Bayesian-Nash Equilibrium and the Revelation Principle

1. Consider the following two person game of incomplete information:

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>$\frac{1}{2}$</td>
<td>0, $\frac{1}{2}$</td>
</tr>
<tr>
<td>B</td>
<td>$\frac{1}{2}$, 0</td>
<td>$-\frac{1}{2}$, $-\frac{1}{2}$</td>
</tr>
</tbody>
</table>

Player 1’s type $\theta_1$ and player 2’s type $\theta_2$ are independently drawn from the uniform distribution on $[0, 1]$.

(a) Derive a pure strategy Bayesian-Nash equilibrium in this game.

It is clear that player 1 should choose T if $\theta_1 > \frac{1}{2}$. We conjecture a strategy for him as follows:

- $T$ if $\theta_1 > \theta_1^*$
- $B$ if $\theta_1 < \theta_1^*$

Similarly, we conjecture the following strategy for player 2:

- $L$ if $\theta_2 > \theta_2^*$
- $R$ if $\theta_2 < \theta_2^*$

We solve for $\theta_1^*$ and $\theta_2^*$ by noting that each player should be indifferent between his two choices at this value of his type:

1: $\theta_1^* (1 - \theta_2^*) + 0 \cdot \theta_2^* = \frac{1}{2} (1 - \theta_2^*) - \frac{1}{4} \cdot \theta_2^*$

$\Leftrightarrow \theta_1^* - \theta_1^* \theta_2^* = \frac{1}{2} - \frac{3}{4} \cdot \theta_2^*$

and

2: $\theta_2^* (1 - \theta_1^*) - 0 \cdot \theta_2^* = \frac{1}{2} (1 - \theta_1^*) - \frac{1}{4} \cdot \theta_1^*$

$\Leftrightarrow \theta_2^* - \theta_1^* \theta_2^* = \frac{1}{2} - \frac{3}{4} \cdot \theta_1^*$

Subtracting the two equations produces

$\theta_1^* - \theta_2^* = - \frac{3}{4} \cdot \theta_2^* + \frac{3}{4} \cdot \theta_1^*$

$\Leftrightarrow \theta_1^* = \theta_2^*$

Substituting into the equation for player 1 produces

$\theta_1^* - \theta_1^{*2} = \frac{1}{2} - \frac{3}{4} \cdot \theta_1^* \Leftrightarrow 0 = 4 \theta_1^{*2} - 7 \theta_1^* + 2 \Leftrightarrow \theta_1^* = \frac{7 \pm \sqrt{49 - 32}}{8} = \frac{7 \pm \sqrt{17}}{8}$

The only solution that is meaningful is

$\theta_1^* = \frac{7 - \sqrt{17}}{8} = \theta_2^*$

It is easy to verify that this indeed defines a Bayesian-Nash equilibrium.
(b) Illustrate the Revelation Principle by defining a revelation game that implements the outcome for the equilibrium that you have just derived.

Given the reports $\theta_1, \theta_2$, the outcome is as follows:

$\theta_1, \theta_2 \leq \frac{7 - \sqrt{17}}{8} : B, R \left( -\frac{1}{4}, \frac{1}{4} \right)$$

$\theta_1 \leq \frac{7 - \sqrt{17}}{8}, \theta_2 > \frac{7 - \sqrt{17}}{8} : B, L \left( \frac{1}{4}, 0 \right)$$

$\theta_1 > \frac{7 - \sqrt{17}}{8}, \theta_2 \leq \frac{7 - \sqrt{17}}{8} : T, R \left( 0, \frac{1}{2} \right)$$

$\theta_1 > \frac{7 - \sqrt{17}}{8}, \theta_2 > \frac{7 - \sqrt{17}}{8} : T, L \left( \theta_1, \theta_2 \right)$

2. Consider the following two-player game of incomplete information:

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</tr>
<tr>
<td>B</td>
<td>$\frac{1}{2}, 0$</td>
<td>$0, -1$</td>
</tr>
</tbody>
</table>

It is common knowledge among the two players that player 1’s type $\theta_1$ and player 2’s type $\theta_2$ are independently drawn from the uniform distribution on $[0, 1]$.

(a) Derive a pure strategy Bayesian-Nash equilibrium in this game.

It is clear that player 1 should choose T if $\theta_1 > \frac{1}{2}$. We conjecture a strategy for him as follows:

$T$ if $\theta_1 > \theta_1^*$

$B$ if $\theta_1 < \theta_1^*$

Similarly, We conjecture the following strategy for player 2:

$L$ if $\theta_2 > \theta_2^*$

$R$ if $\theta_2 < \theta_2^*$

We solve for $\theta_1^*$ and $\theta_2^*$ by noting that each player should be indifferent between his two choices at this value of his type:

1 : $\theta_1^* \left( 1 - \theta_2^* \right) + 1 \cdot \theta_2^* = \frac{1 - \theta_2^*}{2}$

$\Leftrightarrow \theta_1^* - \theta_1^* \theta_2^* = \frac{1}{2} - \frac{3 \theta_2^*}{2}$

and

2 : $\theta_2^* \left( 1 - \theta_1^* \right) + 0 \cdot \theta_1^* = \frac{1}{2} \left( 1 - \theta_1^* \right) - 1 \cdot \theta_1^*$

$\Leftrightarrow \theta_2^* - \theta_1^* \theta_2^* = \frac{1}{2} - \frac{3 \theta_1^*}{2}$

Subtracting the two equations produces

$\theta_1^* - \theta_2^* = -\frac{3 \theta_2^*}{2} + \frac{3 \theta_1^*}{2} \Leftrightarrow$

$0 = -\frac{\theta_2^*}{2} + \frac{\theta_1^*}{2} \Leftrightarrow$

$\theta_1^* = \theta_2^*$

Substituting into the equation for player 1 produces

$\theta_1^* - \theta_1^2 = \frac{1}{2} - \frac{3 \theta_1^*}{2} \Leftrightarrow$

$0 = 2 \theta_1^2 - 5 \theta_1^* + 1$

$\theta_1^* = \frac{5 \pm \sqrt{25 - 8}}{4} = \frac{5 \pm \sqrt{17}}{4}$
The only solution that is meaningful is

$$\frac{5 - \sqrt{17}}{4} \approx 0.104$$

(b) Illustrate the Revelation Principle by defining a revelation game that implements the outcome for the equilibrium that you have just derived.

Given the reports $\theta_1, \theta_2$, the outcome is as follows:

- $\theta_1, \theta_2 \leq \frac{5 - \sqrt{17}}{4}$: \text{B,R} $(0, -1)$
- $\theta_1 \leq \frac{5 - \sqrt{17}}{4}, \theta_2 > \frac{5 - \sqrt{17}}{4}$: \text{B,L} $(\frac{1}{2}, 0)$
- $\theta_1 > \frac{5 - \sqrt{17}}{4}, \theta_2 \leq \frac{5 - \sqrt{17}}{4}$: \text{T,R} $(1, \frac{1}{2})$
- $\theta_1 > \frac{5 - \sqrt{17}}{4}, \theta_2 > \frac{5 - \sqrt{17}}{4}$: \text{T,L} $(\theta_1, \theta_2)$