The Family of VCG Mechanisms

This section generalizes the transfers in the basic VCG mechanism while still retaining the key properties that (i) it is a weak dominant strategy for each agent to report honestly, and (ii) given the honest reports by the agents, an efficient choice is selected for the group in each state. We then discuss a sense in which the VCG mechanisms are the only mechanisms with these two properties.

Definition 113 Given an efficient objective $F$, a mechanism is a VCG mechanism if for reports $\theta^* = (\theta_1, \ldots, \theta_n)$ of the $n$ agents:

1. the mechanism selects the choice $F(\theta^*)$;
2. the transfer $t_i(\theta^*)$ to agent $i$ has the form
   $$t_i(\theta^*) = t_i^{VCG}(\theta^*) + t_i^\tau(\theta^*_{-i})$$
   for some choice of a function $t_i^\tau : \Theta_{-i} \to \mathbb{R}$.

The basic VCG mechanism is thus generalized by adding to its transfer $t_i^{VCG}(\theta^*)$ an additional term $t_i^\tau(\theta^*_{-i})$ that depends only upon the reports $\theta^*_{-i}$ of agents other than agent $i$. This additional term is sometimes called an individualized tax. Because this term does not depend upon agent $i$’s report, it does not affect his incentive to report honestly. This is the content of the following corollary to Theorem 110.

Corollary 114 Honestly reporting his type is a weakly dominant strategy for each agent in any VCG mechanism.

Proof. The introduction of an individualized tax $t_i^\tau(\theta^*_{-i})$ adds the same term to each side of (8), and it therefore does not affect the validity of this inequality. ■

The individualized taxes will be seen below to allow a VCG mechanism to have other properties besides efficiency and honest revelation as weakly dominant strategies. The transfer $t_i^{VCG}(\theta^*) + t_i^\tau(\theta^*_{-i})$ may be simplified through an astute choice of the individualized transfer $t_i^\tau(\theta^*_{-i})$; as will be shown in examples that follow, an efficient mechanism in which honest revelation is a weak dominant strategy for each agent may in fact be a VCG mechanism even though it at first may not appear to be one. The next section concerns instances of the model in which only the VCG mechanisms insure both honest revelation and efficiency in the sense discussed here. It is prudent, however, in all cases to investigate whether or not a mechanism with these properties is in fact a VCG mechanism.

Example 115 A seller has an indivisible item to sell. There are $n$ potential bidders, each of whom privately knows the value $\theta_i \in \mathbb{R}_+$ that he places on the item. This is agent $i$’s type. The set $A$ of alternatives is the simplex

$$A = \left\{ a = (a_1, \ldots, a_n) \bigg| \sum_{j=1}^{n} a_j = 1, \text{ each } a_j \geq 0 \text{ for } j = 1, \ldots n \right\},$$

where $a = (a_1, \ldots, a_n)$ represents a randomized allocation in which each agent $j$ receives the item with probability $a_j$. Agent $i$’s valuation function is

$$v_i(\theta_i, a) = \theta_i \cdot a_i.$$

This example concerns the Vickrey (or second price) auction, which is one way in which the seller can auction the item. The auction proceeds as follows. The agents simultaneously report to the seller their types. The seller assigns the item to the agent who reports the largest type and charges this agent a price equal to the value of the second highest reported type. In the case of ties at the highest value, the item is randomly allocated among those who report the highest type using a fair lottery, with the price remaining as the second-highest (i.e., the highest) among the reported types. An agent who does not receive the item does not make any payment to the seller.
Let $\theta^*_i$ denote the largest type reported among all of the other agents besides agent $i$ and $\chi(\theta^*_i - 1) \geq 1$ denote the number of the other agents who report $\theta^*_i$. The utility function of agent $i$ when $\theta^*_i$ is his type and he reports $\theta^*_i$ is

$$U_i(\theta_i, \theta^*_i, \theta_{i-(n-1)}) = \begin{cases} 
\theta_i - \theta^*_i & \text{if } \theta^*_i > \theta^*_i \\
\theta_i - \theta^*_i + \chi(\theta^*_i - 1) & \text{if } \theta^*_i = \theta^*_i \\
0 & \text{if } \theta^*_i < \theta^*_i
\end{cases} \quad \text{(10)}$$

The three cases are respectively the case in which agent $i$ reports the highest type, the case in which his report is strictly less than the highest reported type.

**Problem:** Show that the Vickrey auction is in the family of VCG mechanisms.

For the sake of this example, the following formula for the transfer $t^V_i(\theta^*_i, \theta^-_i)$ ("V" for Vickrey) to agent $i$ in the second price auction may prove useful:

$$t^V_i(\theta^*_i, \theta^-_i) = \begin{cases} 
-\theta^*_i & \text{if } \theta^*_i > \theta^*_i \\
\theta^*_i - \theta^*_i + \chi(\theta^*_i - 1) & \text{if } \theta^*_i = \theta^*_i \\
0 & \text{if } \theta^*_i < \theta^*_i
\end{cases} \quad \text{(11)}$$

Notice that $t^V_i(\theta^*_i, \theta^-_i)$ is an expected transfer in the case of $\theta_i = \theta^*_i$.

The first step is to reduce the transfer

$$t^V_i(\theta^*_i, \theta^-_i) = \sum_{j \neq i} v_j(\theta^*_j, a) \quad \text{(12)}$$

in the basic VCG mechanism in this auction problem. The value of this transfer depends on the relationship between $\theta^*_i$ and $\theta^*_i$ as it determines whether agent $i$ receives the item with probability one ($\theta^*_i > \theta^*_i$), whether it is randomly allocated among agent $i$ and $\chi(\theta^*_i)$ of the other agents ($\theta^*_i = \theta^*_i$), or whether it is allocated to one of the other agents:

$$t^V_i(\theta^*_i, \theta^-_i) = \begin{cases} 
0 & \text{if } \theta^*_i > \theta^*_i \\
\chi(\theta^*_i) \theta^*_i & \text{if } \theta^*_i = \theta^*_i \\
\theta^*_i + 1 & \text{if } \theta^*_i < \theta^*_i
\end{cases} \quad \text{(13)}$$

Formula (11) implies

$$t^V_i(\theta^*_i, \theta^-_i) - t^V_i(\theta^*_i, \theta^-_i) = \begin{cases} 
-\theta^*_i & \text{if } \theta^*_i > \theta^*_i \\
-\theta^*_i & \text{if } \theta^*_i = \theta^*_i \\
0 & \text{if } \theta^*_i < \theta^*_i
\end{cases} \quad \text{(14)}$$

or

$$t^V_i(\theta^*_i, \theta^-_i) = t^V_i(\theta^*_i, \theta^-_i) - t^V_i(\theta^*_i, \theta^-_i) = -\theta^*_i .$$

The Vickrey auction is thus identified as a VCG mechanism by selecting

$$t^V_i(\theta^*_i, \theta^-_i) = -\theta^*_i = -\max_{j \neq i} \theta^*_j$$

as the individualized tax on agent $i$.

**Efficiency**

The fundamental properties of the family of VCG mechanisms have been introduced. The next steps are to appraise the virtues and failings of this family of mechanisms.

The starting point is the objective of efficiency. While it may seem obvious that the agents should wish to maximize the sum of the valuations in their choice, a concern is the distinction between an agent’s valuation and his utility, i.e., the impact of the transfers to the agents on their welfare. The principal criterion in economics for assessing the welfare of multiple agents is Pareto optimality.
Theorem 118: Given a state \( \theta \) and a constant \( k \in \mathbb{R} \), consider the set \( \mathcal{A}(k) \subset A \times \mathbb{R}^n \) defined as

\[
\mathcal{A}(k) = \left\{ (a, t_1, \ldots, t_n) \mid a \in A \text{ and } \sum_{j=1}^n t_j = k \right\}.
\]

A choice \( a' \in A \) is efficient in the state \( \theta \) if and only if every element of the form

\[
(a', t'_1, \ldots, t'_n) \in \mathcal{A}(k)
\]
is Pareto optimal over the set \( \mathcal{A}(k) \) in the state \( \theta \).

Pareto optimality of \( (a', t'_1, \ldots, t'_n) \) over \( \mathcal{A}(k) \) in the state \( \theta \) means that it is not possible to strictly increase the utilities of some of the agents in this state without strictly decreasing the utilities of some other agents by switching from \( (a', t'_1, \ldots, t'_n) \) to some other element of \( \mathcal{A}(k) \). Intuitively, this means that it is not possible to improve the welfare of the \( n \) agents in an unambiguous fashion, for any change that makes some individuals better off necessarily makes some others worse off. A change can thus be justified in terms of the welfare of the \( n \) agents only by placing greater emphasis on the utilities of some agents and less on the utilities of others.

The equality

\[
\sum_{j=1}^n t_j = k
\]

is a budget constraint in which \( k > 0 \) indicates that a monetary subsidy can be provided to the \( n \) agents, while \( k < 0 \) indicates an aggregate tax on the \( n \) agents. To this point, we have ignored constraints on the aggregate value of the transfers. Without such a constraint, there is no limit to what can be accomplished for the utilities of the agents (i.e., just provide them with larger and larger transfers). Allocation problems thus typically require some form of budget constraint in order to be plausible and budget constraints will play a significant role in the remainder of these notes. Given the budget constraint (15) for whatever value of \( k \) is given, Theorem 118 states that Pareto optimality is achieved only when the choice is efficient. This justifies our focus on efficient choice. The choice transfers that satisfy (15) then serve as a means to redistribute utility among the agents. The postulated form of utility (6) is therefore sometimes referred to as the case of transferrable utility.

Proof. Efficiency \( \Rightarrow \) Pareto optimality. The proof is by contradiction. Given \( (a', t'_1, \ldots, t'_n) \) as assumed in the theorem, let \( (a'', t''_1, \ldots, t''_n) \) have the property that

\[
u_j \left( \theta_j, a'', t''_j \right) \geq u_j \left( \theta_j, a', t'_j \right)
\]

for all agents \( j \), with (16) holding strictly for at least one value of \( j \). Summing over the \( n \) agents and then applying the formula for utility (6) and the budget constraint (15) imply

\[
\sum_{j=1}^n u_j \left( \theta_j, a'', t''_j \right) > \sum_{j=1}^n u_j \left( \theta_j, a', t'_j \right)
\]

\[
\sum_{j=1}^n v_j \left( \theta_j, a'' \right) + \sum_{j=1}^n t''_j > \sum_{j=1}^n v_j \left( \theta_j, a' \right) + \sum_{j=1}^n t'_j
\]

\[
\sum_{j=1}^n v_j \left( \theta_j, a'' \right) + k > \sum_{j=1}^n v_j \left( \theta_j, a' \right) + k
\]

\[
\sum_{j=1}^n v_j \left( \theta_j, a'' \right) > \sum_{j=1}^n v_j \left( \theta_j, a' \right).
\]

This last inequality contradicts the efficiency of \( a' \).

Pareto Optimality \( \Rightarrow \) Efficiency. This proof is also by contradiction. Suppose \( (a', t'_1, \ldots, t'_n) \) is Pareto optimal but not efficient. There exists \( a'' \in A \) such that

\[
\sum_{j=1}^n v_j \left( \theta_j, a'' \right) > \sum_{j=1}^n v_j \left( \theta_j, a' \right).
\]
Define for each agent \( j \) the transfer
\[
t_j^* = t_j + v_j (\theta_j, a') - v_j (\theta_j, a'').
\]
Notice that
\[
u_j (\theta_j, a'', t_j^*) = v_j (\theta_j, a'') + t_j^* \\
= v_j (\theta_j, a') + t_j' \\
= u_j (\theta_j, a', t_j').
\]
Also,
\[
\sum_{j=1}^{n} t_j^* = \sum_{j=1}^{n} \left[ t_j' + v_j (\theta_j, a') - v_j (\theta_j, a'') \right] \\
= \sum_{j=1}^{n} t_j' + \sum_{j=1}^{n} v_j (\theta_j, a') - \sum_{j=1}^{n} v_j (\theta_j, a'') \\
< k.
\]
Define
\[
\varepsilon = \frac{k - \sum_{j=1}^{n} t_j^*}{n}
\]
and consider
\[
(a'', t_1^* + \varepsilon, \ldots, t_n^* + \varepsilon).
\]
It is straightforward to show that
\[
\sum_{j=1}^{n} (t_j^* + \varepsilon) = k
\]
and
\[
u_j (\theta_j, a'', t_j^* + \varepsilon) > u_j (\theta_j, a', t_j')
\]
for each agent \( j \), which contradicts the Pareto optimality of \((a', t_1', \ldots, t_n')\) in \( A(k) \).

**The Form of Utility**

The proof of the Pareto optimality of an efficient choice in Theorem 118 highlights two assumptions, namely the particular form of utility assumed in (6) together with the budget constraint (15). This subsection investigates the form of utility (6) and its significance in our discussion to this point, particularly in relationship to the VCG mechanism. The significance of the budget constraint is discussed in the next subsection.

The form
\[
u_i (\theta_i, a, t_i) = v_i (\theta_i, a) + t_i
\]
is *quasilinear with private values* in the choice \( a \) and the monetary transfer \( t_i \). "Quasilinear" refers to the linearity in the monetary transfer \( t_i \) and the private value assumption is that agent \( i \)'s valuation function \( v_i (\theta_i, a) \) does not depend upon the types \( \theta_{-i} \) of the other agents. The first step here is to understand how these assumptions restrict the choice problem. This is important in understanding how and when this theory can be applied.

**Private Values**

It has been assumed that an agent knows his valuation function \( v_i (\theta_i, \cdot) \) on the different choices in \( A \) once he observes his own type \( \theta_i \). More generally, the valuation function of agent \( i \) may depend upon the types observed by the other agents; agent \( i \) might not fully know the value for him of the different possible choices in \( A \) until the state \( \theta \) is fully realized. This phenomenon commonly arises in auction settings. For instance, the type of a oil firm that bids for the drilling rights on a tract of land in a government auction may consist of the data that the firm collects in order to estimate the future yield of the tract. The firm might change
its estimate, however, if it somehow gained access to the types of competing firms. The firm might change its estimate after participating in the auction because the bids of the other firms may reflect their private information. An extreme case is the common value model, in which an item for sale has exactly the same value to any bidder who buys it. If the value of drilling rights exactly equals the value of oil obtainable through drilling, and if this value is the same for each oil company, then a common value model is appropriate. Financial instruments such as stocks and bonds are often studied using common value models.

For the sake of this discussion, we refer to the case of utility of the form

$$u_i(\theta, a, t_i) = v_i(\theta, a) + t_i$$

as quasilinear utility with nonprivate or interdependent values.

Given the reports (7) of the agents, the basic VCG mechanism extends to the case of quasilinear utility with nonprivate values in a straightforward manner: the choice $$F(\theta^*)$$ is efficient, i.e.,

$$F(\theta^*) \in \arg \max_{\alpha \in A} \sum_{i=1}^{n} v_i(\theta^*, \alpha),$$

and the transfer to the ith agent is

$$t_i^{VCG}(\theta^*) \equiv \sum_{j \neq i} v_j(\theta^*, \alpha).$$

The family of VCG mechanisms is again obtained by adding an individualized tax $$t_i(\theta^*, \theta_{-i})$$ to the transfer of each agent i.

The decision problem of an agent at the interim stage in the case of nonprivate values is problematic: because he does not know $$\theta_{-i}$$, he does not know the function that he should try to maximize in choosing his report $$\theta^*_i$$. There is a sense, however, in which agents retain the incentive to report honestly in a VCG mechanism in this more general formulation. Consider an agent’s incentives ex post, i.e., after all reports have been made and the true state $$\theta$$ is realized. If all agents report their types honestly (whatever those types may be), then agent i maximizes his ex post utility by reporting his type honestly. This is the content of the following theorem.

**Theorem 119** Honest reporting is an ex post Nash equilibrium in any VCG mechanism.

**Proof.** Given the state $$\theta$$, agent i’s utility in a VCG mechanism when he reports $$\theta^*_i$$ and the other agents report honestly is

$$u_i(\theta, F(\theta^*_i, \theta_{-i}), t_i^{VCG}(\theta^*_i, \theta_{-i})) = v_i(\theta, F(\theta^*_i, \theta_{-i})) + \sum_{j \neq i} v_j((\theta^*_i, \theta_{-i}), F(\theta^*_i, \theta_{-i})) + t_i^{VCG}(\theta_{-i})$$

$$= v_i(\theta, a^*) + \sum_{j \neq i} v_j((\theta^*_i, \theta_{-i}), a^*),$$

where

$$a^* = F(\theta^*_i, \theta_{-i}).$$

Choosing $$\theta_i^* = \theta_i$$ insures the selection of a value of $$a^*$$ that maximizes (17). ■

**The Problems of Revelation Mechanisms**

VCG mechanisms are revelation mechanisms, and revelation mechanisms can be impractical if each agent’s type is complicated. It can be implausible that the each agent is capable of reporting his type. The combinatorial auction problem is a very simple and practical problem in which this issue is salient. Consider a single seller who wishes to sell m items to n bidders. A common example concerns the case in which the seller is a government that wishes to sell bandwidth or cell sites to telecommunication companies. A set of alternatives $$A$$ consists of all possible ways of assigning the m items to the n bidders. An agent’s type determines his preferences over the alternatives. We can simplify matters a bit by assuming that each bidder cares only about the set of items that he receives (i.e., he neither benefits nor is hurt by how the items that he does not receive are assigned among the other bidders). This is a restrictive assumption, because
competing firms such as telecommunication companies may care about what the other firms receive as this affects future competition. We'll avoid this complication.

Suppose that each agent's type determines a utility function. This function must specify the agent's utility for each possible subset of the \( m \) items that he may receive. There are \( 2^m \) possible subsets. This can be seen by identifying each subset with an \( m \)-vector of 0s and 1s, with a 0 entry in the \( i \)th slot indicating that the \( i \)th item isn't in the set and a 1 indicating that the \( i \)th item is in the set. The number of possible sets is the number of possible vectors of this form, which is easily calculated to be \( 2^m \). A type of the agent therefore determines a \( 2^m \)-vector of utility values, with each entry corresponding to the utility for a particular subset. Utility can be normalized, but this reduces the dimension of the vector of utilities only to \( 2^m - 1 \). The problem is called the "combinatorial" auction or assignment problem to emphasize the complexity issues. Notice that the problem would only become more complex if an agent also had preferences over the assignment among the other agents of the items that he does not receive.

The point is that the reporting requirements on an agent quickly becomes vast as \( m \) increases. In computer science and electrical engineering, exponential growth in communication or complexity is typically interpreted as a sign of infeasibility. The problem of assigning \( m \) items to \( n \) people is fundamental and arises in many practical settings. The VCG mechanism provides a practical solution only for small \( m \) or with other restrictions on the model.

To elaborate on this last point, researchers typically simplify the combinatorial auction problem by restricting the preferences of the bidders. For example, a common assumption is to assume that an agent has a value for each item and the utility to him of a set of items is simply the sum of the values of the items in the set. This allows his type to be represented by an \( m \)-vector (i.e., the utility for each of the \( m \) items). It assumes away, however, the possibility of complementarities (either positive or negative) among the items. Complementarities are clearly important in such problems as selling cell sites to telecommunication companies. Another approach is to assume that each agent \( i \) is "single-minded" in the sense that he has a desired subset \( A_i \). The subset is typically assumed to be publicly known; the value \( v_i \) that he places on it, however, is his own private information. The agent is single-minded in the sense that: (i) his utility is 0 if the set he receives does not contain \( A_i \); (ii) his utility is \( v_i \) if the set he receives contains \( A_i \). These are just two examples of restricted models that have been considered in the literature. The possibility of reducing communication by having the mechanism operate over time has been considered.