1. (20) Consider in this problem a seller who sells an item through the following auction. There are $n$ bidders. Each bidder submits a bid. The bidder who submits the highest bid wins the item and pays the seller his bid (just like in the first price auction). The bidder who submits the lowest bid also pays his bid to the seller. Think of this as a punishment for being the low bidder; the seller might think that this will lead to an increase in the bids as the bidders wish to avoid being the lowest bidder. This is an all-pay auction in the case of $n = 2$, but not in the case of $n > 2$ (only the highest and the lowest bidders pay their bids to the seller). Ties are resolved with fair lotteries such as coin flips; you can ignore the possibility of ties in your answer.

The reservation values of the $n$ bidders are identically and independently distributed according to the uniform distribution on $[0, 1]$. The purpose of this example is to derive an increasing function $b : [0, 1] \rightarrow [0, 1]$ that defines an equilibrium in this auction: if every other bidder besides bidder $i$ uses the function $b$ to select his bid, then $b(v_i)$ maximizes bidder $i$’s expected profit in the auction for each of his possible reservation values $v_i \in [0, 1]$. Select a bidder $i$, let $v_i$ denote his reservation value, $x$ his bid, and assume all other bidders use the function $b$ to select their bids.

(a) (3) What is the probability that bidder $i$ wins the item with the bid $x$? What is the probability that his bid $b$ is the lowest bid?

He wins with probability equal to $b^{-1}(x)^{n-1}$. His bid $x$ is the lowest with probability $(1 - b^{-1}(x))^{n-1}$ (i.e., all of the other $n-1$ bids are above $b$).

(b) (2) What is bidder $i$’s expected profit function $U(v_i, x)$ as a function of his reservation value $v_i$ and his bid $x$?

$$U(v_i, x) = (v_i - x) b^{-1}(x)^{n-1} - x (1 - b^{-1}(x))^{n-1}$$

(c) (5) For equilibrium, we want $U(v_i, x)$ to be maximized at $x = b(v_i)$. Let $\bar{U}(v_i)$ denote this maximized value of expected profit, $\bar{U}(v_i) = U(v_i, b(v_i))$. Apply the Envelope Theorem to calculate $\bar{U}'(v_i)$. Use your answer to derive a formula for $\bar{U}'(v_i)$.

$$\bar{U}'(v_i) = \frac{\partial}{\partial v_i} U(v_i, x)|_{x=b(v_i)}$$

$$= b^{-1}(x)^{n-1}|_{x=b(v_i)}$$

$$= v_i^{n-1}$$
\[
\begin{align*}
\bar{U}(v_i) &= \frac{v_i^n}{n} + c \\
U(v_i) &= \frac{v_i^n}{n}
\end{align*}
\]

As usual, we argue that \( c = 0 \) by considering the expected profit of bidder \( i \) when \( v_i = 0 \).

(d) (5) Using your answers to b. and c., solve for the strategy \( b(v_i) \).
(The answer may not be pretty). You do not need to check that \( b(v_i) \) is indeed an increasing function by showing that its derivative is positive. In the case of \( n = 2 \), this auction is simply the all-pay auction. You may want to check your answer against the equilibrium we calculated in class for the all-pay auction.

\[
U(v_i, x) = (v_i - x) b^{-1} (x) x^{n-1} - x (1 - b^{-1} (x))^{n-1} \Rightarrow \\
\frac{n^n}{n} = U(v_i) = (v_i - b(v_i)) x_i^{n-1} - b(v_i) (1 - v_i)^{n-1} \\
- \frac{n-1}{n} v_i^n = -b(v_i) \left( v_i^{n-1} + (1 - v_i)^{n-1} \right) \\
b(v_i) = \left( \frac{n-1}{n} \right) \frac{v_i^n}{v_i^{n-1} + (1 - v_i)^{n-1}}
\]

In the case of \( n = 2 \), we have

\[
b(v_i) = \left( \frac{1}{2} \right) \frac{v_i^2}{v_i + (1 - v_i)} = \frac{v_i^2}{2},
\]

which is the equilibrium that we calculated for the all-pay auction.

(e) (5) Does the seller benefit by charging the low bidder his bid in addition to the high bidder, in comparison with simply using the first price sealed bid auction? Answer the question using the Revenue Equivalence Theorem.

A bidder with value \( v_i = 0 \) has an expected utility of zero, and because the strategy \( b \) is increasing, the bidder with the highest value wins the auction. The Revenue Equivalence Theorem thus applies and implies that the expected revenue of the seller is the same in this auction as in the first price sealed bid auction in which only the highest bidder pays his bid. So no, the seller does not benefit from using this auction.

2. (10) Consider the following decision problem with three actions \( A_1, A_2, A_3 \) and three states \( s_1, s_2, s_3 \):
Which action does the decision maker choose if he uses the maxmin criterion? Which action does he choose if minimizes his maximum regret? Show your work.

Maxmin: $A_1$ guarantees at least 2, $A_2$ guarantees 0, and $A_3$ guarantees 1. The decision maker therefore chooses $A_1$ if he uses the maxmin criterion.

Minimizing maximum regret: The regret determined by each action and each state is as follows:

<table>
<thead>
<tr>
<th>action/state</th>
<th>$s_1$</th>
<th>$s_2$</th>
<th>$s_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>0</td>
<td>5</td>
<td>9</td>
</tr>
<tr>
<td>$A_2$</td>
<td>3</td>
<td>8</td>
<td>0</td>
</tr>
<tr>
<td>$A_3$</td>
<td>5</td>
<td>0</td>
<td>10</td>
</tr>
</tbody>
</table>

The maximum regret associated with each action is therefore

<table>
<thead>
<tr>
<th>action/state</th>
<th>max. regret</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>9</td>
</tr>
<tr>
<td>$A_2$</td>
<td>8</td>
</tr>
<tr>
<td>$A_3$</td>
<td>10</td>
</tr>
</tbody>
</table>

and the action $A_2$ minimizes the maximum regret.

3. (20) There are many identical consumers, each with the utility of wealth function $U(w) = \ln (w + 1)$. Each has initial wealth $100$ and each faces a loss of half of it with probability $\pi(e) = \frac{1}{2} - \frac{e}{3}$.

The variable $e$ here equals either 0 or 1; it is a preventative action that an individual can privately take at a cost to his wealth of $10e$ to reduce the likelihood of the loss.

(a) (5) Assuming a competitive market for insurance, what is the market opportunity line if each consumer takes the action $e$?

$$\pi(e) \cdot x + (1 - \pi(e)) \cdot y = \pi(e) \cdot 50 + (1 - \pi(e)) \cdot 100$$

$$\left(\frac{1}{2} - \frac{e}{3}\right) x + \left(\frac{1}{2} + \frac{e}{3}\right) y = \left(\frac{1}{2} - \frac{e}{3}\right) 50 + \left(\frac{1}{2} + \frac{e}{3}\right) 100$$

$$\left(\frac{1}{2} - \frac{e}{3}\right) x + \left(\frac{1}{2} + \frac{e}{3}\right) y = 75 + \frac{50e}{3}$$

(b) (5) What is the individual’s certain wealth in the case of the competitive equilibrium (which is the case of $e = 0$)? We know that the competitive equilibrium involves $e = 0$, in which case the market opportunity line is

$$\frac{x}{2} + \frac{y}{2} = 75$$
Each individual is risk averse and so selects complete insurance,

\[ x = y = 75. \]

(c) (5) Use your answer to b. to determine the price \( p \) per dollar of coverage and the amount \( C \) of coverage for each individual in the competitive equilibrium.

\[
\begin{align*}
\quad x &= 50 + C - pC \\
\quad y &= 100 - pC \\
\end{align*}
\]

We have \( x, y = 75 \), and so

\[
\begin{align*}
25 &= (1 - p)C, \\
25 &= pC \\
2pC &= C \\
p &= \frac{1}{2}, C = 50
\end{align*}
\]

(d) (5) Is the competitive equilibrium efficient? Explain.

Each person’s utility in the competitive equilibrium is \( \ln(76) \). If everyone instead choose \( e = 1 \), the market opportunity line is

\[
\left( \frac{1}{6} \right) x + \left( \frac{5}{6} \right) y = 75 + \frac{50}{3},
\]

and each person’s wealth would be

\[
\ln \left( 76 + \frac{50}{3} - 10 \right) > \ln(76).
\]

No, the competitive equilibrium is not efficient.