1. (12 points) Consider in this problem a marriage market with 4 men and 4 women. The strict preferences of the men and women are as follows:

\[
\begin{align*}
P(m_1) : w_1, w_2, w_3, w_4 & \quad P(w_1) : m_4, m_3, m_2, m_1 \\
P(m_2) : w_4, w_2, w_3, w_1 & \quad P(w_2) : m_2, m_3, m_4, m_1 \\
P(m_3) : w_1, w_2, w_4, w_3 & \quad P(w_3) : m_2, m_3, m_4, m_1 \\
P(m_4) : w_2, w_4, w_3 & \quad P(w_4) : m_1, m_4, m_3, m_2
\end{align*}
\]

(a) (4 points) Determine the matching that is obtained through the deferred acceptance algorithm in which men make the proposals.

\[
\begin{array}{cccccc}
\text{women} & w_1 & w_2 & w_3 & w_4 & \text{women} & w_1 & w_2 & w_3 & w_4 \\
\text{stage 1} & m_1, m_3 & m_4 & m_2 & \Rightarrow & \text{stage 1} & m_3 & m_4 & m_2 \\
\text{stage 2} & m_3 & m_4 & m_2 & \Rightarrow & \text{stage 2} & m_3 & m_4 & m_2 \\
\text{stage 3} & m_3 & m_4 & m_2 & \Rightarrow & \text{stage 3} & m_3 & m_4 & m_2 \\
\end{array}
\]

(b) (4 points) Show in this example that the algorithm used in a. is not strategy-proof.

\[
\begin{align*}
w_4 \text{ ends up with her least favorite man } m_2. \text{ Let’s see what happens if she rejects him in stage 1:} \\
\text{women} & w_1 & w_2 & w_3 & w_4 & \text{women} & w_1 & w_2 & w_3 & w_4 \\
\text{stage 1} & m_1, m_3 & m_4 & m_2 & \Rightarrow & \text{stage 1} & m_3 & m_4 \\
\text{stage 2} & m_3 & m_4, m_2, m_1 & \Rightarrow & \text{stage 2} & m_3 & m_2 \\
\text{stage 3} & m_3 & m_4 & m_2 & \Rightarrow & \text{stage 3} & m_3 & m_2 & m_1 & m_4 \\
\end{align*}
\]

We see that \( w_4 \) can switch from her least favorite man \( m_2 \) to her second favorite man \( m_4 \) by turning down an acceptable man in stage 1. The algorithm is not strategy-proof.

(c) (4 points) Determine the stable matching that is optimal for women. This stable matching is obtained through the deferred acceptance algorithm in which women make the proposals.

\[
\begin{align*}
P(m_1) : w_1, w_2, w_3, w_4 & \quad P(w_1) : m_4, m_3, m_2, m_1 \\
P(m_2) : w_4, w_2, w_3, w_1 & \quad P(w_2) : m_2, m_3, m_4, m_1 \\
P(m_3) : w_1, w_2, w_4, w_3 & \quad P(w_3) : m_2, m_3, m_4, m_1 \\
P(m_4) : w_2, w_4, w_3 & \quad P(w_4) : m_1, m_4, m_3, m_2
\end{align*}
\]

\[
\begin{array}{cccccc}
\text{men} & m_1 & m_2 & m_3 & m_4 & \text{women} & m_1 & m_2 & m_3 & m_4 \\
\text{stage 1} & w_4 & w_2, w_3 & w_1 & \Rightarrow & \text{stage 1} & w_4 & w_2 \\
\text{stage 2} & w_4 & w_2 & w_1, w_3 & \Rightarrow & \text{stage 2} & w_4 & w_2 & w_1 \\
\text{stage 3} & w_4 & w_2 & w_1 & \Rightarrow & \text{stage 3} & w_4 & w_2 & w_1 & w_3 \\
\end{array}
\]

2. (16 points) Consider in this problem a seller who sells an item through the following auction. There are \( n \) bidders. Each bidder submits a bid.
The bidder who submits the highest bid wins the item and pays the seller the average of his bid and the second highest bid. You can ignore the possibility of ties in your answer to this question.

The reservation values of the $n$ bidders are identically and independently distributed according to the uniform distribution on $[0, 1]$. The purpose of this example is to work through the first few steps in deriving an increasing function $b : [0, 1] \rightarrow [0, 1]$ that defines an equilibrium in this auction: if every other bidder besides bidder $i$ uses the function $b$ to select his bid, then $b(v_i)$ maximizes bidder $i$’s expected profit in the auction for each of his possible reservation values $v_i \in [0, 1]$. Select a bidder $i$, let $v_i$ denote his reservation value, $x$ his bid, and assume all other bidders use the function $b$ to select their bids.

(a) (4 points) Let $\gamma$ denote the highest reservation value in the sample of the reservation values of the other $n-1$ traders. What is the cumulative distribution function for $\gamma$ and what is its density?

$$F_{(n-1)}(\gamma) = \gamma^{n-1}$$
$$f_{(n-1)}(\gamma) = (n-1)\gamma^{n-2}$$

(b) (4 points) What is bidder $i$’s profit $u(v_i, x, y)$ as a function of his bid $x$, his reservation value $v_i$ and the value of $y$?

$$u(v_i, x, y) = \begin{cases} v_i - \frac{x + b(y)}{2} & \text{if } x \geq b(y) \\ 0 & \text{if } x < b(y) \end{cases}$$

(c) (4 points) Using your answers to a. and b., what is bidder $i$’s expected profit $U(v_i, x)$ as a function of his bid $x$ and his reservation value $v_i$?

$$U(v_i, x) = \int_0^{b^{-1}(x)} \left( v_i - \frac{x + b(y)}{2} \right) (n-1)\gamma^{n-1} d\gamma$$

Hint: Your answer should be an integral with respect to $\gamma$.

(d) (4 points) Let $U(v_i) = U(v_i, b(v_i))$, where $b(v_i)$ is the strategy that we are hoping to derive. The function $U(v_i)$ specifies the expected utility of buyer $i$ in equilibrium as a function of his reservation value $v_i$. Calculate $U'(v_i)$ using the Envelope Theorem.

$$U'(v_i) = \frac{\partial}{\partial v_i} U(v_i, x) \bigg|_{x=b(v_i)} = \int_0^{b^{-1}(x)} (n-1)y^{n-2}dy = (n-1)\int_0^{b^{-1}(x)} (n-1)\gamma^{n-1} d\gamma \bigg|_{x=b(v_i)} = v_i^{n-1}.$$

3. (8 points) Consider the following decision problem with three actions $A_1, A_2, A_3$ and three states $s_1, s_2, s_3$:

<table>
<thead>
<tr>
<th>action/state</th>
<th>$s_1$</th>
<th>$s_2$</th>
<th>$s_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>4</td>
<td>5</td>
<td>-3</td>
</tr>
<tr>
<td>$A_2$</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$A_3$</td>
<td>5</td>
<td>-2</td>
<td>3</td>
</tr>
</tbody>
</table>
Which action does the decision maker choose if he uses the maxmin criterion? Which action does he choose if minimizes his maximum regret? Show your work.

Maxmin: $A_1$ guarantees at least $-3$, $A_2$ guarantees 0, and $A_3$ guarantees $-2$. The decision maker therefore chooses $A_2$ if he uses the maxmin criterion.

Minimizing maximum regret: The regret determined by each action and each state is as follows:

<table>
<thead>
<tr>
<th>action/state</th>
<th>$s_1$</th>
<th>$s_2$</th>
<th>$s_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>1</td>
<td>0</td>
<td>6</td>
</tr>
<tr>
<td>$A_2$</td>
<td>3</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>$A_3$</td>
<td>0</td>
<td>7</td>
<td>0</td>
</tr>
</tbody>
</table>

The maximum regret associated with each action is therefore

<table>
<thead>
<tr>
<th>action/state</th>
<th>max. regret</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>6</td>
</tr>
<tr>
<td>$A_2$</td>
<td>4</td>
</tr>
<tr>
<td>$A_3$</td>
<td>7</td>
</tr>
</tbody>
</table>

and the action $A_2$ minimizes the maximum regret.

4. (14 points) There are many identical consumers, each with the utility of wealth function $U(w) = \sqrt{w}$. Each has initial wealth $\$100$ and each faces a loss of half of it with probability

$$\pi(e) = \frac{2}{3} - \frac{e}{3}.$$ 

The variable $e$ here equals either 0 or 1; it is a preventative action that an individual can privately take at a cost to his wealth of $20e$ to reduce the likelihood of the loss.

(a) (3 points) Assuming a competitive market for insurance, what is the market opportunity line if each consumer takes the action $e$?

$$\pi(e) \cdot x + (1 - \pi(e)) \cdot y = \pi(e) \cdot 50 + (1 - \pi(e)) \cdot 100$$

$$\left(\frac{2}{3} - \frac{e}{3}\right) x + \left(\frac{1}{3} + \frac{e}{3}\right) y = \left(\frac{2}{3} - \frac{e}{3}\right) \cdot 50 + \left(\frac{1}{3} + \frac{e}{3}\right) \cdot 100$$

$$\left(\frac{2}{3} - \frac{e}{3}\right) x + \left(\frac{1}{3} + \frac{e}{3}\right) y = \frac{200}{3} + \frac{50e}{3}$$

(b) (3 points) What is the individual’s certain wealth in the case of the competitive equilibrium (which is the case of $e = 0$)?

We know that the competitive equilibrium involves $e = 0$, in which case the market opportunity line is

$$\frac{2x}{3} + \frac{y}{3} = \frac{200}{3}$$
Each individual is risk averse and so selects complete insurance,

\[ x = y = \frac{200}{3} \]

(c) (4 points) Use your answer to b. to determine the price \( p \) per dollar of coverage and the amount \( C \) of coverage for each individual in the competitive equilibrium.

\[
\begin{align*}
x & = 50 + C - pC \\
y & = 100 - pC
\end{align*}
\]

We have \( x = y = \frac{200}{3} \), and so

\[
\begin{align*}
50 + C - pC & = 100 - pC \\
\frac{200}{3} & = 100 - 50p \\
50p & = \frac{100}{3} \\
p & = \frac{2}{3} \cdot C = 50
\end{align*}
\]

(d) (4 points) Is the competitive equilibrium efficient? Explain.

Each person’s utility in the competitive equilibrium is \( \sqrt{200} \). If everyone instead choose \( e = 1 \), the market opportunity line is

\[
\left( \frac{1}{3} \right) x + \left( \frac{2}{3} \right) y = \frac{250}{3},
\]

and each person’s wealth would be

\[
\sqrt{\frac{250}{3} - 20} = \sqrt{\frac{190}{3}} < \sqrt{\frac{200}{3}}.
\]

Yes, the competitive equilibrium is efficient.