Problem 3. We have \( g(x) = 3x \).

A. We apply the Samuelson Criterion:

\[
\frac{1}{\sqrt{x}} + \frac{2}{\sqrt{x}} + \frac{9}{\sqrt{x}} = 3
\]

\[
\frac{12}{\sqrt{x}} = 3
\]

\[
x = 16
\]

The cost of this amount of the public good is \( 3 \cdot 16 = 48 \), which leaves \( 100 - 48 = 52 \) units of the private good. The set of Pareto efficient allocations therefore are all \( (x, y_1, y_2, y_3) \) with \( x = 16 \), each \( y_i \geq 0 \), and \( y_1 + y_2 + y_3 = 52 \).

B. To move from \( x = 9 \) to \( x = 16 \) requires \( 3 \cdot (16 - 9) = 21 \) units of the private good. Shared equally, this takes 7 units of the private good from each agent. At the initial allocation, we have

\[
U_1 = 2\sqrt{9} + 18 = 24,
U_2 = 4\sqrt{9} + 15 = 27,
U_3 = 18\sqrt{9} + 40 = 94.
\]

Implementing the efficient level of the public good in this way produces the utilities

\[
U_1 = 2\sqrt{16} + 11 = 19,
U_2 = 4\sqrt{16} + 8 = 24,
U_3 = 18\sqrt{16} + 33 = 97.
\]

We can see that agents 1 and 2 are made worse off in this process, while agent 3 is made better off.

D. We need

\[
U_1 = 2\sqrt{16} + y_1 > 24,
U_2 = 4\sqrt{16} + y_2 > 27,
U_3 = 18\sqrt{16} + y_3 > 94.
\]

For feasibility,

\[
100 - (y_1 + y_2 + y_3) = 3 \cdot 16,
\]

i.e., the amount of the private good \((100 - y_1 + y_2 + y_3)\) that is transferred to the public good is sufficient to produce the amount (16) of the public good. Therefore, we need \( y_1 + y_2 + y_3 = 52 \). We have

\[
U_1 = 8 + y_1 > 24 \iff y_1 > 16,
U_2 = 16 + y_2 > 27 \iff y_2 > 11,
U_3 = 64 + y_3 > 94 \iff y_3 > 30.
\]

In contradiction to what Campbell asks in the question, this cannot be done with \( y_1 + y_2 + y_3 = 52 \!\).

E. We start with

\[
U_1 = 2\sqrt{9} + 18 = 24,
U_2 = 4\sqrt{9} + 15 = 27,
U_3 = 18\sqrt{9} + 40 = 94.
\]
Let’s increase \( x \) from 9 to 10 and make person 3 pay for it. Persons 1 and 3 are clearly made better off. For person 3, we have

\[
U_3 = 18\sqrt{10} + 37 = 93.92.
\]

Well, that didn’t work. Let’s try making person 3 pay \( 2.9 \) of the 3 units of the private good required, with persons 1 and 2 each providing 0.5:

\[
\begin{align*}
U_1 &= 2\sqrt{10} + 17.95 = 24.27, \\
U_2 &= 4\sqrt{10} + 14.95 = 27.6, \\
U_3 &= 18\sqrt{0} + 37.10 = 94.02.
\end{align*}
\]

That works!

**Problem 4.** We have \( g(x) = x \). Let \( \theta \) denote the total amount of the private good.

A. The level \( x \) of the public good is chosen to maximize

\[
U_3(x, \theta - x) = 3\sqrt{x} + \theta - x.
\]

We have the first order condition

\[
0 = \frac{3}{2\sqrt{x}} - 1 \iff \\
2\sqrt{x} = 3 \\
\sqrt{x} = \frac{3}{2} \\
x = \frac{9}{4}.
\]

The second derivative is negative, and so this is a maximum.

B. Yes, it is efficient, because moving to any other allocation would hurt person 3.

C. The Samuelson Condition in this case is

\[
\frac{1}{2\sqrt{x}} + \frac{2}{2\sqrt{x}} + \frac{3}{2\sqrt{x}} = g'(x) = 1 \iff \\
\frac{3}{\sqrt{x}} = 1.
\]

At \( x = 9/4 \), we have

\[
\frac{3}{\frac{9}{4}} = 2 \neq 1,
\]

and so the Samuelson Condition is not satisfied at this allocation.

D. The Samuelson Condition must be satisfied by an *interior* efficient allocation, i.e., an \((x, y_1, y_2, y_3)\) such that \( x, y_1, y_2, y_3 > 0 \). The above allocation is on the boundary of the set of feasible allocations because \( y_1 = y_2 = 0 \).

**Problem 12.** The amount \( x = 12 \) is not feasible in this problem. As in Problem 4 D., the point is that the Samuelson Criterion characterizes the optimal level of a public good *when it is in the interior of the feasible set*. I would guess in this problem that the optimal level of the public good among feasible allocations is the amount closest to \( x = 12 \) but still feasible:

\[
\begin{align*}
\frac{x^2}{2} &= g(x) = 24 \text{ (the total amount of the private good)} \\
x^2 &= 48 \\
x &= 4\sqrt{3}
\end{align*}
\]

**Problem 13.** Let \( R(x) \) denote the report of person 1. If the other people report benefit functions that are constants, then the amount of public good \( x^* \) solves

\[
R'(x^*) = g'(x^*),
\]
because the marginal benefit of each person 2 through \( n \) equals 0. Person 1’s utility is then

\[
U_1 = B(x^*) + \omega_i - \frac{R'_i(x^*)}{R'(x^*)} g(x^*) \\
= B(x^*) + \omega_i - g(x^*).
\]

With his report, person 1 can choose \( x^* \) to maximize \( U_1 \). Reporting honestly insures that the \( x^* \) that maximizes this last expression is chosen.

**Problem 14.** We have

\[
U_i = \alpha_i x + y_i \text{ for each } i, \\
g(x) = \frac{x^2}{2}.
\]

Let’s first verify the statement in the problem. Assume that \( \sum_{i=2}^{n} \alpha_i = 2 \). If \( 2 > \alpha_i > 0 \) and the reported \( \beta_i \)'s of the others sum to 2, then person 1’s utility from reporting honestly is

\[
\alpha_1 (2 + \alpha_1) + \omega_1 - \frac{\alpha_1}{2} (2 + \alpha_1) \\
= (2 + \alpha_1) \cdot \frac{\alpha_1}{2} + \omega_1 \\
= \alpha_1 + \frac{\alpha_1^2}{2} + \omega_1,
\]

and his utility from reporting 0 is

\[
\alpha_1 (2) + \omega_1 - \frac{0}{2} (2) \\
= 2\alpha_1 + \omega_1.
\]

Because \( 2 > \alpha_i > 0 \), \( \alpha_i > \frac{\alpha_1^2}{2} \), and so person 1 is better off by reporting 0 than by reporting honestly.

As to his optimal report, he chooses \( \beta_1 \) to maximize

\[
\alpha_1 (2 + \beta_1) + \omega_1 - \frac{\beta_1}{2} (2 + \beta_1).
\]

The first order condition with respect to \( \beta_1 \) is

\[
0 = \alpha_1 - \frac{2 + \beta_1}{2} - \frac{\beta_1}{2} \\
= \alpha_1 - \beta_1 - 1,
\]

which has \( \alpha_1 - 1 = \beta_1 \). If \( \alpha_1 > 1 \), then this is his optimum report; if \( \alpha \leq 1 \), then \( \beta_1 = 0 \) is his optimum report.

**Problem 15.** Let’s first look at person 1. Assuming that everyone else reports 0, his utility with the report \( \beta_1 \) is

\[
U_1 = 3(\beta_1) + \omega_1 - \frac{\beta_1}{2} (\beta_1).
\]

The derivative with respect to \( \beta_1 \) is

\[
3 - \beta_1,
\]

and so \( \beta_1 = 3 \) is person 1’s best response to the reports by others.

Turning to any person \( i > 1 \), his utility with the report \( \beta_i \) equals

\[
(3 + \beta_i) + \omega_1 - \frac{\beta_1}{2} (3 + \beta_i).
\]

Its derivative with respect to \( \beta_i \) is

\[
1 - \frac{3 + \beta_i}{2} \cdot \frac{\beta_1}{2} = -\frac{1}{2} \cdot \beta_i.
\]
This is negative for all $\beta_i \geq 0$ and so $\beta_i = 0$ maximizes his utility and determines his best response.

**Problem 19.** We have $g(x) = 3x^2/2$. The profit of the firm in the equilibrium is

$$12(4) - \frac{3}{2} \cdot 4^2 = 48 - 24 = 24.$$ 

We first determine the values of $y_1$ and $y_2$. Person 1 has the decision problem

$$\max U_1 = 48\sqrt{x_1 + x_2} + y_1 \text{ s.t. } 12x_1 + y_1 = 212.$$ 

To purchase 4 units of $X$, he spends $4(12) = 48$ dollars. The budget constraint in this case is

$$48 + y_1 = 212 \iff y_1 = 164.$$ 

Producing 4 units of $X$ requires

$$\frac{3}{2} \cdot 4^2 = 24$$

units of $Y$, which are provided by person 1. We have $y_2 = \omega_2 + 12 = 112$.

We need to show three things to verify equilibrium:

- **The firm maximizes its profit at $x = 4$.** Its profit function is
  
  $$12x - \frac{3}{2}x^2,$$

  which implies the first order condition
  
  $$12 - 3x = 0.$$ 

  This is satisfied at $x = 4$, and the second derivative test verifies that it maximizes profit.

- **Person 1 maximizes his utility subject to his budget constraint by purchasing 4 units of the public good.** Substituting using his budget constraint and assuming that person 2 purchases none of the public good, person 1 wishes to maximize

  $$U_1 = 48\sqrt{x} + 212 - 12x.$$ 

  We obtain the first order condition
  
  $$0 = \frac{48}{2\sqrt{x}} - 12,$$

  which has $x = 4$ as its solution. The second derivative test verifies that $x = 4$ maximizes utility.

- **Person 2 maximizes his utility by purchasing none of the public good.** Taking into account that person 1 has purchased 4 units of the public good, person 2 chooses $x_2$ to maximize

  $$\max U_2 = 30\sqrt{4 + x_2} + y_2 \text{ s.t. } 12x_2 + y_2 = 112.$$ 

  Substituting using the budget constraint produces

  $$\max U_2 = 30\sqrt{4 + x_2} + 112 - 12x_2.$$ 

  Person 2’s marginal utility with respect to $x_2$ is then

  $$\frac{30}{2\sqrt{4 + x_2}} - 12 = \frac{15}{\sqrt{4 + x_2}} - 12 < \frac{15}{2} - 12 < 0$$

  for all $x_2 \geq 2$. Utility is therefore maximized at $x_2 = 0$. 


Project $\alpha$ produces a total benefit of 56, project $\gamma$ produces a total benefit of 48, and project $\delta$ produces a total benefit of 53. Project $\alpha$ is therefore selected for the group.

If Soren were not present, $\alpha$ would still be selected. His surtax equals 0.

If Rosie were not present, $\delta$ would be selected. Her surtax is $(15 + 28) - (5 + 33) = 43 - 38 = 5$.

If Edie were not present, G would be selected. Her surtax is $(10 + 25) - (5 + 18) = 35 - 23 = 12$.

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<tr>
<th></th>
<th>Soren</th>
<th>Rosie</th>
<th>Edie</th>
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<tr>
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<td>18</td>
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<tr>
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<td>surtax</td>
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