set consists of his possible values. Let \( v_i \) denote the actual value of bidder \( i \) and \( v_i^* \) his reported value (i.e., his action in the auction); we allow \( v_i^* \neq v_i \) because it must be in bidder \( i \)'s interest to report truthfully and we verify this by evaluating untruthful reports. What defines the auction? We must specify (i) who gets the item and (ii) what payments are made to the seller given the reported values. For a vector of reports

\[
v^* = (v_1^*, v_2^*, ..., v_n^*)
\]

let \( q(v^*) \) denote the probability that the item is given to bidder \( i \) and let \( p_i(v^*) \) denote the expected payment from bidder \( i \) to the seller ("expected" because of randomization in the outcome). Let

\[
q(v^*) = (q_i(v^*))_{1 \leq i \leq n}
\]

and

\[
p(v^*) = (p_i(v^*))_{1 \leq i \leq n}
\]

A revelation auction is defined by specifying the functions \( p(\cdot) \) and \( q(\cdot) \), and we’ll denote a revelation auction by the pair \((q, p)\).

For each \( i \), define

\[
Q_i(v_i^*) = E[q_i(v_i^*, v_{-i})]
\]

i.e., \( Q_i(v_i^*) \) is the probability that bidder \( i \) receives the item when he reports \( v_i^* \) and all other other bidders report truthfully. Similarly,

\[
P_i(v_i^*) = E[p_i(v_i^*, v_{-i})]
\]

is the expected payment of bidder \( i \) when he reports \( v_i^* \) and all other bidders report truthfully. We require in a revelation auction that truthful reporting defines a Bayesian Nash equilibrium, i.e., for each bidder \( i \) and for each \( v_i, v_i^* \),

\[
Q_i(v_i) v_i - P_i(v_i) \geq Q_i(v_i^*) v_i - P_i(v_i^*).
\]

This is the constraint of incentive compatibility (IC). In words, when \( v_i \) is one’s true value, the expected payoff from reporting truthfully is at least as large as the expected profit from any possible "lie" \( v_i^* \). We also require that each bidder \( i \) should participate voluntarily for each of his possible valuations \( v_i \), i.e.,

\[
Q_i(v_i) v_i - P_i(v_i) \geq 0.
\]

(IR)
This is the constraint of individual rationality (IR), i.e., each bidder, after learning his type, would not be hurt by participating in the auction. We don’t model a "NO" option (a decision to not bid) here, but it is effectively addressed with IR. More precisely, this is the constraint of interim individual rationality: participation is voluntary at the moment at which each bidder learns his type but before the auction has operated.

Abstract Auctions

An abstract auction is defined as follows. Let \( A_i \) denote the action set of bidder \( i \). Let \( b_i : [0, 1] \rightarrow A_i \) denote bidder \( i \)'s rule for selecting an action as a function of his value \( v_i \). The outcome of the auction is specified by a function

\[
(\rho, \sigma) : \prod_{i=1}^{n} A_i \rightarrow [0, 1]^n \times \mathbb{R}^n,
\]

\[
\rho = (\rho_i)_{1 \leq i \leq n},
\]

\[
\sigma = (\sigma_i)_{1 \leq i \leq n},
\]

which specifies for each profile of actions \( a = (a_1, ..., a_n) \) and for each agent \( i \) the probability \( \rho_i (a) \) that \( i \) receives the item and his expected payment \( \sigma_i (a) \) to the seller. Let

\[
\left( \prod_{i=1}^{n} A_i, (\rho, \sigma) \right)
\]

denote an abstract auction.

**Theorem 148 (The Revelation Principle)\)** Suppose the rules \( b_1^*, ..., b_n^* \) define a Bayesian Nash equilibrium in the abstract auction \( \left( \prod_{i=1}^{n} A_i, (\rho, \sigma) \right) \), and suppose also that this equilibrium is individually rational (i.e., each bidder's expected payoff is nonnegative conditional on any of his possible values). Then there exists a revelation auction \( (q, p) \) that satisfies IC and IR and with the property that

\[
(q, p) = (\rho, \sigma) \circ (b_1^*, ..., b_n^*),
\]

that is, the revelation auction results in exactly the same outcome when the bidders report honestly as in the equilibrium of the abstract auction.

**The punch line:** We can evaluate the IR outcomes that can arise in equilibrium of any possible auction simply by considering the IR, honest equilibria of revelation auctions.

In a revelation auction that satisfies IC and IR, let

\[
\Pi_i(v) = Q_i(v) v - P_i(v)
\]

denote the equilibrium expected profit of bidder \( i \) given his value \( v \).

**Theorem 149 (8.1, p. 34 of Matthews)\)** Assuming \( Q_i(v), P_i(v) \) are differentiable, the expected profit of bidder \( i \) satisfies

\[
\Pi_i(v) = \Pi_i(0) + \int_0^v Q_i(y) dy,
\]

i.e., it is completely determined by the constant \( \Pi_i(0) \) and the function \( Q_i(\cdot) \).

**Proof.** The constraint of incentive compatibility implies that \( v^* = v \) maximizes

\[
Q_i(v^*) v - P_i(v^*).
\]

The first order condition for this maximization is

\[
Q_i'(v) v - P_i'(v) = 0.
\]
We also have

\[
\Pi_i'(v) = Q_i'(v)v + Q_i(v) - P_i'(v) = Q_i(v)
\]

The equation in the statement of the theorem follows by taking antiderivatives of each side and solving for the constant by which the two antiderivatives may differ.

This is an important result because it captures incentive compatibility in a single equation. The assumption of differentiability isn’t needed, however; in fact, differentiability of \(Q_i(\cdot)\) almost everywhere is a consequence of the following alternative proof.

Here is an alternative proof that doesn’t require the assumption that \(Q_i(v), P_i(v)\) are differentiable:

**Proof.** We use the "revealed preference" argument that we used earlier in the course to analyze \(FPA(0,0)\).\(^6\)

Consider \(v, v^\ast\). When \(v\) is his true value, bidder \(i\) can’t benefit by reporting \(v^\ast\),

\[
\Pi_i(v) = Q_i(v)v - P_i(v) \geq Q_i(v^\ast)v - P_i(v^\ast),
\]

\[
A \geq B
\]

and when \(v^\ast\) is his true value, bidder \(i\) can’t benefit by reporting \(v\),

\[
\Pi_i(v^\ast) = Q_i(v^\ast)v^\ast - P_i(v^\ast) \geq Q_i(v)v^\ast - P_i(v)
\]

\[
C \geq D
\]

We have

\[
B - C \leq A - C \leq A - D
\]

\[
Q_i(v^\ast)(v - v^\ast) \leq \Pi_i(v) - \Pi_i(v^\ast) \leq Q_i(v)(v - v^\ast)
\]

Comparing the first and last term, if \(v > v^\ast\), then

\[
Q_i(v^\ast) \leq Q_i(v)
\]

i.e., \(Q_i(\cdot)\) is (weakly) increasing. A theorem in real analysis states that a nondecreasing function is differentiable and hence continuous almost everywhere. (Matthews assumes differentiability to avoid citing this theorem). From above we have

\[
Q_i(v^\ast) \leq \frac{\Pi_i(v) - \Pi_i(v^\ast)}{(v - v^\ast)} \leq Q_i(v).
\]

Taking the limit as \(v^\ast \to v\) at a value of \(v\) where \(Q_i(\cdot)\) is continuous implies

\[
\frac{d\Pi_i}{dv}(v) = Q_i(v).
\]

Taking antiderivatives of each side and solving for the constant implies

\[
\Pi_i(v) = \Pi_i(0) + \int_0^v Q_i(y)dy
\]

which is the desired result. ■

**Remark 150** Notice that we also learn in this proof that \(Q_i(\cdot)\) is (weakly) increasing in an incentive compatible revelation auction.

One implication of this theorem is that we can solve for the seller’s expected revenue in terms of the allocation rule \(q(\cdot)\) through its determination of the functions \(Q_i(\cdot)\) together with the constants \(\Pi_i(0)\). We have two formulas for \(\Pi_i(v)\),

\[
Q_i(v)v - P_i(v) = \Pi_i(v) = \Pi_i(0) + \int_0^v Q_i(y)dy
\]

\(^6\)I’m surprised that Matthews doesn’t use this revealed preference argument to prove the result because he discusses "revealed preference" at some length in his paper.
The expected payment from bidder $i$ to the seller is
\[
E[P_i(v)] = \int_0^1 P_i(v) f(v) dv
\]
\[
= \int_0^1 \left[ Q_i(v) v - \Pi_i(0) - \int_0^v Q_i(y) dy \right] f(v) dv
\]
\[
= -\Pi_i(0) + \int_0^1 Q_i(v) v f(v) dv - \int_0^1 \int_0^v Q_i(y) f(v) dy dv.
\]

The expected revenue of the seller is therefore
\[
\sum_{i=1}^n E[P_i(v)]
\]
\[
= -\sum_{i=1}^n \Pi_i(0) + \sum_{i=1}^n \left[ \int_0^1 Q_i(v) v f(v) dv - \int_0^1 \int_0^v Q_i(y) f(v) dy dv \right]
\]

For now, this complicated expression may not seem particularly interesting. The notable fact is that the seller’s expected revenue is determined by the allocation rule $q(\cdot)$ and the constants $\Pi_i(0)$. Consequently, any two IC revelation auctions that award the item in the same way in their honest equilibria and that assign the same expected payoff to each bidder of value $v = 0$ necessarily produces the same expected revenue for the seller. More interesting still, we have the following theorem through the revelation principle:

**Theorem 151 (Revenue Equivalence)** Any two equilibria of any two auctions that generate the same allocation rule $q(\cdot)$ and the same conditional expected payoffs $\Pi_i(0)$ for each buyer with value 0 necessarily produce exactly the same expected revenue for the seller.

**Example 152** Consider the equilibria that we found in the 4 auctions, FPA(0,0), SPA(0,0), DA(0,0), and EA(0,0). For good measure, also include the equilibria that you find in the 3 auctions in problems 5, 6, and 7 of Matthews (some computed only in the uniform case). All of these equilibria have the following properties:

- the bidder with the highest value wins the item;
- a bidder with value 0 receives the item with probability 0 and makes no payment to the seller.

The first property implies that each equilibrium of each of these auctions results in exactly the same allocation rule $q(\cdot)$ (i.e., the bidder with the highest value wins). The second property means that $\Pi_i(0) = 0$ for each bidder $i$ in each of these equilibria. Consequently, all of these equilibria of these different auctions generate exactly the same expected revenue for the seller.

**Deriving Equilibria (subsection 8.1 of Matthews, p. 36)** The equation
\[
Q_i(v) v - \Pi_i(0) - \int_0^v Q_i(y) dy = P_i(v)
\]
can help us to derive equilibria in auctions (and not just revelation auctions).

**Example 153** Consider FPA($r,c$). Our previous analysis of the first-price auction was restricted to the case of $r = c = 0$. We search for a function $b : [0,1] \rightarrow \mathbb{R}$ that defines a symmetric Bayesian Nash equilibrium. We’ll assume that
\[
b(v) = NO \text{ for } v < v_0,
\]
i.e., there exists a cutoff $v_0$ so that bidders with low values choose not to enter. We’ll also assume that $b(v_0) = r$ and $b(\cdot)$ is strictly increasing on $[v_0,1]$. These seem like reasonable guesses concerning $b(\cdot)$ and
we’ll justify the guesses if we actually find such a \( b(\cdot) \) that defines an equilibrium. Characterizing the value of \( v_0 \) is part of the problem here.

**Formula for \( b(v) \):** Notice that

\[
Q_i(v) = \begin{cases} 
0 & \text{if } v < v_0 \\
G(v) = F(v)^{n-1} & \text{if } v \geq v_0
\end{cases}
\]

This is because the bidder with the highest value wins. Also, \( \Pi_i(0) = 0 \). Finally, note also that a buyer’s expected payment \( P_i(v) \) is

\[
P_i(v) = \begin{cases} 
b(v)G(v) + c = b(v)F(v)^{n-1} + c & \text{if } v \leq v_0 \\
b(v) & \text{if } v \geq v_0
\end{cases}
\]

Therefore, substituting in the equation above, for \( v \geq v_0 \) we have

\[
F(v)^{n-1}v - b(v)F(v)^{n-1} - c = \Pi_i(v) = \int_{v_0}^{v} F(y)^{n-1}dy \tag{22a}
\]

\[
F(v)^{n-1}v - c - \int_{v_0}^{v} F(y)^{n-1}dy = b(v)F(v)^{n-1}
\]

\[
v - \left( \frac{c + \int_{v_0}^{v} F(y)^{n-1}dy}{F(v)^{n-1}} \right) = b(v) \tag{22b}
\]

\( b(v) \) is increasing for \( v \geq v_0 \): We have

\[
b(v) = v - F(v)^{1-n} \left( c + \int_{v_0}^{v} F(y)^{n-1}dy \right)
\]

and so

\[
b'(v) = 1 - (1-n)F(v)^{-n} \left( c + \int_{v_0}^{v} F(y)^{n-1}dy \right) f(v) - F(v)^{1-n}F(v)^{n-1}
\]

\[
= (n-1)F(v)^{-n} \left( c + \int_{v_0}^{v} F(y)^{n-1}dy \right) f(v) > 0. \tag{23}
\]

\( b(v) \) in the uniform case: To get some insight, we reduce the formula for \( b(v) \) in the uniform case to

\[
b(v) = v - \left( \frac{c + \int_{v_0}^{v} F(y)^{n-1}dy}{F(v)^{n-1}} \right)
\]

\[
= v - \left( \frac{c + \int_{v_0}^{v} y^{n-1}dy}{v^{n-1}} \right)
\]

\[
= v - \left( \frac{c + \frac{v^n - v_0^n}{n}}{v^{n-1}} \right)
\]

\[
= \frac{n-1}{n} v - \left( \frac{nc - v_0^n}{n^{n-1}} \right)
\]

Notice how the bid is decreasing in \( c \). Its dependence on \( r \) is less obvious. We’ll now investigate how \( v_0 \) depends on \( r \).

**Characterizing \( v_0(r,c) \):** It is clear that \( v_0 \geq r \). If \( b(\cdot) \) determines a symmetric Nash equilibrium, then it must be the case that \( \Pi_i(v_0) = 0 \). This is true because a bidder with value \( v_0 - \varepsilon \) stays out and receives a payoff of 0. If \( \Pi_i(v_0) > 0 \), then a bidder with value \( v_0 - \varepsilon \) would want to enter (for \( \varepsilon \) sufficiently small). We of course couldn’t have \( \Pi_i(v_0) < 0 \) in equilibrium, because a bidder can always get a payoff of 0 by staying out.

Equation (22a) implies that

\[
\Pi_i(v_0) = F(v_0)^{n-1}v_0 - b(v_0)F(v_0)^{n-1} - c = 0
\]
We have $b(v_0) = r$, and so

$$ F(v_0)^{n-1}v_0 - rF(v_0)^{n-1} - c = 0 $$

$$ F(v_0)^{n-1}(v_0 - r) - c = 0. $$

At $v_0 = 0$, the left side is negative; at $v_0 = 1$, it is positive so long as $1 > r + c$. The left side is also increasing in $v_0$, and so this equation characterizes a unique value of $v_0$ as long as $1 > r + c$.

Note also now that

$$ b(v_0) = v_0 - F(v_0)^{1-n} \left( c + \int_{v_0}^{v_0} F(y)^{n-1} dy \right) $$

$$ = v_0 - F(v_0)^{1-n} (c) $$

$$ = F(v_0)^{1-n} (F(v_0)^{n-1}v_0 - c) $$

$$ = F(v_0)^{1-n} (F(v_0)^{n-1}r) $$

$$ = r, $$

as desired.

**Remark 154** Think back to our analysis of $SPA(r,c)$. We proved that a bidder in this auction enters only if

$$ \Pi_i(v_0) = F(v_0)^{n-1}v_0 - rF(v_0)^{n-1} - c = 0 $$

i.e., $v_0(r,c)$ is defined in $SPA(r,c)$ by exactly the same formula as in $FPA(r,c)$!

**Claim:** The allocation rule in $FPA(r,c)$ is exactly the same as in $SPA(r,c)$. Consequently, the seller’s expected revenue is the same in each auction.