**Continuing with our digression on order statistics:**

Let’s now turn to \( v_{(n-1)} \), the second largest order statistic in a sample of \( n \) values. What is its cumulative distribution \( F_{(n-1)}(x) \)?

\[
F_{(n-1)}(x) = \Pr \{ v_{(n-1)} \leq x \} = \Pr \{ v_1, v_2, ..., v_n \leq x \} + \Pr \{ v_1, v_2, ..., v_{n-1} \leq x \text{ and } v_n > x \} = F(x)^n + n \cdot (1 - F(x)) \cdot F(x)^{n-1}.
\]

The second line breaks the event in which \( n - 1 \) values lie below \( x \) into two disjoint events, namely, the event in which all \( n \) values lie below \( x \) and the event in which \( n - 1 \) values lie below \( x \) and the \( n \)th value lies above \( x \). The last line calculates the probabilities of these two events. The first term follows from above. The second term effectively devides the event in which one of the \( n \) values lies above \( x \) into \( n \) subevents indexed by which of the \( n \) values lies above \( x \). The term \( 1 - F(x) \) is the probability that the selected value \( v_1 \) lies above \( x \), and the term \( F(x)^{n-1} \) is the probability that the other \( n - 1 \) terms lie below \( x \).

The density \( f_{(n-1)}(x) \) of the \( n - 1 \)th order statistic is therefore

\[
f_{(n-1)}(x) = F'_{(n-1)}(x) = nF(x)^{n-1}f(x) - nF(x)f(x)^{n-1} + n \cdot (1 - F(x)) \cdot (n - 1)F(x)^{n-2}f(x)
\]

\[
= \left[nF(x)^{n-1} - nF(x)f(x)^{n-1} + n \cdot (1 - F(x)) \cdot (n - 1)F(x)^{n-2}\right]f(x)
\]

\[
= n f(x) \cdot (n - 1) \cdot (1 - F(x)) \cdot F(x)^{n-2}.
\]

I interpret this formula as follows:

- \( nf(x)dx \) is the probability that one of the \( n \) values is drawn at (near) the value \( x \);
- In order for this value to be the second largest, one of the remaining \( n - 1 \) values must be above \( x \) (which occurs with probability \( (n - 1) (1 - F(x)) \)), while the remaining \( n - 2 \) values must be below \( x \) (which occurs with probability \( F(x)^{n-2} \)).

**Example 36** Calculate the expected value of the \((n - 1)\)th order statistic \( v_{(n-1)} \) when \( n \) values are independently drawn from the uniform distribution on \([0, 1]\) (i.e., \( F(x) = x \) for \( x \in [0, 1] \)). We can guess that the answer is

\[
1 - \frac{2}{n+1} = \frac{n-1}{n+1}.
\]

To derive this formula, we calculate

\[
\int_0^1 xf_{(n-1)}(x)dx = \int_0^1 xn f(x) \cdot (n - 1) (1 - F(x)) \cdot F(x)^{n-2}dx
\]

\[
= \int_0^1 xn(n - 1)(1 - x) \cdot x^{n-2}dx
\]

\[
= n(n - 1) \int_0^1 x^{n-1} - x^n dx
\]

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0.0.18 Section 3, p. 349  Four Basic Auction Mechanisms

We have already defined the Vickrey auction. The three other forms of auction that we’ll discuss next are the first price sealed bid, the Dutch auction, and the English oral auction.

Definition. The first price sealed bid auction operates as follows. Each bidder simultaneously and privately submits a bid to the seller. The highest bidder is selected to receive the item and pays his own bid as his price.

The English oral auction has an auctioneer who steadily calls out higher and higher prices. Bidders indicate their willingness to pay those prices. The auction stops at the price at which only a single bidder is willing to buy at that price.

Comment. The Japanese "button" auction is a variation of this procedure. A dial or clock that indicates the price is positioned in front of the bidders and the price steadily ascends. Each bidder keeps a "doorbell" button depressed as long as he is willing to pay the current price on the clock. The auction stops and the price stops ascending as soon as there is only one bidder remaining with his button depressed (i.e., when the next to last bidder drops out). The winning bidder pays the price at which the clock stops. The Japanese button auction can be altered by either having one’s button private or in public display. While this does not matter for the private value case that we consider here, it does matter in the case in which a bidder may learn something about the value of the item to him from the behavior of the other bidders.

The English auction is an ascending price auction. The Dutch auction is a descending price auction. An auctioneer begins by calling out a high price and then steadily lowers it until some bidder steps forward and indicates that he is willing to buy at that price.

Comment. In the old days of stereo equipment (the 1980s), there was an electronics store in Chicago that would periodically have a sale that they called a Dutch auction. We might describe it as a "multiunit Dutch auction" because it was a way to sell multiple units of stereo equipment. The starting sale prices on Friday were announced in a big newspaper ad, along with the commitment to lower the price by a certain percentage each day over the weekend. A customer paid a lower price by waiting until Sunday to buy, but he ran the risk that the product he wanted would be sold out by then. You’ll see this occasionally as a gimmick in retailing.

A fundamental problem of auction theory is, "Why is there more than one kind of auction? As the seller, how should I know which one to choose in a particular situation if I’m interested in maximizing my revenue? What principles guide the selection of one form of auction over another?" We first note that some of the differences among auctions are more appearance than reality.
0.0.18.1 Outcome Equivalence.

Definition. Two auctions are outcome equivalent if for any profile of reservation values:
1. each of the two auctions selects the same bidder to receive the item;
2. each bidder pays exactly the same amount in the two auctions.

Theorem 4 The Vickrey auction and the English oral auction are outcome equivalent.

It is a dominant strategy for a bidder in the English oral auction to stay in the auction until the price passes his reservation value. The winning bidder is therefore the bidder with the highest value and he pays the next highest reservation value as his price. This is exactly the same as a Vickrey auction.

There may be practical differences between the two auctions:
- the Vickrey auction operates faster (i.e., in one-shot);
- the price may rise just above the second highest reservation value in a practical implementation of the English oral auction (i.e., it is not until the price passes the second highest value that this bidder drops out of the auction);
- The argument of outcome equivalence depends upon private values: if bidders learn about the value of the item through the behavior of the other bidders, then the two auctions are clearly different, as the English auction provides an opportunity to learn from the behavior of others that the Vickrey auction does not provide.
- Auctioneers in the English oral auction may recognize a phantom or nonexistence bidder as a means of providing one last increase or "bump" in the price above the value at which the second-to-last bidder drops out. While unethical, it can be done. This is clearly not possible if the bids in the Vickrey auction are publicly displayed.

Theorem 5 The Dutch and the first price sealed bid auctions are outcome equivalent.

This will become clearer in the next section as we actually derive a bidder’s equilibrium bidding strategy for each of these two auctions. Imagine that a bidder in the Dutch auction selects before the auction starts the price at which he’ll step up and indicate his willingness to buy the item. Think about the equivalence of two decision problems for a bidder across these two auctions: each bidder must select a bid at which he is willing to buy given that no other bidder has been willing to buy at a higher price. This is as formal of a proof that I can provide at this point in the course.

Example 37 Derivation of an Equilibrium in the First Price Sealed Bid Auction with Uniformly Distributed Private Values
n bidders, with the valuation $v_i$ of bidder $i$ independently drawn from the uniform distribution on $[0, 1]$

We wish to solve for a common rule $b: [0, 1] \rightarrow [0, 1]$ that defines a Bayesian Nash equilibrium.

Assumption: $b$ is strictly increasing and differentiable (the answer we come up with will have this property).

Condition for equilibrium: for each $v_i \in [0, 1], b(v_i) = x$ maximizes

$$U_i(v_i, x) = (v_i - x)(b^{-1}(x))^{n-1}$$

Note: the assumption that $b$ is increasing is used in this formula.

Two distinct ways of deriving a formula for $b(v_i)$ are presented below. The first is straightforward as an approach, the second is more clever and ultimately simpler.

FOC:

$$0 = \frac{\partial}{\partial x} U_i(v_i, x) = -(b^{-1}(x))^{n-1} + (v_i - x)(n-1)(b^{-1}(x))^{n-2}\left(\frac{1}{b'(b^{-1}(x))}\right)$$

Satisfied at $b(v_i) = x$:

$$0 = -(v_i)^{n-1} + (v_i - b(v_i)) \left[ (n-1)(b^{-1}(b(v_i)))^{n-2}\left(\frac{1}{b'(b^{-1}(b(v_i)))}\right) \right]$$

$$0 = -(v_i)^{n-1} + (v_i - b(v_i)) \left[ (n-1)(v_i)^{n-2}\left(\frac{1}{b'(v_i)}\right) \right]$$

$$(v_i - b(v_i)) \left[ (n-1)(v_i)^{n-2} \right] = v_i^{n-1}b'(v_i)$$

$$(n-1)(v_i)^{n-1} = v_i^{n-1}b'(v_i) + b(v_i)(n-1)v_i^{n-2}$$

$$\left(\frac{n-1}{n}\right)\frac{d}{dv_i}(v_i)^n = \frac{d}{dv_i} [v_i^{n-1}b(v_i)]$$

$$\left(\frac{n-1}{n}\right)\frac{d}{dv_i}(v_i)^n = v_i^{n-1}b(v_i) + k$$

Argue: $k = 0$ by substituting $v_i = 0$ into the equation. Reducing, we then obtain

$$b(v_i) = \frac{(n-1)}{n} v_i.$$

The sufficiency of solving the first order condition as a means of maximizing a bidder’s utility. Let’s check this. Suppose all bidders $j$ except bidder $i$ are using the strategy

$$b(v_j) = (n-1)\frac{v_j}{n}.$$  

We need to show that using this strategy maximizes the expected utility of bidder $i$ for each of his possible values $v_i$. Bidder $i$’s expected utility when $v_i$ is his value and $x$ is his bid is

$$(v_i - x)(b^{-1}(x))^{n-1}.$$  

To compute $b^{-1}(x)$, we solve

$$x = \left(\frac{n-1}{n}\right) b^{-1}(x)$$

to obtain

$$\frac{n}{n-1} x = b^{-1}(x).$$
Bidder i’s expected utility is therefore
\[(v_i - x) \left( \frac{n}{n-1} x \right)^{n-1} = (v_i - x) \left( \frac{n}{n-1} \right)^{n-1} x^{n-1}.\]

The first derivative of expected utility is
\[- \left( \frac{n}{n-1} \right)^{n-1} x^{n-1} + (v_i - x) \left( \frac{n}{n-1} \right)^{n-1} (n-1) x^{n-2} = \left( \frac{n}{n-1} \right)^{n-1} x^{n-2} [-x + (v_i - x) (n-1)] = \left( \frac{n}{n-1} \right)^{n-1} x^{n-2} [-nx + (n-1)v_i].\]

This derivative changes from + to − as x increases past x = b(v_i), which verifies that this bid maximizes i’s expected utility.

**Example 38** The next two examples were not covered in class, and you are not responsible for it. It remains here because I don’t want to delete it from my notes. Returning to the bargaining example, there are two people, J and K. J has an asset that he would like to sell to K. J’s reservation value is 2 (i.e., he profits only if he sells it for more than 2). Let v denote K’s reservation value for the asset. J knows that v ∈ [0, 5], but not its probability distribution.

Suppose J makes a "take it or leave it" offer to K: J proposes a price p, which K can accept or reject. We can depict this as an extensive form game:

\[
\begin{array}{c}
\text{J} \\
\nearrow \\
\text{p} \\
\searrow \\
\text{K} \\
\nearrow \\
\text{R} \\
\searrow \\
\text{p-2, v-p} \\
\end{array}
\]

**Maxmin:** When J (the seller) proposes p, one of two things happens:

1. K accepts and J gets p − 2;
2. K rejects and J gets 0.

The smaller (or the "min") of these two is p − 2 < 0 if p < 2 and 0 if p ≥ 2. We therefore have
\[\max_p \min_v u_J(p, v) = 0,
\]
and every p ≥ 2 is a maxmin strategy (i.e., each p ≥ 2 guarantees J a payoff of at least 0, which is the most that J can guarantee himself in this game).

As this example illustrates, a weakly dominated strategy (here, p < 2) cannot be a
maxmin strategy. Maxmin is not particularly helpful in this example in guiding J in the selection of \( p \).

**Minimizing maximum regret:** The regret determined by the price \( p' \) and the value \( v \) is

\[
\max_p u(p, v) - u(p', v).
\]

If \( p' \leq v \), then \( u(p', v) = p' - 2 \) (K pays the price \( p' \)). Regret could come in the form of either (i) selling at a higher price (maximally, the price of \( v \) or from not selling (earning a return of zero). Therefore,

\[
\max_p u(p, v) - u(p', v) = \max_p \{v - 2, 0\} - (p' - 2)
\]

If we now maximize this over \( v \in [p', 5] \), we have

\[
\max_{v \in [p', 5]} \max_p \{v - p', 2 - p'\} = \max_p \{5 - p', 2 - p'\} = 5 - p'.
\]

If \( p' > v \), then \( u(p', v) = 0 \) (K rejects the price \( p' \)). Regret comes in the form of selling at the highest acceptable price of \( v \) (maximally, the price of \( v \) or from not selling (earning a return of zero). Therefore,

\[
\max_p u(p, v) - u(p', v) = \max_p \{v - 2, 0\} - 0 = \max_p \{v - 2, 0\}.
\]

If we now maximize this over \( v \in [0, p'] \), we have

\[
\max_{v \in [0, p']} \max_p \{v - 2, 0\} = \max_p \{p' - 2, 0\}.
\]

In order to minimize his maximum regret, J should therefore choose \( p' \) to minimize

\[
\max_p \{5 - p', p' - 2, 0\}.
\]

The first term is the missed opportunity or magnitude of regret from selling at the price of \( p' \) when a higher price is possible (maximally, 5). The second term is the regret from failing to make the sale because the price was too high. The 0 reflects the fact that J should never end up with a negative payoff. Maximum regret is minimized by choosing \( p' \) to solve \( 5 - p' = p' - 2 \), i.e., \( p' = 7/2 \). See the figure below.

---

**Example 39** A seller has an indivisible item that he may sell to a buyer. The buyer privately knows his reservation value \( v \) for the item and the seller privately knows his cost \( c \) for the item. Both traders know that \( v \) and \( c \) are each in \([0, 1]\). The buyer announces a bid \( b \)
simultaneously as the seller announces an offer \( s \); trade occurs at the price \( p = (b + s) / 2 \) if \( b \geq s \), and it does not occur (with no money changing hands) if \( b < s \). The buyer’s utility is
\[
v - p \text{ if } b \geq s \\
0 \text{ if } b < s
\]
where \( v \) is his value and \( p \) is the price he pays when he trades. Similarly, the seller’s utility is
\[
p - c \text{ if } b \geq s \\
0 \text{ if } b < s
\]
where \( c \) is his cost and \( p \) is the price he receives when he trades. Determine how each trader selects his announcement (bid or offer) using (i) the maxmin criterion and (ii) the minimax regret criterion.

Let’s first consider maxmin. For any bid \( b \) at or below his value \( v \), the minimum possible payoff is 0 in the event of no trade. For bids above his value \( v \), of course, a negative payoff is possible. Consequently, his maxmin bids consists of all bids that are less than or equal to \( v \). A similar argument shows that a seller’s maxmin offers consists of all offers that are greater than or equal to his cost. Maxmin as a decision criterion is not very helpful in this example in terms of guiding the traders or allowing us to predict the outcome of bargaining.

We next consider minimax regret. Consider the buyer with value \( v \) and bid \( b \leq v \). Let \( s \) be the seller’s offer. Here is the regret the buyer may have:

1. \( s \geq v \): The seller’s offer does not present any opportunity for profitable trading and so the regret equals zero.
2. \( b < s < v \): The buyer regrets not bidding \( s \) and making a profit of \( v - s \).
3. \( s = b \): Regret equals zero.
4. \( s < b \): The regret is \( b - s \)

0.0.18.2 Continuing with the problem of deriving an equilibrium in the first price auction, an alternative approach is to apply the Envelope Theorem.

**Theorem 6 (Envelope Theorem)** Consider a differentiable function \( U(x, y) \) for \( x, y \in \mathbb{R} \). Suppose \( y(x) \) is a differentiable function that determines the value of \( y \) that maximizes \( U(x, y) \) for each value of \( x \):
\[
U(x, y(x)) \geq U(x, y) \text{ for all possible } y.
\]
Let \( U(x) \) denote this maximal value of the function \( U \),
\[
U(x) = U(x, y(x)).
\]
Then
\[
U'(x) = \frac{\partial U}{\partial x}(x, y(x)).
\]
This theorem is widely used in microeconomic theory. Its proof is a simple calculation that uses the fact that 
\[
\frac{\partial U}{\partial y}(x, y(x)) = 0
\]
because \( y(x) \) is the value of \( y \) that maximizes \( U(x, y) \):
\[
\overline{U}'(x) = \frac{d}{dx} U(x, y(x)) = \frac{\partial U}{\partial x}(x, y(x)) + \frac{\partial U}{\partial y}(x, y(x)) \cdot y'(x)
\]
\[
= \frac{\partial U}{\partial x}(x, y(x)) + 0 \cdot y'(x)
\]
\[
= \frac{\partial U}{\partial x}(x, y(x)).
\]

The example from which this theorem originated is the relationship between the short-run and the long-run cost curve of a firm. Here, let \( C(x, y) \) denote the minimum possible cost of producing the output \( x \) given the capital \( y \) (you can see from the proof that the theorem applies to both maximization and minimization problems). The short-run is determined by a fixed value of \( y \) (i.e., a given manufacturing facility). The long-run is defined by all inputs being variable, including the capital \( y \). The function \( C(x, y) \) presumes that all other inputs of production are chosen to minimize the cost of producing the output \( x \) for a given value of \( y \). The function \( y(x) \) is the optimal capital for producing the output \( x \), i.e.,
\[
C(x, y(x)) \leq C(x, y)
\]
for all possible \( y \).
The function \( C(x) \) is the long-run cost function,
\[
\overline{C}(x) = C(x, y(x)),
\]
i.e., the cost of producing \( x \) when all inputs can be chosen to minimize this cost. The Envelope Theorem in this case states
\[
\overline{C}'(x) = \frac{\partial C}{\partial x}(x, y(x)),
\]
i.e., the long-run marginal cost at an output \( x \) equals the short-run marginal cost for the short-run cost curve corresponding to the optimal capital for producing \( x \). Graphically, this corresponds to the tangency at each output \( x \) of the long-run cost curve and the particular short-run cost curve determined by the capital \( y(x) \) that is optimal for producing \( x \). This reflects the fact that the long-run cost curve is the lower envelope of all possible short-run cost curves.
We next apply the Envelope Theorem to the determination of an equilibrium bidding strategy in the first price sealed bid auction. Individual \( i \)’s expected utility when he bids \( x \) and all others use an increasing function \( b(\cdot) \) to determine their bids is

\[
U_i(v_i, x) = (v_i - x) \left( b^{-1}(x) \right)^{n-1}.
\]

We want \( b(v_i) \) to be the value of the bid \( x \) that maximizes bidder \( i \)’s expected utility. The Envelope Theorem implies

\[
\frac{d}{dv_i} U_i(v_i, b(v_i)) = \frac{\partial}{\partial v_i} U_i(v_i, x) \text{ evaluated at } x = b(v_i) = (b^{-1}(x))^{n-1} \text{ evaluated at } x = b(v_i) = v_i^{n-1}
\]

and so

\[
U_i(v_i, b(v_i)) = \frac{v_i^n}{n} + c
\]

We can conclude that \( c = 0 \) as a consequence of increasing strategies and the fact that a buyer with value \( v_i = 0 \) therefore has an expected payoff of zero, i.e., \( U_i(0, b(0)) = 0 \).

We have another formula:

\[
U_i(v_i, b(v_i)) = (v_i - b(v_i)) v_i^{n-1} = \frac{v_i^n}{n}
\]

and so

\[
v_i - b(v_i) = \frac{v_i}{n}
\]

or

\[
b(v_i) = \frac{(n - 1)}{n} v_i,
\]

which is the same answer we obtained before.

Notice that the winning bidder makes a profit by virtue of his underbidding in the equilibrium

\[
b(v_i) = \frac{(n - 1)}{n} v_i.
\]

The seller thus fails to extract all of the surplus or profits from the auction. These are the rents that the bidders earn by virtue of having private information about their preferences (as given by their reservation values).

**0.0.18.3 An Equilibrium of the First Price Sealed Bid Auction for an Arbitrary Distribution \( F \).**

We now assume that the reservation values of the \( n \) bidders are independently and identically distributed according to the distribution \( F \) on the interval \([v, \bar{v}]\). The probability that any bidder’s value \( v_i \) is less than the number \( x \) is \( F(x) \). We also assume that \( F \) has a density function \( f \) that is nonzero on \([v, \bar{v}]\), i.e.,

\[
F'(x) = f(x) > 0 \text{ for } x \in [v, \bar{v}].
\]

We’ll repeat the above steps, replacing the uniform distribution with the arbitrary distribution \( F \) and the interval \([0, 1]\) with \([v, \bar{v}]\). Individual \( i \)’s expected utility when he bids \( x \) and all others use an increasing function \( b(\cdot) \) to determine their bids is

\[
U_i(v_i, x) = (v_i - x) F \left( b^{-1}(x) \right)^{n-1}.
\]
We want \( b(v_i) \) to be the value of the bid \( x \) that maximizes bidder \( i \)'s expected utility. The Envelope Theorem implies

\[
\frac{d}{dv_i} U_i(v_i, b(v_i)) = \frac{\partial}{\partial v_i} U_i(v_i, x) \text{ evaluated at } x = b(v_i) = F(b^{-1}(x))^{n-1} \text{ evaluated at } x = b(v_i) = F(v_i)^{n-1}
\]

and so

\[
U_i(v_i, b(v_i)) = \int_{v_i}^{v_i} F(v)^{n-1} dv + c
\]

where \( v \) is a dummy variable. We can conclude that \( c = 0 \) as a consequence of increasing strategies and the fact that a bidder with reservation value \( v_i = v \) therefore has an expected payoff of zero, i.e., \( U_i(v_i, b(v_i)) = 0 \). Unfortunately, we cannot simplify this further in the case of a general distribution \( F \).

We have another formula:

\[
U_i(v_i, b(v_i)) = (v_i - b(v_i)) F(v_i)^{n-1} = \int_{v_i}^{v_i} F(v)^{n-1} dv.
\]

and so

\[
v_i - b(v_i) = \frac{1}{F(v_i)^{n-1}} \int_{v_i}^{v_i} F(v)^{n-1} dv
\]

or

\[
b(v_i) = v_i - \frac{1}{F(v_i)^{n-1}} \int_{v_i}^{v_i} F(v)^{n-1} dv.
\]

We assumed in this derivation that \( b(\cdot) \) is an increasing function. Let’s verify that the above answer actually has this property.

\[
b'(v_i) = 1 - \frac{F(v_i)^{n-1} \cdot F(v_i)^{n-1} - \int_{v_i}^{v_i} F(v)^{n-1} dv \cdot (n - 1) F(v_i)^{n-2} f(v_i)}{F(v_i)^{2n-2}}
\]

\[
= 1 - 1 + \frac{\int_{v_i}^{v_i} F(v)^{n-1} dv \cdot (n - 1) f(v_i)}{F(v_i)^n}
\]

\[
= \frac{\int_{v_i}^{v_i} F(v)^{n-1} dv \cdot (n - 1) f(v_i)}{F(v_i)^n}
\]

\[
> 0,
\]

and so it is strictly increasing for all \( F \).