November 12

p. 436: 3 (omit part C.), 4, 12, 13, 14, 15 (note the information provided immediately before problem 14), 19

p. 452: 4 (part A only)

With benefit taxation, an agent may benefit from misreporting; he can benefit by reducing his tax burden by dishonestly reporting his marginal benefit from the public good. I find Campbell’s discussion of this taxation scheme on p. 434 to be confusing.

Example 71  Let $U_1 = x + y_1$, $U_2 = 2x + y_2$, $U_3 = 5x + y_3$ and $g(x) = x_2/2$. Each agent is endowed with 24 units of the private good. The Samuelson Criterion implies that the efficient level $x^*$ of the public good solves:

$$1 + 2 + 5 = x^*$$

and so $x^* = 8$. We’ll focus in this example on agent 1, whose utility in the case of honest reporting is

$$U_1 = 8 + 24 - \frac{1}{8} \cdot \frac{1}{2} \cdot 8^2 = 32 - 4 = 28.$$  

Assuming honest reporting by agents 2 and 3, assume now that agent 1 reports $B_1'(x) = 0$. The level of the public good that is chosen is

$$2 + 5 = 7,$$

and agent 1 receives a utility of

$$U_1 = 7 + 24 - \frac{0}{7} \cdot \frac{1}{2} \cdot 7^2 = 31 > 28.$$  

While lying reduces the amount of the public good to an inefficient level, it benefits agent 1 because he reduces his tax burden to zero.

Example 72  Ex. 1.6, p. 435. Nash equilibrium of the benefit tax mechanism. Let’s try to address every agent’s incentive to misreport by finding a Nash equilibrium among reports in the benefit tax mechanism. There are $n$ individuals, each with utility function $U_i(x, y_i) = x + y_i$ that the individual knows privately. We have $g(x) = x^2/2$ and each agent is endowed with $\omega_i$ units of the private good. We’ll assume that each agent $i$ is known to have a utility function of the form $U_i^{\alpha_i} = \alpha_i x + y_i$. The number $\alpha_i$ is agent $i$’s marginal benefit from the public good. We’ll consider the game in which agents are asked to report their $\alpha_i$’s. Based on these reports, Samuelson Criterion is used to identify the optimal level of the public good, which is then paid for using benefit taxation. We’ll look for a Nash equilibrium in the reported values.

If each agent $i$ reports $B_i(x) = \alpha_i$, then the Samuelson Criterion selects $x^*$ as the level of the public good, where

$$\sum_{i=1}^{n} B_i'(x^*) = g'(x^*) \iff \alpha_1 + \ldots + \alpha_n = x^*.$$  

(18)

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The optimal level of the public good based upon the true value of $\alpha_i = 1$ is therefore $n$.

Let’s now solve for a Nash equilibrium in the reports. Let

$$\beta = \sum_{j \neq i}^{n} \alpha_j$$

denote the sum of the reports of the other agents besides $i$. Given these other reports and his own report $\alpha_i$, agent $i$’s utility is

$$U_i = x^* + y_i$$

$$= (\alpha_i + \beta) + \left(\omega_i - \frac{\alpha_i}{\alpha_i + \beta} \cdot g(\alpha_i + \beta)\right)$$

$$= (\alpha_i + \beta) + \left(\omega_i - \frac{\alpha_i}{\alpha_i + \beta} \cdot \frac{(\alpha_i + \beta)^2}{2}\right).$$

Here, we use (18) to determine that $x^* = \alpha_i + \beta$ and $\sum_{i=1}^{n} B_i(x^*) = \alpha_i + \beta$. Let’s simplify this:

$$U_i = (\alpha_i + \beta) + \left(\omega_i - \frac{\alpha_i}{\alpha_i + \beta} \cdot \frac{(\alpha_i + \beta)^2}{2}\right)$$

$$= (\alpha_i + \beta) + \left(\omega_i - \frac{\alpha_i(\alpha_i + \beta)}{2}\right).$$

In a best response to the $\beta$ determined by the reports of other agents, agent $i$ chooses $\alpha_i$ to maximize this expression:

$$0 = \frac{\partial U_i}{\partial \alpha_i} = 1 - \frac{1}{2}((\alpha_i + \beta) + \alpha_i) \Leftrightarrow$$

$$\frac{1}{2}(2\alpha_i + \beta) = 1 \Leftrightarrow$$

$$\alpha_i = 1 - \frac{\beta}{2}.$$

We have

$$\frac{\partial^2 U_i}{\partial \alpha_i^2} < 0,$$

and so this choice of $\alpha_i$ is indeed a best response to $\beta$.

For a Nash equilibrium, we therefore need $\alpha_1, \ldots, \alpha_n$ such that, for each $i$,

$$\alpha_i = 1 - \frac{\sum_{j \neq i}^{n} \alpha_j}{2}.$$

This implies

$$\frac{\alpha_i}{2} = 1 - \frac{\sum_{j=1}^{n} \alpha_j}{2},$$

i.e., $\alpha_i$ has the same value for all $i$. The only possible Nash equilibrium is a symmetric
equilibrium. Calling this value $\alpha$, we have

$$\frac{\alpha}{2} = 1 - \frac{n\alpha}{2} \iff \frac{n + 1}{2} \alpha = 1 \iff \alpha = \frac{2}{n + 1}.$$  

The level of the public good in this Nash equilibrium is

$$n\alpha = \frac{2n}{n + 1} < 2,$$

even though the optimal level of the public good is $n$. The difference between the equilibrium level of the public good and the optimal level thus grows larger as the number of agents increases.

### 0.0.29 The Pivotal Mechanism

The pivotal mechanism uses an approach similar to the second price auction to make honest reporting of preferences for the public good into a dominant strategy for each agent. It works by imposing a surtax on each individual that equals the cost/benefit that his participation imposes on the rest of the group. We first develop it in the simple case of two possible alternatives for the public good, $F$ and $G$, each of which consists of (i) a level ($x_F$ or $x_G$) of the public good and (ii) a tax ($c_i(F)$ or $c_i(G)$) specified for each of the $n$ agents. We are thus now including the funding of the public good as part of the alternative; $F$ and $G$, for instance, could each specify the same level of the public good but propose different methods of funding the project. We may impose the constraint that the chosen product is fully funded, i.e.,

$$\sum_{i=1}^{n} c_i(F) = g(F) \text{ and } \sum_{i=1}^{n} c_i(G) = g(G).$$

This, however, will play an significant role in our discussions of incentives below. We use

$$V_i(F) = B_i(F) - c_i(F)$$

and

$$V_i(G) = B_i(G) - c_i(G)$$

to simplify our notation in the discussion below. We have

$$U_i(F) = V_i(F) + \omega_i.$$

In our previous discussion, the taxes were critical in their influence on the incentives of the agents for non-truthful reporting. The pivotal mechanism will levy surtaxes on the agents so as to incentivize them to report honestly. Recall that the taxes to fund a project are included as part of the definition of the project itself; we’re now considering surtaxes in the sense that they are additional taxes not to pay for the project but to incentivize honest reporting of preference information. Formally, with the addition of the surtaxes, honest reporting of one’s preferences for the public good is a dominant strategy for each agent. The pivotal mechanism operates as follows:
Each agent $i$ reports a pair of numbers $(V_i(F), V_i(G))$. This amounts to reporting his ranking of $F$ relative to $G$.

$F$ is chosen for the group if $\sum_{i \in N} V_i(F) > \sum_{i \in N} V_i(G)$, and otherwise $G$ is chosen (let’s ignore the possibility of ties).

Surtax on agent $j$: We assess here the impact of agent $j$’s report on the welfare of the other $n - 1$ agents. Suppose $F$ is chosen.

- If $\sum_{i \not= j} V_i(F) > \sum_{i \not= j} V_i(G)$, then $F$ would have been chosen even without agent $j$’s report of $(V_j(F), V_j(G))$ (i.e., agent $j$’s report was not pivotal in the decision). Consequently, no surtax is imposed on agent $j$.

- If $\sum_{i \not= j} V_i(F) < \sum_{i \not= j} V_i(G)$, then agent $j$’s report of $(V_j(F), V_j(G))$ cost the other $n - 1$ agents a total of $\sum_{i \not= j} V_i(G) - \sum_{i \not= j} V_i(F)$ by causing $F$ to be chosen instead of $G$. Require $j$ to pay this amount as a surtax so that he bears responsibility for the impact of his report on the welfare not only of himself but also the others.

- Proceed similarly if $G$ is chosen.

A person is "pivotal" if the outcome depends upon his report. In the pivotal mechanism, an individual pays a surtax equal to the damages that he imposes on others in the event that he is pivotal. Notice that if $F$ is chosen over $G$, and if the taxes fully fund the cost of the project, then

$$\sum_{i=1}^{n} V_i(F) > \sum_{i=1}^{n} V_i(G)$$

$$\sum_{i=1}^{n} (B_i(F) - c_i(F)) > \sum_{i=1}^{n} (B_i(G) - c_i(G))$$

$$\sum_{i=1}^{n} B_i(F) - \sum_{i=1}^{n} c_i(F) > \sum_{i=1}^{n} B_i(G) - \sum_{i=1}^{n} c_i(G)$$

$$\sum_{i=1}^{n} B_i(F) - g(F) > \sum_{i=1}^{n} B_i(G) - g(F).$$

The pivotal mechanism thus selects the project that is Pareto efficient.

Example 73 This is example 2.1 on p. 444. There are three individuals and two projects.

<table>
<thead>
<tr>
<th></th>
<th>Soren</th>
<th>Rosie</th>
<th>Edie</th>
</tr>
</thead>
<tbody>
<tr>
<td>Project F</td>
<td>10</td>
<td>19</td>
<td>30</td>
</tr>
<tr>
<td>Project G</td>
<td>15</td>
<td>10</td>
<td>40</td>
</tr>
<tr>
<td>surtax</td>
<td>0</td>
<td>0</td>
<td>4</td>
</tr>
</tbody>
</table>

We’ll consider the case in which people report honestly, which will be shown below to be a dominant strategy for each agent. Project $F$ produces a total benefit of $10 + 19 + 30 = 59$
Let’s investigate what agent necessarily be truthful, and this is what establishes the dominance of truthful reporting.

Project $G$ is therefore selected for the group.

If Soren were not present, the total benefit of project $F$ to Rosie and Edie is 49 and the total benefit of project $G$ is 50. Soren’s participation therefore does not affect the group choice, he has no impact on the welfare of Rosie and Edie, and his surtax is therefore 0.

If Rosie were not present, the total benefit from project $F$ to Soren and Rosie would be 29 and the total benefit of project $G$ is 45. Project $G$ would still be chosen for Soren and Edie, Rosie’s participation has no impact upon their welfare, and so her surtax equals 0.

If Edie were not present, the total benefit from project $F$ to Soren and Rosie would be 29 and the total benefit of project $G$ is 45. Project $G$ would still be chosen for Soren and Edie, Rosie’s participation has no impact upon their welfare, and so her surtax equals 0.

Can it pro...
and his utility with the dishonest report is $V_j(G)$. We have

$$V_j(F) - (V_{-j}(G) - V_{-j}(F)) - V_j(G) = \sum_{i \in N} V_i(F) - \sum_{i \in N} V_i(G) > 0,$$

and so lying is not profitable for agent $j$.

- $V_j(F) < V_j(G)$ and $V_{-j}(F) > V_{-j}(G)$: In this case, agent $j$ prefers $G$, but does not pay a surcharge with his honest report. His utility is $V_j(F)$ and his utility if he instead lied and caused $G$ to be chosen would be $V_j(G) - (V_{-j}(F) - V_{-j}(G))$ because by implementing $G$ he would earn a surtax. We have

$$V_j(F) - [V_j(G) - (V_{-j}(F) - V_{-j}(G))] = \sum_{i \in N} V_i(F) - \sum_{i \in N} V_i(G) > 0,$$

and so lying is not profitable for him.

- $V_j(F) < V_j(G)$ and $V_{-j}(F) < V_{-j}(G)$: This case is not possible, because it contradicts $\sum_{i \in N} V_i(F) > \sum_{i \in N} V_i(G)$ (i.e., the assumption that $F$ is chosen over $G$).

### 0.0.29.1 Defects of the Pivotal Mechanism

Because the surcharges are always nonnegative, the pivotal mechanism can only run a monetary surplus (again, notice that the tax burden for funding the project on agent $i$ is included in his $V_i$ function). We’ll now show that returning this surplus to the agents may disrupt the incentive to report honestly if agents take their shares of the returned surplus into account in making their reports.

**Example 74** Example 2.2, p. 446. Let’s return to this example and assume that the three individuals know that any surplus will be shared equally among them.

<table>
<thead>
<tr>
<th></th>
<th>Soren</th>
<th>Rosie</th>
<th>Edie</th>
</tr>
</thead>
<tbody>
<tr>
<td>Project $F$</td>
<td>10</td>
<td>19</td>
<td>30</td>
</tr>
<tr>
<td>Project $G$</td>
<td>15</td>
<td>10</td>
<td>40</td>
</tr>
<tr>
<td>surtax</td>
<td>0</td>
<td>0</td>
<td>4</td>
</tr>
</tbody>
</table>

Recall that $G$ is chosen over $F$ and the mechanism runs a surplus equal to 4. Consider a report by Rosie of $V_{1R}(F) = 24$ and $V_{1R}(G) = 10$.

<table>
<thead>
<tr>
<th></th>
<th>Soren</th>
<th>Rosie</th>
<th>Edie</th>
</tr>
</thead>
<tbody>
<tr>
<td>Project $F$</td>
<td>10</td>
<td>24*</td>
<td>30</td>
</tr>
<tr>
<td>Project $G$</td>
<td>15</td>
<td>10</td>
<td>40</td>
</tr>
<tr>
<td>surtax</td>
<td>4</td>
<td>0</td>
<td>9</td>
</tr>
</tbody>
</table>

The * indicates the lie. The total reported benefit of $F$ is 64 and the total reported benefit of $G$ is 65, and so Rosie’s lie does not alter the selection of $G$ over $F$. The surtaxes, however, change:

- If Soren were not present, $F$ would be chosen over $G$. Soren is now taxed $(24 + 30) - (10 + 40) = 4$ to account for the harm he does to Rosie and Edie.
If Rosie were not present, G would still be chosen over F. Rosie pays no surtax.

If Edie were not present, then F would be chosen over G. Edie bears a surtax equal to \((10 + 24) - (15 + 10) = 9\) for the harm that she imposes on Soren and Rosie.

We see that Rosie profits by lying: her lie does not cause her to bear a surtax, but it does insure that she shares in the distribution of a larger surplus (9 versus 4). Honesty is thus no longer a dominant strategy if the agents anticipate the return of the monetary surplus.

Campbell makes the point that while the pivotal mechanism implements the optimal level of the public good, the allocation including all taxes is not Pareto optimal: each individual who pays a positive surtax is strictly worse off than in the Pareto optimal allocation in which public good is exactly paid for by the people.

Example 75  Example 2.3, p. 447. Participation can be harmful. Consider the following table:

<table>
<thead>
<tr>
<th></th>
<th>Soren</th>
<th>Rosie</th>
<th>Edie</th>
</tr>
</thead>
<tbody>
<tr>
<td>Project F</td>
<td>20</td>
<td>30</td>
<td>40</td>
</tr>
<tr>
<td>Project G</td>
<td>30</td>
<td>25</td>
<td>31</td>
</tr>
<tr>
<td>surtax</td>
<td>0</td>
<td>1</td>
<td>5</td>
</tr>
</tbody>
</table>

Campbell asks us to consider the case in which F is the status quo and G is a newly available option. The pivotal mechanism determines that G should not replace F, but both Rosie and Edie must pay surtaxes to make this determination. They in effect pay and are made worse off with no change in the status quo.

Example 76  Example 2.4, p. 447. Two-person manipulation of the pivotal mechanism. Consider the following tables.

<table>
<thead>
<tr>
<th></th>
<th>Soren</th>
<th>Rosie</th>
<th>Edie</th>
</tr>
</thead>
<tbody>
<tr>
<td>Project F</td>
<td>20</td>
<td>300</td>
<td>300</td>
</tr>
<tr>
<td>Project G</td>
<td>40</td>
<td>25</td>
<td>25</td>
</tr>
<tr>
<td>surtax</td>
<td>10</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Soren</th>
<th>Rosie</th>
<th>Edie</th>
</tr>
</thead>
<tbody>
<tr>
<td>Project F</td>
<td>20</td>
<td>300</td>
<td>300</td>
</tr>
<tr>
<td>Project G</td>
<td>40</td>
<td>25</td>
<td>25</td>
</tr>
<tr>
<td>surtax</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

We can see that with truthful reports G is chosen over F. A majority (i.e., Rosie and Edie), however, prefer F to G. Rosie and Edie can each increase their utility by 5 by exaggerating their values for F over G. While not vulnerable to manipulation by individuals, the pivotal mechanism is susceptible to manipulation by groups of individuals.