**Reny’s Result**

Reny creates a single argument that serves to prove both Arrow’s Theorem and a result that implies the Gibbard-Satterthwaite Theorem as a corollary. It thus serves to emphasize the close connection between the two results. Here is the result for social choice functions that will be proven:

**Theorem 90** If there are at least 3 alternatives and \( f : \mathcal{L}^N \rightarrow A \) satisfies unanimity and monotonicity, then \( f \) is dictatorial.

**Proof of the Gibbard-Satterthwaite Theorem**

The result follows from the following lemma.

**Lemma 91** If \( f : \mathcal{L}^N \rightarrow A \) is strategy-proof and onto, then \( f \) satisfies unanimity and monotonicity.

The Gibbard-Satterthwaite Theorem follows from this lemma and the theorem proven above.

We now prove the lemma.

**Proof.** We’ll start with monotonicity. We start with the case in which only one agent’s preferences are changed. Suppose that:

1. \( f(L_i, L_{-i}) = a \)
2. \( a \) does not fall relative to any alternative in moving from \( L_i \) to \( L_i' \).
3. \( f(L_i', L_{-i}) = b \neq a \)

Strategy proofness implies that \( a \) is ranked above \( b \) according to \( L_i \). By 2., this means \( a \) is ranked above \( b \) according to \( L_i' \). But this contradicts strategy proofness when the profile is \((L_i', L_{-i})\), and so \( f(L_i', L_{-i}) = a \).

Turning to the general case, suppose that:

1. \( f(L_1, ..., L_N) = a \)
2. \( a \) does not fall relative to any alternative in moving from \( L_i \) to \( L_i' \) for each value of \( i \).

By applying the above argument \( N \) times, we can show that \( f(L_1', ..., L_N') = a \), which verifies monotonicity.

Turning to unanimity, choose \( a \in A \). Because \( f \) is onto, there exists \( L_1, ..., L_N \) such that

\[
f(L_1, ..., L_N) = a.
\]

Applying monotonicity (which we’ve just established for \( f \)), \( a \) remains the social choice if (i) we elevate \( a \) in each ranking to the top spot, and then (ii) change the ranking of the other alternatives however we wish. Consequently, \( a \) is selected whenever each agent ranks it at the top. Because \( a \) is arbitrary, \( f \) therefore satisfies unanimity.

**Proof of Reny’s Result**

**Theorem 92** If there are at least 3 alternatives and \( f : \mathcal{L}^N \rightarrow A \) satisfies monotonicity and unanimity, then \( f \) is dictatorial.

The proof is broken down into 5 steps.

**Proof. Step 1.** We’re trying to show that \( f \) is dictatorial and so we’ll go looking for the dictator. We will find an instance in which some agent \( i \)’s ranking of \( a \) or \( b \) as the top choice determines whether \( a \) or \( b \) is chosen (i.e., agent \( i \) is pivotal). This agent will be shown to be the dictator.

We consider \( a \neq b \) and start with a ranking for each agent in which \( a \) is at the top and \( b \) is at the bottom. Unanimity dictates that \( a \) is the group choice. Starting with agent 1, we alter agent 1’s ranking by successive moving \( b \) upward past each alternative. By unanimity, this does not change the group choice as long as \( b \) remains below \( a \).
A change may occur, however, when \( \beta \) replaces \( \alpha \) at the top of 1’s ranking (unanimity no longer binds). Let \( L_1 \) denote the ranking in which \( \alpha \) is first and \( \beta \) is second, with \( L_1' \) denoting the preferences in which \( \beta \) is first and \( \alpha \) is second. Let \( L_i \) denote the ranking of any other agent in which \( \alpha \) is ranked first and \( \beta \) is ranked last. We know that

\[
f(L_1, \ldots, L_N) = a. \tag{2}
\]

My claim is that monotonicity implies

\[
f(L_1', \ldots, L_N) = a \text{ or } b,
\]

i.e., changing 1’s ranking from \( L_1 \) to \( L_1' \) results in either \( a \) or \( b \) but not some third outcome. The argument is by contradiction. Suppose instead that

\[
f(L_1', \ldots, L_N) = c \neq a, b.
\]

In moving from \( (L_1', \ldots, L_N) \) to \( (L_1, \ldots, L_N) \), the relative ranking of \( c \) compared to another alternative does not change. Monotonicity thus implies \( f(L_1, \ldots, L_N) = c \) which contradicts (2).

If \( f(L_1, \ldots, L_N) = b \), we’re done with step 1. Otherwise, leave 1’s ranking at \( L_1' \) and go through the same process with agent 2. Monotonicity implies that the group choice doesn’t change until \( \beta \) replaces \( \alpha \) at the top of 2’s ranking. The same argument as above again shows that moving \( \beta \) past \( \alpha \) into the top spot either does not affect the group choice or changes it to \( b \). (This follows from monotonicity).

If the group choice changes to \( b \) when \( \beta \) moves into the top spot of 2’s ranking, then we’re done with step 1. If it doesn’t, then we move on to the next agent.

There must exist some agent \( n \) at which moving \( \beta \) past \( \alpha \) into the top spot changes the social choice from \( b \) to \( a \). If not, then we would or way through the entire set of \( N \) agents, ending up with all \( N \) agents ranking \( b \) at the top and yet \( a \) being the social choice. This would contradict Pareto efficiency. The situation we have identified can be depicted as follows, with \( n \) as the pivotal agent and \( L_i \) again representing a ranking of agent \( i \):

\[
\begin{array}{cccccccc}
L_1 & \cdots & L_{n-1} & L_n & L_{n+1} & \cdots & L_N \\
\beta & \cdots & b & a & a & \cdots & a \\
a & \cdots & a & b & \cdots & \rightarrow a \\
\vdots & \vdots & \vdots & \vdots & \vdots & \\
\end{array}
\tag{#1}
\]

\[
\begin{array}{cccccccc}
L_1 & \cdots & L_{n-1} & L_n & L_{n+1} & \cdots & L_N \\
\beta & \cdots & b & b & a & \cdots & a \\
a & \cdots & a & a & \cdots & \rightarrow b \\
\vdots & \vdots & \vdots & \vdots & \vdots & \\
\end{array}
\tag{#2}
\]

Comment: If the theorem is correct (i.e., there is a dictator), then we have found him (agent \( n \)). Now we have to prove it.

**Step 2.** We wish to argue that the following figures are correct:

\[
\begin{array}{cccccccc}
L_1 & \cdots & L_{n-1} & L_n & L_{n+1} & \cdots & L_N \\
\beta & \cdots & b & a & \vdots & \cdots & \vdots \\
\vdots & \vdots & \vdots & b & \rightarrow a \\
\end{array}
\tag{#1a}
\]

\[
\begin{array}{cccccccc}
L_1 & \cdots & L_{n-1} & L_n & L_{n+1} & \cdots & L_N \\
\beta & \cdots & b & b & \vdots & \cdots & \vdots \\
\vdots & \vdots & \vdots & a & \rightarrow b \\
\end{array}
\tag{#2a}
\]

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We’ve altered the rankings of agents 1 through n − 1 by moving a downwards to the bottom position. The preferences of agents n + 1 through N have been altered by moving a down to just above the bottom position. We need to show these changes do not alter the social choice.

Consider first the change from (#2) to (#2a). We will then use (#2a) to establish (#1a) and we’ll use (#1a) below. Notice that b does not fall relative to any other alternative in the move from (#2) to (#2a). Monotonicity thus implies that b must be the social choice in (#2a).

In order to establish that the social choice is a in (#1a), let’s consider the change in preferences in moving from (#2a) to (#1a). The only change is that a and b switch places for agent n. Applying an argument made in Step 1, monotonicity implies that the social choice in (#1a) is either a or b (not some third alternative c). If it equaled b, then monotonicity would also imply that the social choice in (#1) would have to be b, which is a contradiction. Therefore, the social choice in (#1a) must be a.

**Step 3.** For \(c \neq a, b\), consider

\[
\begin{array}{cccccc}
L_1 & \cdots & L_{n-1} & L_n & L_{n+1} & \cdots & L_N \\
: & : & a & : & : & \rightarrow a & (#3) \\
: & : & c & : & : & \\
c & \cdots & c & : & : & \\
b & \cdots & b & : & a & \cdots & a \\
a & \cdots & a & b & \cdots & b
\end{array}
\]

This situation can be obtained from (#1a) by (i) moving b downward for agents 1 through n − 1; (ii) moving c downward for agents 1 through n − 1 and n + 1 through N; (iii) moving c upward in the ranking for agent n. Because a does not fall in any agent’s ranking, the social choice remains equal to a.

**Step 4.** The next situation is obtained from (#3) by interchanging the rankings of a and b for agents n + 1 through N:

\[
\begin{array}{cccccc}
L_1 & \cdots & L_{n-1} & L_n & L_{n+1} & \cdots & L_N \\
: & : & a & : & : & \rightarrow a & (#4) \\
: & : & c & : & : & \\
c & \cdots & c & : & : & \\
b & \cdots & b & : & b & \cdots & b \\
a & \cdots & a & a & \cdots & a
\end{array}
\]

Monotonicity implies that the social choice in (#4) must be either a or b. (Suppose it is some other alternative d. In moving from (#4) to (#3), d does not fall relative to any alternative...). Suppose the outcome is b in (#4). Move c upwards to the top spot in every agent’s ranking. Pareto efficiency would then imply that c is the social choice, but monotonicity would imply that it remains equal to b. Therefore, the social choice in (#4) must be a.

**Step 5.** From (#4), we can change \(L_1, \ldots, L_{n-1}, L_{n+1}, \ldots, L_N\), and \(L_n\) however we wish without changing the social choice from a. This is true by monotonicity. The alternative a does not fall relative to any alternative, and so the social choice must remain at a.

We’ve thus proven that when agent n ranks a at the top, a must be the social choice regardless of the rankings of the other agents.

The choice of a was arbitrary, however, and so we’ve proven that for any alternative there exists some agent who "dictates" that alternative as the social choice whenever he ranks it at the top.

We can’t have more than one dictator, however: What would happen if agent n "dictates" a and some other agent "dictates" b? Consequently, the agent n that we’ve identified must be the dictator in the social choice.
Exercise 93 Reny explicitly assumes that each agent’s ordering of the alternatives is strict, i.e., no agent is indifferent between two alternatives. This is captured by his specification of $\mathcal{L}$ rather than $\mathcal{L}^*$. Where do his proofs require the strict ranking of alternatives? Note: This assumption is not needed in the standard statement of the Gibbard-Satterthwaite Theorem.


This paper originated in Maskin’s doctoral dissertation circa 1977. It is the critical paper in support of his Nobel Prize. It thus may surprise you to learn that it was not published until 1999 on the invitation of the editor of RESTUD. For a number of years, it was simply cited in manuscript form or as "in press" at the Mathematics of Operations Research. The reason that it was not published for so many years is that it had a number of troublesome errors, despite being well-known and very influential. It certainly inspired a lot of other work. The errors were corrected over time by Maskin and others.

Maskin addresses the following question: Which social choice rules $\phi$ (or correspondences) can result from the Nash equilibria of games? Specifically, given the social choice rule, does there exist a game so that (i) every alternative that is selected according to $\phi$ results from some Nash equilibrium of the game, and (ii) every Nash equilibrium of the game results in an alternative that agrees or is consistent with $\phi$? This in a fundamental way addresses whether or not $\phi$ is incentive compatible in the Nash sense. It is a natural next step following the Gibbard-Satterthwaite Theorem: given how severely restricted a social choice function $f$ must be in order to be implemented in dominant strategies, it makes sense to lower our sights a bit and seek Nash implementation instead (i.e., a less demanding notion of incentive compatibility). The work has found application in the theory of contracts, where the Nash property is interpreted as insuring that an agent will willingly meet his obligations under a contract if all other agents choose to meet their obligations (i.e., the contract is "self-enforcing"). Notice the special issue of RESTUD in which it appeared. (Mention treaties to address global warming.)

I wish to cover the paper not only because of its significance but also to highlight the importance of monotonicity, which is so crucial in Reny's paper. It is also notable that unlike Gibbard-Satterthwaite, Maskin does not require unrestricted preferences. It thus has a much broader range of applications.

The Model

$A$ - set of alternatives (not necessarily finite)
$\mathcal{R}_A$ (set of all orderings of the elements of $A$)
$n$ - number of agents
$\mathcal{R}_i \subset \mathcal{R}_A$
$\mathcal{R} = \mathcal{R}_1 \times \ldots \times \mathcal{R}_n$ - the set of profiles of admissible preferences
$R \in \mathcal{R}$

social choice rule (SCR) $- f : \mathcal{R} \rightarrow A$
$\rightarrow\rightarrow$ denotes correspondence
$a \in f(R)$ - $a$ is $f$-optimal for $R$

Nash Implementation

Given strategy sets $S_1, \ldots, S_n$, an $n$-person game form is a mapping

$$g : S_1 \times \ldots \times S_n \rightarrow A$$

Maskin mentions both pure and mixed strategy Nash equilibrium.

$\mu = (\mu_1, \ldots, \mu_n)$: a vector of mixed strategies, where the probability of outcome $g(s_1, \ldots, s_n)$ is $\mu_1(s_1) \ldots \mu_n(s_n)$

Question: How does an agent calculate his "expected payoff" if payoffs have not been specified? Do mixed strategy equilibria make any sense in this case? Do we invoke the fact that an agent is indifferent among those strategies to which he assigns positive probability in a mixed strategy equilibrium?

I'll restrict attention to pure strategy Nash equilibria in the following discussion.

Definition 94 A game form $g$ implements the social choice rule $f$ in Nash equilibrium iff:
1. any f-optimal alternative can be achieved with a pure strategy Nash equilibrium: for $\forall R \in \mathcal{R}$ and $\forall a \in f(R)$, $\exists (s_1, ..., s_n) \ni$

   \[ g(s_1, ..., s_n) = a \text{ and} \]
   
   \[ aR_ig(s'_i, s_{-i}) \text{ for all } i \text{ and } s'_i \in S_i \]

2. all Nash equilibria "agree" with $f$: for $\forall R \in \mathcal{R}$, if $(s_1, ..., s_n)$ is a Nash equilibrium given the preference profile $R$, then

   \[ g(s_1, ..., s_n) \in f(R) \]

   for all $(s_1, ..., s_n)$ in the support of $\mu$.

Notice in 2. that we don’t consider the average or expected outcome according to $\mu$, for $A$ may not have an additive structure. A msne is required to always agree with $f(R)$ in the sense given above.

This is the same definition as before, but we’ll restate it in the interest of clarity. For $R_i \in R_A$ and $a \in A$, define the lower contour set $L(a, R_i)$ of $a$ given $R_i$ as

\[ L(a, R_i) = \{ b \in A | aR_ib \} \]

In words, this is the set of all alternatives that are ranked as either worse or indifferent to $a$ given the preferences $R_i$.

**Definition 95** $f : \mathcal{R} \rightarrow A$ is monotonic if for any $R, R' \in \mathcal{R}$ and $a \in f(R)$, the condition

\[ L(a, R_i) \subset L(a, R'_i) \quad (\star) \]

holding for each $i$ implies that $a \in f(R')$.

The condition $(\star)$ holding for each $i$ means that $a$ does not fall relative to any alternative when preferences change from $R_i$ to $R'_i$ (just as in our earlier definition). This is a way to avoid using two "if...then" statements combined in one, which is confusing.

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