Answers – First Exam  
Information Economics (490), Fall 2012

1. (7) Bob’s utility of wealth function is $U(w) = \sqrt{w}$. He has $1000 and he faces a possible loss of $600 with probability $1/4$. An insurance company offers him a policy that provides $1 of coverage for $0.30. Let $C$ denote the dollar value of the coverage that Bob purchases.

(a) (3) What is the market opportunity line? Is it a fair odds line?

The low state is

$$x = 400 - .3C + C = 400 + .7C,$$

and the high state is

$$y = 1000 - .3C$$

Solving both equations for $C$,

$$\frac{x - 400}{.7} = C = \frac{1000 - y}{.3}$$

$$\frac{3x - 120}{.7} = -.7y + 700$$

$$\frac{.3x + .7y}{.7} = 820$$

No, it is not a fair odds line.

(b) (4) What is Bob’s expected utility if he buys $C$ dollars of coverage? How much coverage does he choose to buy?

His expected utility is

$$\mathbb{E}(U) = \frac{\sqrt{400 + .7C}}{4} + \frac{3\sqrt{1000 - .3C}}{4}$$

We maximize this with respect to $C$ by setting the derivative equal to zero:

$$0 = \frac{0.7}{8\sqrt{400 + .7C}} - \frac{0.9}{8\sqrt{1000 - .3C}}$$

$$\frac{9}{\sqrt{1000 - .3C}} = \frac{7}{\sqrt{400 + .7C}}$$

$$\frac{81}{1000 - .3C} = \frac{49}{400 + .7C}$$

$$32, 400 + 56.7C = 49, 000 - 14.7C$$

$$71.4C = 16, 600$$

$$C \approx 232.50$$

The second derivative test implies that this value of $C$ maximizes Bob’s expected utility.
2. (8 points) There are two agents, each of whom owns an item and each of whom wishes to acquire at most one item. The two items are identical. Each agent \( i \) privately knows his value \( v_i \). The utility of each agent \( i \) is

\[
v_i \cdot \delta + x_i
\]

where \( x_i \) is any monetary transfer he receives, \( \delta = 1 \) if he acquires the item of the other agent, \( \delta = -1 \) if he sells his item to the other agent, and \( \delta = 0 \) if he neither acquires an item nor sells his item. Gains from trade exist whenever the two values are not the same. A reallocation of the items is efficient if the agent with the smaller value gives up his item to the agent with the higher value.

Suppose the agents make reports \( v_1^*, v_2^* \) of their values, and the agent with the smaller reported value gives up his item to the other agent. Consider the following monetary transfer to agent 1:

\[
x_1(v_1^*, v_2^*) = \begin{cases} 
-v_2^* & \text{if } v_1^* \geq v_2^* \\
2^* & \text{if } v_1^* < v_2^*
\end{cases}
\]

Carefully show that honestly reporting his value is a dominant strategy for agent 1.

Agent 1’s utility is

\[
\begin{align*}
&v_1 - v_2^* \text{ if } v_1^* > v_2^* \\
v_2^* - v_1 \text{ if } v_1^* < v_2^*
\end{align*}
\]

We compare agent 1’s utility with a report of \( v_1^* > v_1 \), for all possible values of \( v_2^* \):

<table>
<thead>
<tr>
<th>report ( v_2^* )</th>
<th>( v_3^* &lt; v_1 )</th>
<th>( v_1 &lt; v_2^* &lt; v_1^* )</th>
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The table shows that \( v_1 \) weakly dominates \( v_1^* > v_1 \). We next compare the report of \( v_1 \) with a report of \( v_1^* < v_1 \):

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This table shows that the honest report weakly dominates underreporting.

3. (15 points) Consider the following 3 stage bargaining game in which $1 is to be split between players 1 and 2. Notice that player 1 makes the first and second offers while player 2 makes the final offer. The numbers \( x_1, x_2, x_3 \) are real numbers in the unit interval \([0,1]\). The first number \( x_i \) always indicates what player 1 gets while the second number represents what 2 gets. You can think about player 2 in the third stage proposing a share \( 1 - x_3 \) for himself if you want instead of proposing the share \( x_3 \).
to be given to player 1 (as depicted in the diagram). The key point is that at each stage the designated player proposes a division of the dollar. Each player $i$ has discount factor $\delta_i \in (0, 1)$.

(a) (5) Determine the subgame perfect equilibrium of this game.

- stage 3: 1 accepts any offer, 2 proposes $x_3 = 0$
- stage 2: 2 accepts iff $1 - x_2 \geq \delta_2$, 1 offers $x_2 = 1 - \delta_2$
- stage 3: 2 accepts iff $1 - x_1 \geq (\delta_2)^2$, 1 offers $x_2 = 1 - (\delta_2)^2$

(b) (5) Devise a Nash equilibrium in which the dollar is split in stage $t = 2$, with player 1 receiving $2/3$ of the dollar and player 2 receiving $1/3$ of the dollar. Be sure to completely specify each player’s strategy. Note that the "1/3" and "2/3" are portions of the dollar and not discounted payoffs.

- stage 1: $x_1 = 1$, 2 chooses R in all cases
- stage 2: $x_2 = \frac{2}{3}$, 2 chooses A if and only if $1 - x_2 \geq \frac{1}{3}$
- stage 3: $x_3 = 0$, 1 chooses R in all cases

(c) (5) Explain why your answer to b. is not subgame perfect. *Hint:* Your answer needn’t be a lengthy discourse on subgame perfection; focus on criticizing your answer to b.
One reason of several: in stage 2, player 2 would reject positive offers $1 - x_2 < \frac{1}{3}$ so that he instead receives 0 in stage 3. This is not a best response for him, and consequently the strategies do not define Nash equilibria in all subgames.

4. (20) Two identical, quantity-setting firms produce a homogeneous good with zero costs. For $i = 1, 2$, let $x_i$ be the quantity produced by firm $i$ and let $X = x_1 + x_2$. The market inverse demand function is $P(X) = A - X$ where $A$ is a constant.

(a) (2) What is the profit function of each firm?
$$\pi_i(x_1, x_2) = (A - (x_1 + x_2)) x_i$$

(b) (5) Find the Nash equilibrium of this game. What is the profit of each firm in this equilibrium?
$$0 = \frac{\partial \pi_i}{\partial x_i} = (A - (x_1 + x_2)) - x_i$$

We have two first order conditions:
$$0 = (A - (x_1 + x_2)) - x_1$$
$$0 = (A - (x_1 + x_2)) - x_1$$

These imply $x_1 = x_2 = A/3$. The second derivative of profit is
$$\frac{\partial^2 \pi_i}{\partial x_i^2} = -x_i - 1 < 0$$
and so these values of $x_1$ and $x_2$ are a Nash equilibrium.
Firm $i$’s profit is
$$\left(A - \frac{2A}{3}\right)\frac{A}{3} = \frac{A^2}{9}.$$ 

(c) (5) What total output $X$ maximizes the total profit of the two firms? Assuming that the two firms produce equal amounts of this total output, what is the profit of each firm?

We wish to maximize
$$(A - (x_1 + x_2))(x_1 + x_2) = (A - X)X$$

The first order condition is
$$0 = (A - X) - X \Rightarrow X = \frac{A}{2}$$

The second derivative test verifies that this value of $X$ maximizes total profit. The maximized total profit is
$$\frac{A^2}{4}.$$
and so each firm earns a profit of

\[ \frac{A^2}{8}. \]

(d) Suppose that the game is repeated infinitely many times. Let \( \delta \) be the discount factor of each firm. Consider the following pair of trigger strategies: Each firm produces half of the profit-maximizing output from c. in each period provided that no firm has ever deviated. Once a firm deviates, the two firms begin to produce the Nash equilibrium quantities from b. in each period. What conditions on \( A \) and \( \delta \) insure that these trigger strategies form a subgame perfect Nash equilibrium in the repeated game?

We first need to calculate how much a firm can profit by deviating when the other firm produces \( \frac{A}{4} \). We maximize

\[
\pi_i \left( x_i, \frac{A}{4} \right) = \left( A - \left( x_i + \frac{A}{4} \right) \right) x_i
\]

by deriving the first order condition

\[ 0 = \frac{\partial \pi_i}{\partial x_i} \left( x_i, \frac{A}{4} \right) = \left( \frac{3A}{4} - x_i \right) - x_i = \frac{3A}{4} - 2x_i \Rightarrow x_i = \frac{3A}{8}. \]

The second derivative test verifies that this is a maximum. The corresponding profit is

\[
\left( \frac{3A}{4} - \frac{3A}{8} \right) \frac{3A}{8} = \frac{9A^2}{64}
\]

We then have

\[
\pi_i = \frac{9A^2}{64},
\]

\[
\pi^* = \frac{A^2}{8},
\]

\[
\pi^{NE} = \frac{A^2}{9},
\]

and the bound we derived in class reduces to

\[
\delta \geq \frac{d_i - \pi^*}{d_i - \pi^{NE}} = \frac{\frac{9A^2}{64} - \frac{A^2}{8}}{\frac{9A^2}{64} - \frac{A^2}{9}} = \frac{\frac{9}{64} - \frac{1}{8}}{\frac{9}{64} - \frac{1}{9}} = \frac{\frac{72}{64} - \frac{64}{64}}{\frac{61}{64} - \frac{64}{64}} = \frac{8}{\frac{17}{9}} = \frac{9}{17}
\]