Final Exam Answers  
Economics 490  
Fall Semester, 2013  
Prof. Steven Williams

Rules for the exam:

- You can consult all course materials, including your notes, past homework assignments, and everything posted on the course webpage. Do not consult other books besides our course textbook and do not consult any other sources, such as online sites.

- Your work should be your own. Do not consult with anyone else.

- Submit questions about the exam to me by email. "Is this correct?" is not a legitimate question!

- Exams are due by 4:30 p.m. on Wednesday, December 18 (the scheduled ending time if we were to have an in-class final exam). Turning in your exam in advance of this deadline will be appreciated by me as it will allow me to complete my grading sooner. If you do not type your exam, then please insure that your writing is legible. If you submit your exam to me electronically, then please insure that the format you send me is readable.

1. (15 points) There are two players. Player 1 privately observes a parameter \( \theta_1 \in [0, 1] \) and player 2 privately observes a parameter \( \theta_2 \in [0, 1] \). The players then play the following game, where the values of \( \theta_1 \) and \( \theta_2 \) determine the payoffs:

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>( \frac{1}{4}, \frac{1}{4} )</td>
<td>( \frac{1}{4}, \frac{1}{4} )</td>
</tr>
<tr>
<td>B</td>
<td>( \frac{1}{2}, 0 )</td>
<td>( 1-\theta_1, 1-\theta_2 )</td>
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A strategy for a player in this game is a function that specifies an action (T or B for player 1, L or R for player 2) for each of his possible parameters in \([0, 1]\).

(a) (5 points) Determine a dominant strategy for each player.

- Player 1:
  \[
  \begin{cases} 
  B & \text{if } \theta_1 \leq \frac{3}{4} \\
  T & \text{if } \theta_1 > \frac{3}{4} 
  \end{cases}
  \]  
- Player 2:
  \[
  \begin{cases} 
  R & \text{if } \theta_2 \leq \frac{1}{2} \\
  L & \text{if } \theta_2 > \frac{1}{2} 
  \end{cases}
  \]

(b) (5 points) Illustrate the revelation principle by constructing a revelation game in which: (i) honest reporting is a dominant strategy for each player; (ii) when each player reports honestly, the result of the game coincides with the outcome in the dominant strategy equilibrium you found in 1 a.  

Hint: A similar problem was worked in class.

For reported \( \theta^*_1, \theta^*_2 \), the outcome of the game is as follows:

- \( \theta^*_1 \leq \frac{3}{4}, \theta^*_2 \leq \frac{1}{2} \): B, R
\( \theta_1^* > \frac{3}{4}, \theta_2^* \leq \frac{1}{2} : \text{T}, \text{R} \)
\( \theta_1^* \leq \frac{3}{4}, \theta_2^* > \frac{1}{2} : \text{B}, \text{L} \)
\( \theta_1^* > \frac{3}{4}, \theta_2^* > \frac{1}{2} : \text{T}, \text{L} \)

(c) (5 points) Explain in several sentences (only a few) the significance of the revelation principle. Please do not ask me to elaborate on what I mean by this question!

The revelation principle implies that the outcomes of all possible dominant strategy equilibria in all possible games can be studied by limiting one’s attention to revelation games in which honesty is a dominant strategy for each player. Revelation games in which honest revelation is a dominant strategy are thus "canonical forms" for all games with all dominant strategy equilibria. This can greatly simplify the study of all possible games with dominant strategies.

2. (10 points) There are four alternatives \( W, X, Y, \) and \( Z \) and three voters \((1, 2, 3)\). The preferences of the voters are given by the following table:

<table>
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<th>1</th>
<th>2</th>
<th>3</th>
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</thead>
<tbody>
<tr>
<td>( Z )</td>
<td>( Y )</td>
<td>( W )</td>
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<tr>
<td>( W )</td>
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<td>( X )</td>
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<td>( Y )</td>
<td>( X )</td>
<td>( X )</td>
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</table>

(a) (5 points) Show that the preferences of these three voters are single-peaked. 

Hint: You can answer the question with a suitable graph of the utility functions of the voters.
(b) (5 points) Determine the unique majority winner of the four alternatives whose existence is guaranteed by Sen’s Theorem. 

Z beats every other alternative in a one-on-one majority vote.

3. (17 points) There is a private good $Y$, a public good $X$ and $n = 3$ agents. For an amount $x$ of the public good and an amount $y_i$ of the private good, agent $i$’s utility is 

$$U_i(x, y_i) = i \cdot \sqrt{x} + y_i.$$ 

Each agent $i$ starts with an initial endowment of $\omega_i = 20$ units of the private good. Let $g(x) = x$ be the amount of the private good that is required to produce $x$ units of the public good.

(a) (5 points) What is the Pareto efficient level $x^*$ of the public good in this problem? 

Applying the Samuelson Criterion, $x^*$ satisfies 

$$\sum_{i=1}^{3} B_i'(x) = g'(x),$$ 

which in this case reduces to 

$$\frac{1}{2\sqrt{x}} + \frac{2}{2\sqrt{x}} + \frac{3}{2\sqrt{x}} = 1 \Leftrightarrow \frac{3}{\sqrt{x}} = 1 \Leftrightarrow x^* = 9.$$
(b) (6 points) Suppose each agent $i$ determines on his own an amount $x_i$ of his private good to contribute toward the production of the public good. This is a voluntary contribution game. Determine a Nash equilibrium levels of contributions for the three agents. Let $x$ denote the total contributions, i.e., $x = x_1 + x_2 + x_3$. The utilities of the three agents are as follows:

\[
U_1(x_1, x_2, x_3) = \sqrt{x} + 20 - x_1,
U_2(x_1, x_2, x_3) = 2\sqrt{x} + 20 - x_2,
U_3(x_1, x_2, x_3) = 3\sqrt{x} + 20 - x_3.
\]

This implies the following first order conditions for Nash equilibrium:

\[
\begin{align*}
0 &= \frac{\partial U_1}{\partial x_1} = \frac{1}{2\sqrt{x}} - 1, \\
0 &= \frac{\partial U_2}{\partial x_2} = \frac{2}{2\sqrt{x}} - 1, \\
0 &= \frac{\partial U_3}{\partial x_3} = \frac{3}{2\sqrt{x}} - 1.
\end{align*}
\]

Clearly, there is no $x$ that satisfies all three of these equations. Each is necessary for a Nash equilibrium that is in the interior of the cube $0 \leq x_1, x_2, x_3 \leq 20$, so we'll look for an equilibrium on the boundary. Consider $x_1 = x_2 = 0$ and $x_3 = 9/4$. This value of $x_3$ is optimal for agent 3 given the contributions $x_1 = x_2 = 0$ of the other two agents. Both agent 1 and agent 2 have negative marginal utilities at these contributions, and so neither would benefit by increasing his contribution above zero (they'd like to choose negative contributions, but that isn't possible). So this is a Nash equilibrium.

(c) (6 points) We consider only 3 possible levels of the public good in part c. of this question, $x'$, $x''$, and $x'''$, each of which is to be funded by sharing its cost equally among the three agents. Define the pivotal mechanism in this problem. You can ignore the complications that may be caused by ties. I am particularly interested in the precise formula for the surtax in this problem.

Each agent is asked to report a triple $(V_i^*(x'), V_i^*(x''), V_i^*(x'''))$ of values for the for the three alternatives. Based on these reports, the provider of the public good selects $x^*$ as whichever of $x', x'', x'''$ that produces the largest value of $V_1(x) + V_2(x) + V_3(x)$. Assume that

\[
V_1^*(x') + V_2^*(x') + V_3^*(x') > V_1^*(x'') + V_2^*(x'') + V_3^*(x''), V_1^*(x''') + V_2^*(x''') + V_3^*(x''')
\]

so that $x'$ is chosen over $x''$ and $x'''$. The two other possible cases are addressed by simply interchanging the roles of $x'$, $x''$, and $x'''$ in the discussion that follows. For $j \neq i, k$, Agent $j$ pays a surcharge equal to

\[
\max \left\{0, \left[ V_i^*(x'') + V_k^*(x''') \right] - \left[ V_i^*(x') + V_k^*(x') \right], \left[ V_i^*(x''') + V_k^*(x'') \right] - \left[ V_i^*(x') + V_k^*(x') \right] \right\}.
\]
4. (8 points) (problem 1, p. 412, Ch. 7 Sec. 3 of Campbell’s text) Each of the questions defines a social choice rule. In each case determine whether truthful revelation is a dominant strategy. If it is not, demonstrate that fact with a simple example. If truthful revelation is a dominant strategy, then prove your claim by means of a simple, informal – but convincing – argument. Please read each statement carefully. Hint: Focus on the possibility that an individual’s report is pivotal in determining the group choice. Your answers to these questions are expected to be concise and to the point.

(a) (2 points) There are two feasible alternatives, $x$ and $y$, and there are ten individuals. The rule selects the majority winner if there is one, and if there is a tie the rule selects the alternative that is preferred by person 1.

This is strategy-proof. The issue for any individual is the possibility that his vote might affect the group choice; in this case, he wants to report honestly, while in other cases, his vote doesn’t matter.

(b) (2 points) There are two feasible alternatives, $x$ and $y$, and there are $n$ individuals. The rule selects $y$ if all persons declare that they prefer $x$ to $y$; otherwise, the rule selects $x$.

This is not strategy proof. Suppose that all persons except person 1 submits the preferences $x > y$, and person 1 truly prefers $x$ to $y$. He should submit $y > x$ to insure that $x$ is selected for the group.

(c) (2 points) There are two feasible alternatives, $x$ and $y$, and there are $n$ individuals. The rule selects $y$ if all persons declare that they prefer $y$ to $x$; otherwise, the rule selects $x$.

This is strategy-proof. The issue for any individual is the possibility that his vote might affect the group choice, which is only in the case in which everyone else reports that he prefers $y$ to $x$. The wants to report honestly in this case, while in other cases, his vote doesn’t matter.

(d) (2 points) There are two possible alternatives, $x$ and $y$, and there are six individuals. The rule selects the alternative that gets the most votes, but individuals 1, 2, and 3 are each allowed to cast three votes for their preferred alternatives, and individuals 4, 5, and 6 may each cast only two votes for their preferred alternatives.

This is strategy proof: in any case in which an individual’s votes matters, he wants to assign them to his most favored alternative. If his votes cannot influence the outcome, then he is not hurt by voting for his most favored alternative.