Final Exam Answers
Economics 490
Fall Semester, 2011
Prof. Steven Williams

Rules for the exam:

• You can consult all course materials, including your notes, past homework assignments, and everything posted on the course webpage. Do not consult other books other than our course textbook and do not consult any other sources, such as online sites.

• Your work should be your own. Do not consult with anyone else.

• If you have questions, then submit them to me by email. "Is this correct?" is not a legitimate question!

• You can turn in your exam by placing it in my mailbox in room 214 of David Kinley Hall. Exams are due by 5 p.m. on Friday, December 16 (the last day of final exams). Turning in your exam earlier will be greatly favored.

• Answers to the exam will be posted on the course website when I have completed the grading.

1. (15 points) There are four alternatives \( W, X, Y, \) and \( Z \) and five voters \((1, 2, 3, 4, 5)\). The preferences of the voters are given by the following table:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
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</thead>
<tbody>
<tr>
<td>Z</td>
<td>X</td>
<td>X</td>
<td>W</td>
<td>W</td>
<td></td>
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<tr>
<td>X</td>
<td>W</td>
<td>W</td>
<td>Y</td>
<td>X</td>
<td></td>
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<tr>
<td>W</td>
<td>Z</td>
<td>Y</td>
<td>X</td>
<td>Z</td>
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<tr>
<td>Y</td>
<td>Y</td>
<td>Z</td>
<td>Z</td>
<td>Y</td>
<td></td>
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</tbody>
</table>

(a) (10 points) Show that the preferences of these five voters are single-peaked. Hint: You can answer the question with a suitable graph of the utility functions of the voters.
(b) (5 points) Determine the unique majority winner of the four alternatives whose existence is guaranteed by Sen’s Theorem.

X beats every other alternative in a one-on-one majority vote.

2. (15 points) Consider a marriage market with three men and four women. The strict preferences of the men and women are as follows:

<table>
<thead>
<tr>
<th>m1</th>
<th>m2</th>
<th>m3</th>
</tr>
</thead>
<tbody>
<tr>
<td>w1</td>
<td>w2</td>
<td>w1</td>
</tr>
<tr>
<td>w2</td>
<td>w1</td>
<td>w2</td>
</tr>
<tr>
<td>w3</td>
<td>w4</td>
<td>w3</td>
</tr>
<tr>
<td>w4</td>
<td>w3</td>
<td></td>
</tr>
</tbody>
</table>

or alternatively as

<table>
<thead>
<tr>
<th>w1</th>
<th>w2</th>
<th>w3</th>
<th>w4</th>
</tr>
</thead>
<tbody>
<tr>
<td>m2</td>
<td>m2</td>
<td>m3</td>
<td>m1</td>
</tr>
<tr>
<td>m3</td>
<td>m3</td>
<td>m2</td>
<td>m3</td>
</tr>
<tr>
<td>m1</td>
<td>m2</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

or alternatively as

\[
\begin{align*}
P(m_1) : w_1, w_2, w_3, w_4 & \quad P(w_1) : m_2, m_3, m_1 \\
P(m_2) : w_2, w_1, w_4, w_3 & \quad P(w_2) : m_2, m_3 \\
P(m_3) : w_1, w_2, w_3 & \quad P(w_3) : m_3, m_2 \\
P(w_4) : m_1, m_3, m_2 & \quad P(w_4) : m_1, m_3, m_2
\end{align*}
\]

(a) (5 points) Apply the deferred acceptance (or Gale-Shapley) algorithm to determine the M-optimal stable matching:

Men make the proposals:

stage 1:
w_1 \ w_2 \ w_3 \ w_4 \quad P(m_1) : w_2, w_3, w_4 \\
m_3 \ m_2 \quad P(m_2) : w_1, w_4, w_3 \\
P(m_3) : w_2, w_3 \\

stage 2:

w_1 \ w_2 \ w_3 \ w_4 \quad P(m_1) : w_3, w_4 \\
m_3 \ m_2 \quad P(m_2) : w_1, w_4, w_3 \\
P(m_3) : w_2, w_3 \\

stage 3:

w_1 \ w_2 \ w_3 \ w_4 \quad P(m_1) : w_4 \\
m_3 \ m_2 \quad P(m_2) : w_1, w_4, w_3 \\
P(m_3) : w_2, w_3 \\

stage 4:

w_1 \ w_2 \ w_3 \ w_4 \quad P(m_1) : \\
m_3 \ m_2 \ m_1 \quad P(m_2) : w_1, w_4, w_3 \\
P(m_3) : w_2, w_3 \\

(b) (5 points) Apply the deferred acceptance (or Gale-Shapley) algorithm to determine the W-optimal stable matching:

Women make the proposals:

P(m_1) : w_1, w_2, w_3, w_4 \quad P(w_1) : m_2, m_3, m_1 \\
P(m_2) : w_2, w_1, w_4, w_3 \quad P(w_2) : m_2, m_3 \\
P(m_3) : w_1, w_2, w_3 \quad P(w_3) : m_3, m_2 \\
P(w_4) : m_1, m_3, m_2 \\

stage 1:

P(w_1) : m_3, m_1 \\
w_4 \ w_2 \ w_3 \quad P(w_2) : m_3 \\
m_1 \ m_2 \ m_3 \quad P(w_3) : m_2 \\
P(w_4) : m_3, m_2 \\

stage 2:

P(w_1) : m_1 \\
w_4 \ w_2 \ w_1 \quad P(w_2) : m_3 \\
m_1 \ m_2 \ m_3 \quad P(w_3) : m_2 \\
P(w_4) : m_3, m_2 \\

stage 3:

P(w_1) : m_1 \\
w_4 \ w_2 \ w_1 \quad P(w_2) : m_3 \\
m_1 \ m_2 \ m_3 \quad P(w_3) : \\
P(w_4) : m_3, m_2
(c) (5 points) Explain using your answer to a)-b) how you know that there is at least one woman who never gets married in any possible stable matching.

$w_3$ is not matched in the W-optimal matching, which means that being single is her favorite achievable match. If she were married to man $m'$ in some stable matching, then she would strictly prefer being single to man $m'$. This contradicts stability.

Alternatively, some of you noticed that the M-optimal and W-optimal stable matchings are the same. These are the best and the worst stable matchings for each side of the market. Consequently, this is the only stable matching in this problem, and woman $w_3$ is unmarried.

3. (10 points) (Problem 3, p. 495 of Campbell). There are six students, $A$, $B$, $C$, $D$, $E$ and $F$, and three colleges, $X$, $Y$, and $Z$, each with room for two students. The test scores and the student preferences are given in the tables below. Work out the matching determined by the CODA if college $X$ uses only the quantitative score and colleges $Y$ and $Z$ each use only the verbal score.

<table>
<thead>
<tr>
<th>student</th>
<th>quant. score</th>
<th>verbal score</th>
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<tbody>
<tr>
<td>$A$</td>
<td>90</td>
<td>80</td>
</tr>
<tr>
<td>$B$</td>
<td>85</td>
<td>90</td>
</tr>
<tr>
<td>$C$</td>
<td>80</td>
<td>70</td>
</tr>
<tr>
<td>$D$</td>
<td>75</td>
<td>60</td>
</tr>
<tr>
<td>$E$</td>
<td>70</td>
<td>75</td>
</tr>
<tr>
<td>$F$</td>
<td>65</td>
<td>85</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
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</thead>
<tbody>
<tr>
<td>X</td>
<td>X</td>
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<td>X</td>
<td>X</td>
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</tbody>
</table>

stage 1: $X$ offers admission to $A$ and $B$. $Y$ and $Z$ offer admission to $B$ and $F$. We end up with

$X : A, B$

$Y :$

$Z : F$

stage 2: $Z$ offers admission to $A$, $Y$ offers admission to $A$ and $E$. We end up with

$X : A, B$

$Y : E$

$Z : F$

stage 3: $Z$ offers admission to $E$, $Y$ offers admission to $C$. We end up with

$X : A, B$

$Y : E, C$

$Z : F$

stage 4: $Z$ offers admission to $C$. We end up with
stage 5: Y offers admission to D.

stage 6: 

stage 7:

stage 8:

stage 9:

stage 10:

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