1. (10 points) A seller can produce an indivisible item that he may sell to a buyer. The cost of producing the item is $c$; the seller’s utility is $p - c$ if he produces the item and sells it to the buyer at the price of $p$ and it is zero if the item is not produced and sold. The value of the item to the buyer is $v$. The buyer’s utility is $v - p$ if he buys the item at the price $p$ and it is zero if he does not buy the item. The buyer privately knows his value $v$ and the seller privately knows his cost $c$.

Consider the following procedure for trading in which an intermediary provides a subsidy for trading. The seller proposes an offer $s$ and the buyer proposes a bid $b$; trade occurs if $b \geq s$, in which case the buyer pays $s$ to the intermediary and the seller receives $b$ from the intermediary. No money is exchanged when $b < s$. Determine a dominant strategy outcome of this procedure and demonstrate for one of the traders that his strategy is dominant for him.

It is a dominant strategy for the buyer to set $b = v$ and for the seller to set $s = c$. We can see this for the buyer. Let $c^*$ denote the report of the seller. The buyer’s utility is therefore

\[
\begin{align*}
    v - c^* & \text{ if } b \geq c^* \Leftrightarrow b - c^* \geq 0 \\
    0 & \text{ if } b < c^* \Leftrightarrow b - c^* < 0
\end{align*}
\]

The buyer’s bid $b$ determines whether he gets $v - c^*$ or zero. It would be best for him if he received $v - c^*$ if and only if it is nonnegative. The bid of $b = v$ insures that he receives this optimal possible payoff.

2. (15 points) Consider the following game with two players.

(a) (5 points) Find a subgame perfect Nash equilibrium.

This is derived through backwards induction.
(b) (10 points) Devise a Nash equilibrium that is not subgame perfect in which player 2 receives the payoff of 3. Explain why this equilibrium is not subgame perfect.

Player’s 2’s "threat" to play L at the left-hand node is not credible: it is not a best choice for him at this node. The strategies therefore do not define a Nash equilibrium in the bottom left subgame, as required for subgame perfection.

3. (25 points) There are two identical firms that produce the same product. Each firm $i$ can choose an amount $q_i$ to produce at a cost equal to $q_i^2$. The market price $p$ for the good is determined by the total production $q_1 + q_2$ by the formula $p = 25 - (q_1 + q_2)$. Each firm $i$’s profit function is therefore

$$\pi_i(q_1, q_2) = [25 - (q_1 + q_2)]q_i - q_i^2.$$ 

(a) (10 points) Find a Nash equilibrium pair of outputs for the two firms. What is the profit of each firm in this Nash equilibrium?

Given the output of the other firm, the first order condition for maximizing firm $i$’s profit is

$$0 = \frac{\partial \pi_i(q_1, q_2)}{\partial q_i} = -q_i + [25 - (q_1 + q_2)] - 2q_i$$

$$= -3q_i + [25 - (q_1 + q_2)].$$

For $i = 1, 2$, we therefore have the first order conditions

$$0 = -3q_1 + [25 - (q_1 + q_2)]$$
$$0 = -3q_2 + [25 - (q_1 + q_2)].$$

Subtracting implies

$$0 = -3q_1 + 3q_2 \Leftrightarrow q_1 = q_2.$$ 

Substitution into firm 1’s first order condition implies

$$0 = -3q_1 + [25 - (2q_1)]$$

$$= -5q_1 + 25 \Rightarrow q_1 = q_2 = 5.$$

We have

$$\frac{\partial^2 \pi_i(q_1, q_2)}{\partial q_i^2} = -4q_i$$

and so the second derivative test indeed verifies that $q_1 = q_2 = 5$ is a Nash equilibrium. A firm’s profit in the Nash equilibrium is

$$\pi_i(5, 5) = [25 - 10]5 - 25$$
$$= 75 - 25 = 50.$$ 

(b) (15 points) Each firm’s profit when each produces 4.5 is

$$\pi_i(4.5, 4.5) = [25 - 9] 4.5 - (4.5)^2 = 51.75.$$  

I am providing this to you as a fact that may prove useful below. Suppose they play the infinite repetition of the game defined in a.

i. (5 points) If firm 2 produces 4.5, then how much should firm 1 produce in order to maximize its profit? What is the value of this maximal profit?

$$0 = \frac{\partial \pi_1(q_1, 4.5)}{\partial q_1} = -3q_1 + [25 - (q_1 + 4.5)]$$

$$= -4q_1 + 20.5 \Rightarrow q_1 = 5.125.$$  

Again, by the second derivative test, this maximizes firm 1’s profit. It’s profit is then

$$\pi_1(5.125, 4.5) = [25 - (9.625)] 5.125 - (5.125)^2$$

$$= (25 - 9.625 - 5.125) 5.125$$

$$= 52.53125.$$  

ii. (5 points) Derive a bound on the firms’ discount factors that is sufficient to insure that there is an equilibrium in the infinitely repeated game in which each firm receives a profit equal 51.75 in each period. You may use the formula

$$\delta_i \geq \frac{d_i - \pi_i^*}{d_i - \pi_i}$$

that was derived in class.

Here, we have $d_i = 52.53125$, $\pi_i^* = 51.75$ and $\pi_i = 50$, which implies the bound

$$\delta_i \geq \frac{52.53125 - 51.75}{52.53125 - 50} = 0.309.$$  

iii. (5 points) Define the trigger strategy of each firm that results in the outcome equilibrium outcome in b ii..  

Firm i’s strategy is: Choose $q_i = 4.5$ to start the game and as long as $(4.5, 4.5)$ is chosen by the firms in each stage. If it is ever not the choices in some stage, then switch to $q_i = 5$ for ever after.