Exercise. Calculate an equilibrium in the all-pay auction in the case of \( n \) bidders whose reservation values are drawn from the uniform distribution on \([0,1]\). What happens to a bidder’s equilibrium bid as \( n \to \infty \)?

We seek an increasing function \( B: [0,1] \to \mathbb{R} \) such that a bidder maximizes his expected utility by using \( B \) to choose his bid when he assumes that the other bidders are also using \( B \). Bidder \( i \)'s expected utility when \( v_i \) is his value and he bids \( x \) is

\[
U(v_i, x) = v_i \cdot (B^{-1}(x))^{n-1} - x,
\]

where \( (B^{-1}(x))^{n-1} \) is the probability that the other bidder bids less than \( x \). Let \( U(v_i) \) denote his equilibrium expected utility when he bids \( B(v_i) \),

\[
U(v_i) = U(v_i, B(v_i)) = v_i^n - B(v_i).
\]

The Envelope Theorem implies

\[
U'(v_i) = \frac{\partial}{\partial v_i} U(v_i, x) \text{ at } x = B(v_i) = (B^{-1}(x))^{n-1} \text{ at } x = B(v_i) = v_i^{n-1}.
\]

Therefore,

\[
U(v_i) = \int_0^{v_i} v^{n-1} dv + c = \frac{v_i^n}{n} + c.
\]

We know that a bidder with value \( v_i = 0 \) wins with probability equal to zero. He clearly should not bid a positive amount in this case because he’ll have to pay it! Therefore, \( U(0) = 0 \) and \( c = 0 \), and so

\[
U(v_i) = \frac{v_i^n}{n}.
\]

We now solve for \( B(v_i) \) by equating the two formulas for \( U(v_i) \):

\[
v_i^n - B(v_i) = U(v_i) = \frac{v_i^n}{n} \Rightarrow B(v_i) = \frac{n-1}{n} v_i^n.
\]

As to the limit as \( n \to \infty \),

\[
\lim_{n \to \infty} \frac{n-1}{n} v_i^n = \lim_{n \to \infty} v_i^n - \frac{v_i^n}{n} = 0 \text{ for } v_i < 1.
\]

If there are a large number of bidders, a bidder expects that he will not be the one with the highest value and therefore his bid goes to zero as he avoids making a payment with little chance of winning. We can in fact also see that

\[
\frac{d}{dn} \left[ \frac{n-1}{n} v_i^n \right] = \frac{d}{dn} \left[ \frac{n-1}{n} e^{n \ln v_i} \right] = \frac{v_i^n}{n^2} + \frac{n-1}{n} v_i^n \cdot \ln v_i = \frac{v_i^n}{n} \left( \frac{1}{n} + (n-1) \ln v_i \right).
\]

For \( v_i \in (0,1) \), we have \( \ln v_i < 0 \), and so the equilibrium bid is decreasing in \( n \) for all \( v_i \), assuming \( n \) is sufficiently large.

Exercise. Calculate an equilibrium in the all-pay auction in the case of \( n = 2 \) bidders whose reservation values are drawn from the distribution \( F \) on \([0,1]\). Generalize your result to an arbitrary number \( n \) of bidders. What happens to the equilibrium bidding strategy as \( n \to \infty \)?
We seek an increasing function \( B : [0, 1] \to \mathbb{R} \) such that a bidder maximizes his expected utility by using \( B \) to choose his bid when he assumes that the other bidders are also using \( B \). Bidder \( i \)'s expected utility when \( v_i \) is his value and he bids \( x \) is
\[
U(v_i, x) = v_i \cdot F(B^{-1}(x)) - x,
\]
where \( F(B^{-1}(x)) \) is the probability that the other bidder bids less than \( x \). Let \( U(v_i) \) denote his equilibrium expected utility when he bids \( B(v_i) \),
\[
U(v_i) = U(v_i, B(v_i)) = v_i \cdot F(v_i) - B(v_i).
\]
The Envelope Theorem implies
\[
U'(v_i) = \frac{\partial}{\partial v_i} U(v_i, x) \text{ at } x = B(v_i) = (B^{-1}(x))^{n-1} \text{ at } x = B(v_i) = F(v_i).
\]
Therefore,
\[
U(v_i) = \int_0^{v_i} F(v_i) \, dv + c.
\]
We know that a bidder with value \( v_i = 0 \) wins with probability equal to zero. He clearly should not bid a positive amount in this case because he'll have to pay it! Therefore, \( U(0) = 0 \) and \( c = 0 \), and so
\[
U(v_i) = \int_0^{v_i} F(v_i) \, dv.
\]
We now solve for \( B(v_i) \) by equating the two formulas for \( U(v_i) \):
\[
v_i \cdot F(v_i) - B(v_i) = U(v_i) = \int_0^{v_i} F(v_i) \, dv \Rightarrow B(v_i) = v_i \cdot F(v_i) - \int_0^{v_i} F(v_i) \, dv.
\]
Turning to the case of \( n \) bidders, we seek an increasing function \( B : [0, 1] \to \mathbb{R} \) such that a bidder maximizes his expected utility by using \( B \) to choose his bid when he assumes that the other bidders are also using \( B \). Bidder \( i \)'s expected utility when \( v_i \) is his value and he bids \( x \) is
\[
U(v_i, x) = v_i \cdot F(B^{-1}(x))^{n-1} - x,
\]
where \( F(B^{-1}(x))^{n-1} \) is the probability that the other bidder bids less than \( x \). Let \( U(v_i) \) denote his equilibrium expected utility when he bids \( B(v_i) \),
\[
U(v_i) = U(v_i, B(v_i)) = v_i F(v_i)^{n-1} - B(v_i).
\]
The Envelope Theorem implies
\[
U'(v_i) = \frac{\partial}{\partial v_i} U(v_i, x) \text{ at } x = B(v_i) = F(B^{-1}(x))^{n-1} \text{ at } x = B(v_i) = F(v_i)^{n-1}.
\]
Therefore,
\[
U(v_i) = \int_0^{v_i} F(v_i)^{n-1} \, dv + c.
\]
We know that a bidder with value \( v_i = 0 \) wins with probability equal to zero. He clearly should not bid a positive amount in this case because he'll have to pay it! Therefore, \( U(0) = 0 \) and \( c = 0 \), and so
\[
U(v_i) = \int_0^{v_i} F(v_i)^{n-1} \, dv.
\]
We now solve for $B(v_i)$ by equating the two formulas for $U(v_i)$:

$$v_i F(v_i)^{n-1} - B(v_i) = U(v_i) = \int_0^{v_i} F(v) v_i F(v_i)^{n-1} dv \Rightarrow B(v_i) = v_i F(v_i)^{n-1} - \int_0^{v_i} F(v)^{n-1} dv.$$  

As in the uniform case, as $n \to \infty$,

$$\lim_{n \to \infty} v_i F(v_i)^{n-1} - \int_0^{v_i} F(v_i)^{n-1} dv = 0 \text{ for } v_i < 1.$$