December 8

0.0.38.3 A Lemma

Lemma 28 For the efficient allocation rule $q(v, c)$, suppose the pricing rule $p(v, c)$ satisfies IC. Then:

1. 
   \begin{align}
   \Pi_b (v) & = \Pi_b (v) + \int_{v}^{x} Q_b (y) dy, \\
   \Pi_s (c) & = \Pi_s (\tau) + \int_{x}^{c} Q_s (x) dx,
   \end{align}
   \tag{23}

   where $x$ and $y$ are dummy variables.

2. If IR is also satisfied by $q(v, c)$ and $p(v, c)$, then $q(v, c)$ satisfies the equation
   \begin{align}
   \Gamma & = \Pi_b (\omega) + \int_{\omega}^{v} \int_{x}^{c} Q_b (y) g(y) dy dv + \Pi_s (\tau) + \int_{\tau}^{c} \int_{x}^{c} Q_s (x) f(c) dc dx.
   \end{align}
   \tag{24}

   The $x$ and the $y$ in (24) are dummy variables. Notice that from 2. it follows that IR is satisfied by an IC mechanism if and only if
   \begin{align}
   \Pi_b (\omega), \Pi_s (\tau) \geq 0.
   \end{align}

   We can think of $\omega$ as the least favorable value for the buyer and $\tau$ as the least favorable cost for the seller. Given IC, IR thus binds as a constraint at the "worst-off" types of the two traders.

   Notice also that the formulas in (23) are derived directly from the constraint of incentive compatibility. IC is quite restrictive as a constraint; I sometimes refer to it as a double continuum of constraints, because each pair of a true value and a reported value defines its own inequality. IC is so restrictive as a constraint that it determines each trader’s profit function $\Pi_b (v), \Pi_s (\tau)$ up to a constant ($\Pi_b (\omega)$ or $\Pi_s (\tau)$).

0.0.38.4 The Inefficiency Result

I want to demonstrate the meaning and the value of (24) while it is fresh in our minds. For its value, let’s turn to the main theorem of Myerson and Satterthwaite. The efficient allocation rule $q(v, c)$ requires that

\begin{align}
q(v, c) = \begin{cases}
1 & \text{if } v > c \\
0 & \text{if } v \leq c
\end{cases}.
\end{align}

If $\tau \leq \omega$ so that the interval $(\omega, \tau)$ lies below $(\omega, \tau)$, then $q(v, c) = 0$ for all $v, c$ is efficient; this can be implemented in an IC, IR and ex post budget-balanced way by allowing the buyer and the seller to trade at a fixed price $p$ chosen between $\tau$ and $\omega$ (the probability of trade is zero). If $\tau \leq \omega$ so that the interval $(\omega, \tau)$ lies above $(\omega, \tau)$, then $q(v, c) = 1$ for all $v, c$ is efficient; this can be implemented in an IC, IR and ex post budget-balanced way by allowing the buyer and the seller to trade at a fixed price $p$ chosen between $\tau$ and $\omega$ (the
probability of trade is one). If \((\varphi, \tau) \cap (\omega, \tau) \neq \emptyset\), however, then it is not certain whether or not trade should occur or should not occur (as it is in the previous two cases). In this case, the following theorem holds.

**Theorem 29** If \((\varphi, \tau) \cap (\omega, \tau) \neq \emptyset\), then

\[
\Gamma - \int_\omega \int_\omega Q_b(y) g(y) dy dv - \int_\omega \int_\omega Q_s(x) f(c) dx dc < 0
\]

in the case of the efficient allocation rule \(q(v, c)\). Consequently, no mechanism \((q, p)\) exists that is IC and IR and in which \(q(v, c)\) is the ex post efficient trading rule.

Myerson and Satterthwaite prove this theorem by directly evaluating these integrals in the case of the ex post efficient allocation rule (it involves integration by parts as means of repackaging the integrals until the sign of left side of (25) is clear). A simpler and more insightful proof was subsequently developed that we’ll now consider. It involves a deeper understanding of the formula

\[
\Gamma = \Pi_b(\omega) + \int_\omega \int_\omega Q_b(y) g(y) dy dv + \Pi_s(\tau) + \int_\omega \int_\omega Q_s(x) f(c) dx dc.
\] (26)

Let’s interpret it as follows. The term \(\Gamma\) is the expected gains from trade. As they are derived directly from the constraint of incentive compatibility and the assumption of the efficient allocation rule, formulas (23) for \(\Pi_b(\omega)\) and \(\Pi_s(\tau)\) describe how the traders must be compensated as functions of their types in order to induce them to reveal those types. If the mechanism is to be individually rational and budget-balanced, then the expected gains from trade must be sufficiently large to reward the traders with the expected utilities they require in order to induce them to reveal their types. This produces the inequality (24).

Notice that the formula (26) is determined by the densities \(f, g\) and the efficient allocation rule up to the constants \(\Pi_b(\omega)\) and \(\Pi_s(\tau)\). We can calculate all possible values of the right side of (26) using any using any family of efficient and incentive compatible mechanism that we choose with the knowledge that the answer does not depend upon our choice. In terms of the traders’ expected utilities, mechanisms can only differ in their determination of constants \(\Pi_b(\omega)\) and \(\Pi_s(\tau)\).

A VCG mechanism is efficient and incentive compatible. Its allocation rule is the efficient allocation rule, and the fact that it is dominant strategy incentive compatible means that the ex ante expected utilities of the two traders in this mechanism must therefore satisfy (23). By varying the individualized transfers, we can vary the values of \(\Pi_b(\omega)\) and \(\Pi_s(\tau)\) however we wish. If it is possible that some efficient and IC mechanism satisfies to satisfy (24), then some VCG mechanism will exist that satisfies it. This narrows our search from all IC, efficient mechanisms to the family of VCG mechanisms with constant individualized transfers. We will now prove the theorem by showing that no VCG mechanism can satisfy (24).

Given the individualized transfers \(t_b\), the buyer’s expected utility in a VCG mechanism
is
\[ \int_{\underline{v}}^{\overline{v}} \int_{\underline{c}}^{\overline{c}} [(v - c) + t_b] q(v, c) f(c) g(v) dc dv = t_b + \Gamma. \]

The buyer receives \( t_b \) in every state, and he receives the item \((v)\) and pays \( c \) when trade occurs \((q(v, c) = 1)\). For efficiency, \( q(v, c) = 0 \) when \( v < c \). Similarly, the seller’s ex ante expected utility is
\[ t_s + \int_{\underline{c}}^{\overline{c}} \int_{\underline{v}}^{\overline{v}} (v - c) q(v, c) f(c) g(v) dc dv = t_s + \Gamma. \]

In words, each trader’s ex ante expected utility is his individualized transfer plus the ex ante expected gains from trade. Notice that nothing would be gained here by allowing \( t_b \) to depend on the seller’s reported cost \( c \) and \( t_s \) on the buyer’s reported value \( v \), as this dependence would be integrated out.

The question of whether or not (24) holds can now be reinterpreted as follows: can \( t_b \) and \( t_s \) be chosen so that
\[ \Gamma = (t_b + \Gamma) + (t_s + \Gamma) \]
and individual rationality is satisfied? The right side is what the sum of what the traders receive in the VCG mechanism. This reduces to
\[ \Gamma = -(t_b + t_s), \]
\[ (27) \]
i.e., the individualized taxes must be sufficient to cover the deficit \( \Gamma \) of the basic VCG mechanism. We haven’t yet dealt with IR, which we’ll need to complete the impossibility proof.

Consider first the special case of \([\underline{v}, \overline{v}] = [\underline{c}, \overline{c}]\). In the VCG mechanism we have
\[ \Pi_b(v) = \int_{\underline{c}}^{\overline{c}} (v - c) q(v, c) f(c) dc + t_b = t_b \]
and
\[ \Pi_s(c) = \int_{\underline{c}}^{\overline{c}} (v - c) q(v, c) g(v) dv + t_s = t_s, \]
\[ (29) \]
and so IR implies
\[ t_b, t_s \geq 0. \]

We’ve effectively bounded above the amount that the traders can be taxed. Substitution into (27) implies
\[ \Gamma = -(t_b + t_s) \leq 0, \]
which does not hold.

The restriction
\[ [\underline{v}, \overline{v}] = [\underline{c}, \overline{c}] \]
makes it easy to solve for \( \Pi_b(v) \) and \( \Pi_s(c) \). The argument works in the general case; it’s more delicate and reveals the role of the relative positions of the intervals \([\underline{v}, \overline{v}]\) and \([\underline{c}, \overline{c}]\).
We have from (28) and (29) and the constraint of IR that

\[ t_b + t_s \geq - \int \mathbb{R} (v - c) q(v, c) f(c) dc - \int \mathbb{R} (v - \tau) q(v, c) g(v) dv \quad \Leftrightarrow \]

\[ - (t_b + t_s) \leq \int \mathbb{R} (v - c) q(v, c) f(c) dc + \int \mathbb{R} (v - \tau) q(v, c) g(v) dv < \Gamma. \]

The sum

\[ \int \mathbb{R} (v - c) q(v, c) f(c) dc + \int \mathbb{R} (v - \tau) q(v, c) g(v) dv \]

is the expected gains from trade for a buyer with type \( v \) plus the expected gains from trade for a seller with type \( \tau \). This sum is clearly less than \( \Gamma \), the total expected gains from trade, which incorporates all of the gains from trade and not just those of two types of traders. Efficiency is thus inconsistent with incentive compatibility, individual rationality, and budget balance.

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5 We’re using here the assumption that \((\mathbb{R}, \tau) \cap (\mathbb{R}, \tau) \neq \emptyset\).