September 22

**Exercises: p. 122-124, problems 6, 7, 8 and 9  
 p. 135, problems 4 and 5**

The **market opportunity line** depicts in the $x$, $y$ plane the different combinations of outcomes $x$ and $y$ that are available to the individual at the prevailing market prices, depending on how much of an asset he chooses to purchase. A market opportunity line is a **fair odds line** if it can be written in the form

$$\pi x + (1 - \pi)y = \theta$$

for some constant $\theta$, where $\pi$ denotes the actual probability of occurrence of $x$ and $1 - \pi$ is the probability of occurrence of $y$.

Campbell is defining this term as a prelude to graphically depicting optimal investments and insurance.

Let’s go back to Problem 2 above.

An individual has a utility-of-wealth function $U(w) = \ln(w + 1)$ and a current wealth of $20$. How much of this wealth will the person use to purchase an asset that yields zero with probability $1/2$ and with probability $1/2$ returns $4$ for every dollar invested?

Again, we’ll use $C$ for the amount that the individual chooses to invest. If the asset yields zero, the individual receives

$$x = 20 - C,$$

and if the asset succeeds, he receives

$$y = 20 - C + 4C = 20 + 3C.$$

Solve each equation for $C$:

$$20 - x = C = \frac{y - 20}{3} \Rightarrow$$

$$60 - 3x = y - 20$$

$$80 = 3x + y.$$

We can check that this is correct by considering $C = 0$ (which produces $x = y = 20$) and $C = 20$ (which produces $x = 0$ and $y = 80$). The market opportunity line thus depicts the range of risky positions that the individual can take on given the asset available for purchase. This is not a fair odds line because it cannot be rewritten in the form

$$\frac{x}{2} + \frac{y}{2} = \theta$$

for some constant $\theta$.

**Example 6.6, p. 120.** An individual with current wealth of $100 will have 70% of it stolen with a probability of 0.3. He can purchase insurance for 40 cents per dollar of coverage. Determine the market opportunity line.

Let $C \leq 70$ be the amount of coverage that he buys. If he is not robbed, his wealth is

$$y = 100 - 0.4C.$$ 

If he is robbed, then his wealth is

$$x = 30 - (0.4)C + C = 30 + (0.6)C.$$ 

Solving for $C$ in both equations produces

$$\frac{100 - y}{0.4} = C = \frac{x - 30}{0.6}.$$ 

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Cross-multiplying produces
\[
0.6(100 - y) = 0.4(x - 30) \\
60 + 12 = (0.4) x + (0.6) y \\
72 = (0.4) x + (0.6) y
\]

Let’s check our answer by considering a few points on the line. With \( C = 0 \), we have \( x = 30 \) and \( y = 100 \). This satisfies the equation. With \( C = 70 \) we have \( x = y = 100 - (0.4)70 = 72 \), which also satisfies the equation. Notice that this is not a fair odds line because \( x \) doesn’t occur with probability 0.70.

Notice that the probability of the low outcome occurring plays no role in the definition of the market opportunity line. The market opportunity line purely reflects the opportunities for purchases available in the market (e.g., from an insurance company or a stock broker). Of course, the market prices may reflect the probability of loss, but the market opportunity line is purely defined by the market prices themselves.

### 0.0.11 Section 7 Insurance

There are two aspects of the market for insurance that we will address later in the course:

1. **moral hazard**: Individuals may take preventative action to diminish the likelihood of losses, and having insurance that covers losses may make one less diligent in taking such actions.

2. **adverse selection**: Different individuals may face different probabilities of losses, and
posting prices for insurance attracts those who are most likely to benefit from coverage (i.e., those with the greatest likelihood of losses). This is an adverse selection for an insurance company from the entire pool of individuals who may benefit from coverage.

We’re going to ignore both of these issues in this chapter for simplicity and to establish the following benchmark result.

**Theorem 2 (Complete Insurance Theorem)** A risk averse individual faced with fair odds will maximize his expected utility at the point on the fair odds line at which $x = y$.

A risk aversion person, faced with the option to buy insurance at prices that reflect the true probabilities of the outcome, will choose to insure himself against losses completely by opting for a point $x = y$ (i.e., the individual bears no risk because his wealth is the same regardless of the outcome).

**Proof.** Let $0 < \pi < 1$ denote the probability of a loss (i.e., the bad or low outcome) and $1 - \pi$ the probability of no loss (the good or high outcome). The assumption is that the market opportunity line is a fair odds line, i.e., it can be rewritten in the form

$$\pi x + (1 - \pi) y = \theta,$$

where $\theta$ is a constant. The value of $\theta$ is determined by the individual’s wealth, the market prices, etc. We don’t need to do this derivation here; we’ll just take it as a constant.

The individual’s problem is to maximize

$$EU(x, y) = \pi U(x) + (1 - \pi) U(y)$$

subject to the constraint

$$\pi x + (1 - \pi) y = \theta$$

and $x, y \geq 0$. We can use the fair odds line to substitute for $y$ in terms of $x$ and reduce this to a problem in the single variable $x$:

$$y = \frac{\theta - \pi x}{1 - \pi},$$

$$V(x) = \pi U(x) + (1 - \pi) U\left(\frac{\theta - \pi x}{1 - \pi}\right)$$

I’ve added the notation $V(x)$ to simplify the discussion below. We have the first order condition

$$0 = V'(x) = \pi U'(x) + (1 - \pi) U'\left(\frac{\theta - \pi x}{1 - \pi}\right) \cdot \left(\frac{\theta - \pi x}{1 - \pi}\right)$$

$$= U'(x) - U'\left(\frac{\theta - \pi x}{1 - \pi}\right)$$

$$= U'(x) - U'(y)$$

Risk aversion implies $U'' < 0$, i.e., $U'$ is strictly decreasing. The last equation therefore requires that $x = y$ in order to satisfy the first order condition. Finally,

$$V''(x) = U''(x) - U''(y) \cdot \frac{dy}{dx} < 0,$$
and so the second derivative test therefore implies that this point is indeed the maximizer of expected utility.

It is also possible to prove this result by forming the Lagrangian

$$EU(x, y) = \pi U(x) + (1 - \pi) U(y) - \lambda [\pi x + (1 - \pi) y - \theta].$$

Campbell uses the technique of substituting for $y$ with $x$ to keep the math as elementary as possible.

**Example 7.2, p. 127. Insurance without fair odds.** An individual’s utility function is $U(w) = \sqrt{w}$ and she faces the following situation:

<table>
<thead>
<tr>
<th>state</th>
<th>probability</th>
<th>wealth</th>
</tr>
</thead>
<tbody>
<tr>
<td>no accident</td>
<td>0.7</td>
<td>100</td>
</tr>
<tr>
<td>accident</td>
<td>0.3</td>
<td>30</td>
</tr>
</tbody>
</table>

She can buy a dollar’s worth of coverage for $0.40. How much insurance does she buy?

We figured out the market opportunity line above:

$$(0.4)x + (0.6)y = 72.$$ The number 0.4, 0.6 sum to 1 but do not reflect the true probabilities of 0.3 and 0.7. This market opportunity line is not a fair odds line. We therefore cannot conclude that the individual purchases coverage of $C = 70$, i.e., $x = y = 72$.

What is the optimal coverage $C$ for this person? Her expected utility is

$$EU(C) = (0.3)\sqrt{x} + (0.7)\sqrt{y}$$

Taking the derivative and setting it equal to zero produces

$$0 = (0.3) \cdot \frac{1}{2} \frac{(0.6)}{\sqrt{30 + (0.6)C}} + (0.7) \cdot \frac{1}{2} \frac{-(0.4)}{\sqrt{100 - (0.4)C}}$$

$$\Rightarrow 0.18 \sqrt{100 - (0.4)C} = 0.28 \sqrt{30 + (0.6)C}$$

$$\Rightarrow \left(\frac{0.18}{0.28}\right)^2 (100 - (0.4)C) = 30 + (0.6)C$$

$$\Rightarrow \left(\frac{9}{14}\right)^2 (100 - (0.4)C) = 30 + (0.6)C$$

$$\Rightarrow \frac{81}{196} (100 - (0.4)C) = 30 + (0.6)C$$

$$8100 - 32.4C = 5880 + 117.6C$$

$$2220 = 150C$$

$$14.8 = C$$
Going back above, we can see that the second derivative of expected utility is

\[
(0.3) \cdot \frac{1}{2} \cdot (0.6) \cdot \left(-\frac{1}{2}\right) (30 + (0.6)C)^{-3/2} \cdot (0.6) +
\]

\[
(0.7) \cdot \frac{1}{2} (-0.4) \cdot \left(-\frac{1}{2}\right) (100 - (0.4)C)^{-3/2} (-0.4) < 0
\]

and so \( C = 14.8 \) is indeed the maximal value. We can then demonstrate that \( x \neq y \) by calculating

\[
x = 30 + (0.6) (14.8) = 38.88
\]

and

\[
y = 100 - (0.4)(14.8) = 94.08.
\]

There is another way to solve this problem using the market opportunity line. The individual wishes to maximize

\[
(0.3)\sqrt{x} + (0.7) \sqrt{y}
\]

subject to the constraint that \( x \) and \( y \) are on the market opportunity line

\[
(0.4) x + (0.6) y = 72.
\]

We substitute for \( y \) in the objective

\[
(0.3)\sqrt{x} + (0.7) \sqrt{72 - (0.4)x}
\]

Set the derivative with respect to \( x \) equal to 0:

\[
0 = (0.3) \cdot \frac{1}{2\sqrt{x}} - (0.7) \frac{(0.4)}{(0.6)} \frac{2\sqrt{72 - (0.4)x}}{(0.6)}.
\]

Cancel the \( 1/2 \) and simplify:

\[
0 = (0.3) \cdot \frac{1}{\sqrt{x}} - (0.7) \frac{2\sqrt{72 - (0.4)x}}{(0.6)} \iff
\]

\[
\frac{1}{\sqrt{72 - (0.4)x}} = \frac{9}{14} \sqrt{x}
\]

Squaring each side produces

\[
\frac{(0.6)\sqrt{72 - (0.4)x}}{5832 - 32.4x} = \frac{81}{196x}
\]

\[
5832 - 32.4x = 117.6x
\]

\[
5832 = 150x
\]

\[
38.88 = x
\]

This is the same value of \( x \) that we obtained above. From the market opportunity line we can calculate

\[
(0.4) 38.88 + (0.6) y = 72 \Rightarrow y = 94.08.
\]

The premium paid is

\[
0.4C = 100 - 94.08 \Rightarrow C = 14.8.
\]
Perfectly Competitive Insurance Markets

The point of section 7.4 is to argue that perfect competition among insurance companies produces prices for coverage that makes every individual’s market opportunity line into a fair odds line. The assumptions are extreme, but again, it serves as a benchmark. There are assumed to be many individuals all of whom share the same utility of wealth function and all of whom face the same probability of a loss. All individuals have the same wealth and the potential loss is the same for each person. All individuals purchase the same policy. The fair odds line in this case is equivalent to a zero expected profit condition on the insurance companies. The idea is that with many competing insurance companies, the price of each policy is driven by competition down to a price at which the firms earn no profit on average. This amounts to the fair odds line.

Don’t worry any more about section 7.4.

We’ll now show that the insurance company earns zero profit on a nontrivial policy if and only if the market opportunity line is a fair odds line. By "nontrivial", I mean that the policy is not vacuous (i.e., the case of zero insurance at no cost). To use some notation coming up in the next section, let \( z \) denote the individual’s initial wealth and \( a \) his wealth in the event of an accident. The accident occurs with probability equal to \( \pi \). Let’s represent a market opportunity line as

\[
p x + (1-p) y = \theta
\]

for some \( p \in [0,1] \). The market opportunity line is a fair odds line if and only if \( p = \pi \). We can solve for \( \theta \) in the market opportunity line by considering the case of zero insurance, which is always an opportunity for the customer:

\[
p x + (1-p) y = pa + (1-p) z.
\]

With this background, let’s now calculate the profit of the insurance company when the individual purchases a policy that places him at the point \((x, y)\) on the market opportunity line. The insurance company collects a premium \( z - y \). Let \( I \) denote the amount that the insurance company pays to the individual in the event of an accident. We have

\[
x - a = I - (z - y).
\]

The left side is the gain to the customer from having insurance in the case of an accident. This gain equals the amount \( I \) that the policy pays out minus the cost of the policy \( z - y \). The insurance company’s expected profit is

\[
-\pi I + (z - y) = \pi [-(x - a) - (z - y)] + (z - y) = -\pi (x - a) + (1 - \pi) (z - y) = [\pi a + (1 - \pi) z] - [\pi x + (1 - \pi) y].
\]

The first line begins with the expected payout \(-\pi I\) of the policy and adds to it the premium \( z - y \). From (6), we can see that this last term equals zero if the market opportunity line is a fair odds line. If the market opportunity line is not a fair odds line,
then the insurance company’s profit is

\[ [\pi a + (1 - \pi) z] - [\pi x + (1 - \pi) y] \]

\[ = [p a + (1 - p) z] - [p x + (1 - p) y] + \left[ (\pi - p) a + (p - \pi) z \right] - \left[ (\pi - p) x + (p - \pi) y \right] \]

\[ = \left[ (\pi - p) a + (p - \pi) z \right] - \left[ (\pi - p) x + (p - \pi) y \right] \]

\[ = (\pi - p) [(y - x) - (z - a)]. \]

Because \( \pi \neq p \), this last expression equals zero only if \( z = y \) and \( a = x \), which is the case in which no policy is sold.