Example 12  Matching Pennies

\[
\begin{array}{ccc}
1/2 & H & T \\
H & 1, -1 & -1, 1 \\
T & -1, 1 & 1, -1 \\
\end{array}
\]

In this example, there is no Nash equilibrium in pure strategies. Each player has two pure strategies – H and T. No pair consisting of these two strategies defines a Nash equilibrium.

There does exist a Nash equilibrium in mixed strategies. A mixed strategy refers to a specific randomization over one’s pure strategies. Suppose player 2 flips his coin, i.e., he plays each of his pure strategies with probability equal to 1/2. Let \( p \) denote the probability that player 1 chooses \( H \) and so \( 1 - p \) is the probability that he chooses \( T \). Player 1’s expected payoff is therefore

\[
p \left( \frac{1}{2} - \frac{1}{2} \right) + (1 - p) \left( -\frac{1}{2} + \frac{1}{2} \right) = 0,
\]

and so every mixed strategy produces an expected payoff of zero for player 1. Every mixed strategy is therefore a best response to player 2’s coin flip. Similarly, suppose \( p = 1/2 \) and player 1 flips his coin. Every mixed strategy is a best response for player 2. Consequently, \( p = 1/2, q = 1/2 \) are best responses to each other, and so they define a mixed strategy Nash equilibrium. We’ve illustrated an important feature of mixed strategy Nash equilibrium, namely, a player’s method of randomization makes his opponents indifferent among all of their pure strategies that they play with positive probability in the mixed strategy Nash equilibrium.

Example 13  Find the Nash equilibria in the following game.

\[
\begin{array}{ccc}
1/2 & L & R \\
T & 3, 2 & -1, 3 \\
B & 1, 1 & 0, -1 \\
\end{array}
\]

We note first that there are no pure strategy Nash equilibria in this game. We therefore search for a mixed strategy Nash equilibrium. Letting \( p \) denote the probability that player 1 chooses \( T \), \( p \) must satisfy

\[
L : 2p + 1(1 - p) = 3p + (-1)(1 - p) : R,
\]

which equates the expected payoffs to player 2 from his pure strategies \( L \) and \( R \). This implies

\[
p + 1 = 4p - 1 \iff p = \frac{2}{3}.
\]

Letting \( q \) denote the probability that player 2 chooses \( L \), \( q \) must satisfy

\[
T : 3q + (-1)(1 - q) = q + (0)(1 - q) : B,
\]

\[
q + 1 = 3q \iff q = \frac{1}{2}.
\]
which equates the expected payoffs to player 1 from his pure strategies $T$ and $B$. This implies

$$4q - 1 = q \iff q = \frac{1}{3}.$$ 

These values of $p$ and $q$ define a mixed strategy Nash equilibrium: if player 1 chooses $T$ with probability $2/3$, then every mixed strategy for player 2 produces the same expected payoff for him and hence choosing $L$ with probability $1/3$ is a best response; similarly, if player 2 chooses $L$ with probability $1/3$, then every mixed strategy for player 1 produces the same expected payoff for him and hence choosing $T$ with probability $2/3$ is a best response.

**Example 14 (The Braess Paradox)** There are 4000 motorists who drive each morning from the point labeled Start to the point labeled Finish. There are two possible routes, one through $A$ and one through $B$. The routes $A \rightarrow$ Start and $B \rightarrow$ Start can be thought of as bridges or limited capacity roads. The travel time for each motorist on each of these routes is $m/100$ minutes, where $m$ is the total number of motorists who choose that particular route. Travel time thus increases linearly in the number of motorists who choose a particular route. The routes $A \rightarrow$ Finish and $B \rightarrow$ Finish are high capacity, modern roads that are each sufficiently large to handle all 4000 motorists without increasing the travel time. The travel time on these routes is 45 minutes, regardless of how many motorists travel on the route.

![Diagram of the Braess Paradox](image)

We assume that each motorist wishes to minimize his total travel time from Start to Finish, taking into account the travel pattern determined by the routes chosen by all other motorists. We thus interpret this as a game with 4000 players in which player chooses either the top route through $A$ or the bottom route through $B$. Each motorist therefore has two possible strategies. A motorist will change his route in favor of a shorter trip. We thus look for a distribution of motorists across the two routes that forms a Nash equilibrium, i.e., no motorist can benefit by changing routes, given the choices of every other motorist.
We first characterize a property of a Nash equilibrium. We seek a number $m_A$ of motorists for the top route and a number $m_B$ of motorists for the bottom route, where

$$m_A + m_B = 4000.$$

The travel time on the top route is

$$\frac{m_A}{100} + 45,$$

and the travel time on the bottom route is

$$\frac{m_B}{100} + 45.$$

A motorist who switches from the top route to the bottom route changes his travel time from

$$\frac{m_A}{100} + 45$$

to

$$\frac{m_B + 1}{100} + 45$$

because he adds a motorist on the bottom route. For the driver on the top route to have no incentive to switch, it must be the case that

$$\frac{m_B + 1}{100} + 45 \geq \frac{m_A}{100} + 45 \iff m_B + 1 \geq m_A.$$

Similarly, for a driver on the bottom to have no incentive to switch, it must be the case that

$$\frac{m_A + 1}{100} + 45 \geq \frac{m_B}{100} + 45 \iff m_A + 1 \geq m_B.$$

For a Nash equilibrium, it is necessary that no driver want to switch, i.e., both of these inequalities hold. Therefore,

$$m_B + 1 \geq m_A \geq m_B - 1.$$

The number $m_A$ equals either $m_B - 1$, $m_B$, or $m_B + 1$. Recall that there are 4000 motorists, and so

$$m_A + m_B = 4000.$$

If $m_A = m_B - 1$, then

$$m_A + m_B = 2m_B - 1 = 4000$$

which contradicts $m_B$ being a whole number. Similarly, $m_A = m_B + 1$ is not possible, and so

$$m_A = m_B = 2000$$

is the only possibility for a Nash equilibrium. It is clear that this indeed is a Nash equilibrium distribution of motorists, for a driver who changes routes strictly increases his travel time. Each motorist’s travel time in the only Nash equilibrium is

$$\frac{2000}{100} + 45 = 65$$

minutes.

There’s no paradox yet, but here it comes! Suppose next that in the interest of improving traffic flow a one-way route is added from $A$ to $B$:
For simplicity, we'll assume that travel time on the route $A \rightarrow B$ equals zero. How does the addition of this "shortcut" change the travel time of motorists in a Nash equilibrium? We claim that the route $\text{Start} \rightarrow A \rightarrow B \rightarrow \text{Finish}$ is the unique dominant strategy of every motorist in this new game of choosing one's route. To verify this, we select a motorist and consider his choice of a route given that the choices of the other 3999 motorists determine a value of $x_A$ along the route $\text{Start} \rightarrow A$ and a value $x_B$ along the route $B \rightarrow \text{Finish}$. It is not necessarily the case that $x_A + x_B = 3999$, because some of the other motorists may take the shortcut $A \rightarrow B$ and thus count among both the numbers $x_A$ and $x_B$. It is true, however, that $x_A, x_B \leq 3999$ and $x_A + x_B \geq 3999$.

The selected motorist now has 3 possible routes with 3 possible travel times:

<table>
<thead>
<tr>
<th>route</th>
<th>travel time</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{Start} \rightarrow A \rightarrow \text{Finish}$</td>
<td>$\frac{x_A + 1}{100} + 45$</td>
</tr>
<tr>
<td>$\text{Start} \rightarrow A \rightarrow B \rightarrow \text{Finish}$</td>
<td>$\frac{x_A + 1}{100} + \frac{m_A + 1}{100}$</td>
</tr>
<tr>
<td>$\text{Start} \rightarrow B \rightarrow \text{Finish}$</td>
<td>$\frac{x_B + 1}{100} + 45$</td>
</tr>
</tbody>
</table>

The "+1" indicates the congestion that the selected motorist creates by adding himself to a particular route. We have

$$\frac{x_A + 1}{100} + \frac{x_B + 1}{100} \leq \frac{x_A + 1}{100} + \frac{4000}{100} = \frac{x_A + 1}{100} + 40 < \frac{x_A + 1}{100} + 45,$$

and similarly,

$$\frac{x_A + 1}{100} + \frac{x_B + 1}{100} \leq \frac{4000}{100} + \frac{x_B + 1}{100} = 40 + \frac{x_B + 1}{100} < \frac{x_B + 1}{100} + 45.$$

The route $\text{Start} \rightarrow A \rightarrow B \rightarrow \text{Finish}$ is therefore fastest for the selected motorist regardless of the decisions of the other motorists and the values of $x_A$ and $x_B$ that are determined by these decisions. It is therefore a dominant strategy for each motorist.

The unique dominant strategy equilibrium outcome is therefore that all motorists select the route $\text{Start} \rightarrow A \rightarrow B \rightarrow \text{Finish}$. The driving time of each motorist is then

$$\frac{4000}{100} + \frac{4000}{100} = 80$$

minutes, which is strictly more than the 65 minutes required before the shortcut $A \rightarrow B$ was introduced. This is the Braess paradox, namely, adding a shortcut can increase the
average travel time. Conversely, the average travel time may be decreased by closing a road!

More generally, we can apply this to congestion in any kind of network in which users choose their own routes. Adding a link in a network can diminish the performance of the network while deleting a link can improve performance. This depends upon the assumption that network users array themselves as in a Nash equilibrium.

Pareto efficiency, dominance, and Nash equilibrium all utilize the idea of whether or not a change makes someone better off. It is perhaps easiest to keep these ideas distinct by identifying the different perspectives with which of these ideas are applied.

1. Dominance and Nash equilibria concern an individual player looking only at his own payoffs and making changes only to his own strategy (not the strategies of other players) in order to advance his own interests. This is the essence of noncooperative behavior.

2. Pareto efficiency examines the consequences to the payoffs of all players from changing any and all of their strategies. While dominance and Nash equilibrium reflect the perspectives of individual players concerned only about their own payoffs, Pareto efficiency is concerned with the well-being of every player. Pareto efficiency is perhaps best understood as reflecting the perspective of a well-meaning outsider (e.g., an economic consultant, or a benevolent king) who wishes the best for the entire group of players.

**Example 15** What is Pareto efficient in the context of the Braess Paradox? Consider the original game:

There are 4000 motorists who drive each morning from the point labeled Start to the point labeled Finish. There are two possible routes, one through A and one through B. The routes Start → A and B → Finish can be thought of as bridges or limited capacity...
roads. The travel time for each motorist on each of these routes is \( m / 100 \) minutes, where \( m \) is the total number of motorists who choose that particular route. Travel time thus increases linearly in the number of motorists who choose a particular route. The routes \( A \rightarrow \text{Finish} \) and \( \text{Start} \rightarrow B \) are high capacity, modern roads that are each sufficiently large to handle all 4000 motorists without increasing the travel time. The travel time on these routes is 45 minutes, regardless of how many motorists travel on the route.

The alternatives here are the specifications of routes for all 4000 motorists. This determines \( m_A \) and \( m_B \) with \( m_A + m_B = 4000 \).

1. The alternatives \( m_A = 4000, m_B = 0 \) and \( m_A = 0, m_B = 4000 \) are Pareto dominated: moving a person from the route with 4000 motorists to the other route reduces the travel time of every motorist.

2. Consider a specification of routes for the motorists that determine \( m_A, m_B > 0 \). My claim is that any such specification is Pareto efficient. Suppose not, i.e., suppose there exists a reassignment of motorists that determines \( m_A^*, m_B^* \) and with the property that every motorist’s travel time is not larger in the reassignment and some motorist’s travel times is strictly less in the reassignment. Obviously, \( m_A^* \neq m_A \) and \( m_B^* \neq m_B \). Because

\[
m_A^* + m_B^* = 4000 = m_A + m_B, \tag{2}
\]

it must be the case that either (i) \( m_A^* > m_A \) and \( m_B^* < m_B \) or (ii) \( m_A^* < m_A \) and \( m_B^* > m_B \). We first consider case (i) in which \( m_A^* > m_A \) and \( m_B^* < m_B \); the argument in case (ii) then follows simply by interchanging the roles of \( A \) and \( B \) in the argument for case (i). Notice that in case (i) it must be true that \( m_A < 4000 \), i.e., we’re not in the situation addressed in point 1. above. Consider a motorist who in the initial assignment takes route \( A \) (recall that \( m_A \geq 1 \)). Because \( m_A^* > m_A \), it must be the case that in the reassignment he takes route \( B \), for otherwise, his travel time would increase. We therefore have

\[
m_A \leq m_B^* \tag{3}
\]

so that the capacity \( m_B^* \) on route \( B \) after the reassignment is sufficiently large to take all of these motorists from route \( A \) and

\[
m_A \geq m_B^* \tag{4}
\]

so that these reassigned motorists do not increase their travel times. Consequently, \( m_A = m_B^* \) and therefore \( m_B = m_A^* \) because of (2). In the reassignment, everyone on the \( A \) route switches to the \( B \) route and everyone on the \( B \) route switches to the \( A \) route. In this case, however, no one’s travel time is strictly decreased through the reassignment, which contradicts our hypothesis. Therefore, the initial assignment is Pareto efficient.

The analysis of Pareto efficiency in the case in which a road from \( A \) to \( B \) is added is complicated and I have not yet figured it out.

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**Example 16** There are two divisions of a firm (1 and 2) that would benefit from a research project conducted by the headquarters (HQ) of the firm. The value of the project
to division \(i\) is \(v_i\) and the division receives 0 if the project is not provided. Providing the project would cost HQ an amount \(c\). HQ is concerned about the welfare of the entire firm. It therefore would like to provide the project if and only if

\[ v_1 + v_2 \geq c. \]

Each division knows the value that it would receive from the project if it is provided. HQ, however, does not know these values and must ask the divisions to report them. If a division anticipates how its report will influence the outcome, then it may choose to misrepresent the benefit that it would receive from the project. In the discussion below, let \(v_i^*\) denote a reported benefit by division \(i\).

Suppose HQ decides to charge division \(i\) the amount

\[ p_i = c - v_i^*, \]

when \(v_1^* + v_2^* \geq c\). Here, \(v_i^*\) again denotes the report of the other division \(-i\). No money is exchanged when the project is not provided. This may not seem particularly intuitive as a rule for the price, but it has several virtues that will be made apparent through this example.

1. What is division \(i\)'s payoff as a function of its report and the report of the other division?

\[ u_i(v_i^*, v_{-i}^*) = \begin{cases} v_i + v_{-i}^* - c & \text{if } v_1^* + v_2^* \geq c \\ 0 & \text{otherwise} \end{cases} \]

2. Show that reporting honestly (i.e., \(v_i^* = v_i\)) dominates any report that is less than \(v_i\). Show next that honest reporting dominates any report that is greater than \(v_i\). Conclude that the honest report is a dominant strategy for division \(i\).

The division’s report determines whether or not it receives \(v_i + v_{-i}^* - c\) or 0. Ideally, division \(i\) would like to receive \(v_i + v_{-i}^* - c\) if and only if it is nonnegative. Honest reporting insures this outcome and therefore is a dominant strategy.

Following the question, compare the report of \(v_i\) to \(v_i^* < v_i\). The following table presents the payoff to division \(i\) given its report and the report of division \(-i\):

<table>
<thead>
<tr>
<th>report (v_{-i}^*)</th>
<th>(c - v_i^*)</th>
<th>(c - v_i^* &lt; v_{-i}^* \leq v_i^*)</th>
<th>(c - v_i^* &lt; v_{-i}^* &lt; c - v_i^*)</th>
<th>(c - v_i^* \leq v_{-i}^*)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(v_i)</td>
<td>0</td>
<td>(v_i + v_{-i}^* - c &gt; 0)</td>
<td>(v_i + v_{-i}^* - c)</td>
<td>(v_i + v_{-i}^* - c)</td>
</tr>
<tr>
<td>(v_i^* &lt; v_i)</td>
<td>0</td>
<td>0</td>
<td>(v_i + v_{-i}^* - c)</td>
<td>(v_i^* + v_{-i}^* - c)</td>
</tr>
</tbody>
</table>

Notice that the payoff from honest reporting is either the same or strictly more than from reporting \(v_i^* < v_i\). Honest reporting thus weakly dominates under-reporting.

Now compare the report of \(v_i\) to \(v_i^* > v_i\). The following table presents the payoff to division \(i\) given its report and the report of division \(-i\):

<table>
<thead>
<tr>
<th>report (v_{-i}^*)</th>
<th>(c - v_i^*)</th>
<th>(c - v_i^* &lt; v_{-i}^* \leq c - v_i)</th>
<th>(c - v_i^* &lt; v_{-i}^* &lt; c - v_i)</th>
<th>(c - v_i^* \leq v_{-i}^*)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(v_i)</td>
<td>0</td>
<td>(v_i + v_{-i}^* - c &gt; 0)</td>
<td>(v_i + v_{-i}^* - c)</td>
<td>(v_i^* + v_{-i}^* - c)</td>
</tr>
<tr>
<td>(v_i^* &gt; v_i)</td>
<td>0</td>
<td>(v_i + v_{-i}^* - c &lt; 0)</td>
<td>(v_i^* + v_{-i}^* - c)</td>
<td>(v_i^* + v_{-i}^* - c)</td>
</tr>
</tbody>
</table>

Notice that the payoff from honest reporting is either the same or strictly more than from reporting \(v_i^* > v_i\). Honest reporting thus weakly dominates over-reporting.

3. Does division \(i\) have any other dominant strategies? Explain (your answer to 2. may be helpful).

No – this is shown by the above tables.
4. Assuming that each division uses its unique dominant strategy, show that the HQ provides the project exactly when it should be provided.

With honest reporting, the project is provided if and only if \( v_1 + v_2 \geq c \), which is exactly when it should be provided.

5. How much of a deficit does HQ incur in following this procedure?

HQ collects \( 2c - (v_1 + v_2) \) from the divisions and funds the project at a cost of \( c \). Its deficit is therefore equal to the negative of the benefit to the firm as a whole from the project, \( -(v_1 + v_2 - c) \).