Exercises: p. 400-401, problems 4, 7 and 8
review Condorcet triple

0.0.16.5 Restrictions on Preferences

The Gibbard-Satterthwaite Theorem assumes that (i) there are three or more alternatives, and (ii) the preferences of each voter are unrestricted in the sense that all possible rankings of the alternatives are feasible for each voter. In such a case, the only possible non-manipulable voting procedure is a dictatorship. In this section, we discuss cases in which the possible preferences of the group members may be restricted in some way. Preferences are often restricted in economic settings (e.g., everyone may prefer "more" to "less").

Example 44 Here's an example that motivates restrictions on preferences. Let's imagine three alternatives, L, M and H. L might correspond, for instance, to low taxes and small government, M to medium taxes and medium size government, and H to high taxes and large government. We can characterize Ron Paul’s preferences as

$L > M > H,$

and Rachel Maddow’s as

$H > M > L.$

A moderate democrat might have the preferences

$M > H > L,$

and a moderate republican (do they exist any more?) might have the preferences

$M > L > H.$

The moderates in each case rank M as the top choice, with the difference between moderate republicans and democrats concerning whether they would rather have L or H if they cannot have M.

Who has the preferences

$H > L > M,$

or

$L > H > M$?

Would a person who cannot have his first choice of big government H really prefer small government over medium government? If these rankings simply do not occur among the population, then we say that preferences are restricted. It will turn out that it may be possible to accomplish things through voting when preferences are restricted that cannot be accomplished when preferences are unrestricted.

Note that the implausibility of these last two rankings is in reference to this particular interpretation of L, M, and H. If L represents "Luigi’s Italian Restaurant," M represents "Manny’s BBQ," and H represents "Hunan Dynasty Chinese Food," and if the group members were trying pick a place to go to dinner together, then we may not have any reason to rule out the last two preferences as unreasonable or unlikely.
Example 45 (Single Peaked Preferences)  A common restriction on preferences is to assume that they are single peaked. This means that the alternatives can be ordered on a line and a numerical representation of preferences so that utility has a "single peak". For instance, suppose there are 4 alternatives X, Y, Z, W that are ordered in this way from left to right. The following graph depicts two utility functions that represent single-peaked preferences:

The following two graphs depict utility functions that do not represent single-peaked preferences:
The reason for the name "single peaked" is clear from these pictures. In the preceding example, we exclude $M < H < L$, which might be represented graphically as

Excluding this case and $M < L < H$ is sufficient to insure that the remaining orderings are all single peaked.

**Definition 1** The possible preferences of a group of voters are **value-restricted** if Condorcet cycles do not exist for any three alternatives. In other words, there do not alternatives $X, Y, \text{ and } Z$ such that all three orderings

\[ X > Y > Z, \]
\[ Y > Z > X, \]
\[ Z > X > Y. \]

are possible among the group members.

**Remark 1** If the set of preferences are single-peaked, then the preferences are necessarily value-restricted. Value-restricted is more general (less restrictive) in the sense that it
only requires that no three elements and three individuals in the group form a Condorcet triple.

The preferences in the example concerning low, medium and high levels of government are value-restricted if we rule out the possibility of any voter ranking M as his worst choice.

**Theorem 9 (Sen’s Majority Rule Theorem, p. 397)** If the number of individuals is odd, the number of alternatives is finite, preference rankings are strict (no indifference among alternatives), and preferences are value-restricted, then there exists a unique alternative that defeats any other alternative in majority rule. We call this alternative the unique majority winner.

The first and third assumptions avoid the problem of dealing with ties, which would complicate the proof. The result still holds in the case of an even number of voters if an explicit tie-breaking rule is adopted.

There are two points to be noted about a unique majority winner. First, the existence of a unique majority winner guarantees a strategy-proof (dominant strategy) voting method for identifying this alternative, namely successively choosing between all pairs of candidates through majority rule. Regardless of the way in which the pairwise runoffs are organized, the procedure always ends up selecting the unique majority winner, and each voter has the incentive in each pairwise election to honestly vote for his favorite of the two candidates.

We also noted in class a second point. If a unique majority winner does not exist among the alternatives, then any alternative that is selective will be second-best to some other alternative in the rankings of a majority of the group members. The group will thus be discontent with any alternative that is selected. This is a difficult situation for the group and for the selected alternative (e.g., if it is a candidate selected to lead the group).

**Example 46** Let’s travel back in time several years and restrict attention to three candidates for the Republican Party nomination for president: Romney, Gingrich, and Santorum. Is it necessarily the case that one of these three candidates could necessarily defeat each of the other two in a one-on-one election? The theorem states a condition under which this is necessarily the case: if there do not exist three voters 1, 2, and 3 in the electorate who rank the candidates as in a Condorcet triple,

1: R > G > S,
2: G > S > R,
3: S > R > G,

then there must exist a unique majority winner.

Suppose that we could order the candidates as points along a line,
There are no units on this line. We might think of the line as listing the candidates in order of increasing conservatism on social issues, etc. If all voters order the candidates in this way, and if preferences are single-peaked, then the following orderings are possible among voters:

\[
R > G > S,
G > R > S,
G > S > R,
S > G > R.
\]

Notice that I have left off

\[
S > R > G,
R > S > G.
\]

In other words, no voter ranks the middle candidate as worst. In this case, we can be certain that there exists a unique majority winner among the three candidates.

**Example 47**  The Economics Department two years ago conducted a national search to hire a new department head. A committee was formed by the Dean of LAS to review the applicants. There were three finalists who were brought to campus. Considering the faculty of the department as the electorate, suppose that a majority winner did not exist. This implies that for whichever finalist X is selected, there exists some other finalist Y that a majority of the faculty would prefer to X! The Dean made the final decision on whom to appoint; if a majority winner did not exist, then she surely could not please the faculty with her choice.

Interestingly, a member of faculty wished to have meeting in advance of Dean’s final decision to "aggregate" the views of the faculty members and make a collective recommendation to the Dean. Other members of the faculty opposed this, however, on several grounds:

1. Some of us were aware that a majority winner may not exist. The candidate who the Dean selected would therefore know from this meeting that if he accepted the position then he may not have the support of a majority of the faculty. That would certainly discourage a person from accepting the position.

2. The point of the meeting was also to bind the Dean by telling her who the department wanted her to appoint. Some of us thought that this was unfair to the Dean and disrespectful of her role in the appointment process.

In the end, no vote was taken at the meeting.

**Example 48**  A member of the economics department once proposed that the Director
of Graduate Studies for the department be approved by a majority of the department's graduate students. This would insure that whichever faculty member served in this role would have the endorsement or support of a majority of the graduate students. This is different from Sen's Theorem because it involves approval/disapproval of each potential candidate for the job, as opposed to pairwise running each two faculty members against each other in an election. The problem, however, is clear: How do we know that there exists some faculty who would be approved by at least half of the graduate students?

The preferences in the example concerning low, medium and high levels of government are value-restricted if we rule out the possibility of any voter ranking M as his worst choice.

**Theorem 10 (Sen’s Majority Rule Theorem, p. 397)**  
*If the number of individuals is odd, the number of alternatives is finite, preference rankings are strict (no indifference among alternatives), and preferences are value-restricted, then there exists a unique alternative that defeats any other alternative in majority rule. We call this alternative the unique majority winner.*

**Proof.** [Proof of Sen’s Majority Rule Theorem] The theorem is proven in two steps.

**Step 1.** We show that there cannot exist a triple of alternatives $X$, $Y$, and $Z$ where $X$ defeats $Y$ in a majority vote, $Y$ defeats $Z$ in a majority vote, and $Z$ defeats $X$ in a majority vote. Suppose that such a triple exists. We will obtain a contradiction to value-restricted preferences by showing that there must be some individual with the preferences $X > Y > Z$, some individual with the preferences $Y > Z > X$, and some individual with the preferences $Z > X > Y$.

Letting $A$ denote the set of people who prefer $X$ to $Y$ and $B$ the set who prefer $Y$ to $Z$. Because both $A$ and $B$ contain more than half of the people, we must have $A \cap B \neq \phi$, that is, there is some person who ranks the alternatives as $X > Y > Z$.

The same argument applied to the sets of people who prefer $Y$ to $Z$ and $Z$ to $X$ implies that there is at least one person with the ranking $Y > Z > X$.

You can see where we’re headed. Because $Z$ defeats $X$ in a majority vote, and $X$ defeats $Y$ in a majority vote, there must be some individual with the ranking $Z > X > Y$.

This contradicts the assumption of value-restricted preferences.

**Step 2.** We now prove the desired result by induction on the number $m$ of alternatives. It’s clear that there is a majority winner if $m = 2$. Suppose that if the set of alternatives contains $m \geq 2$ alternatives, then there is necessarily a unique majority winner. Suppose now that the set $A$ of alternatives has $m + 1$ elements. Pick any element $X \in A$ and let $B = A \setminus \{X\}$. The induction assumption implies that there exists a unique alternative $Y \in B$ that defeats any other alternative in $B$ in majority rule.
How does $Y$ compare to $X$? If $Y$ defeats $X$, then $Y$ is clearly the unique majority winner in the set $A$, and we’re done with the proof.

Suppose $X$ defeats $Y$. If $X$ also defeats every other element of $B$, then $X$ is the unique majority winner in the set $A$. Again, we’re done.

If not, then there is some alternative $Z \in B$ such that $Z$ defeats $X$. We’re assuming here that $X$ defeats $Y$, and $Y$ defeats $Z$ because it defeats every other alternative in $B$. This contradicts the conclusion of step 1 above and so we are done.

0.0.17 Manipulation by Groups

We discussed manipulation by an individual (i.e., acting as though his ranking of the alternatives is different from the true ranking as a way of influencing the outcome of a voting procedure in his favor, assuming that everyone else acts according to his true preferences). We can also consider manipulation by a group, namely, a coordinated effort by a group of people to act according to "false" preferences in order to influence the outcome, assuming that everyone outside the group acts according to his true preferences. It is good if a voting procedure is resistant to manipulation by individuals; an even stronger property is to be resistant to manipulation by any group.

A side note: as department head, I learned first-hand how vulnerable approval voting can be to manipulation by groups. A group of faculty with a shared agenda would collude and mark their ballots consistently with one another to insure that particular individuals were on the advisory committee and particular individuals were left off the committee.

Theorem 11 (Nonmanipulability of Majority Rule for Value-Restricted Preferences, p. 399)

Suppose that the number of individuals is odd, the number of alternatives is finite, preferences are strict and value restricted. Then majority rule cannot be manipulated by any group.

Proof. We know from Sen’s Theorem that there exists a unique majority choice $X$ based upon the true preferences of the group members. Suppose there is a subgroup $C$ whose members prefer $Y$ to $X$. $X$ defeats $Y$ in a majority vote of all members, and so the set $C$ consists of less than half of the group, and there are more people outside $C$ who prefer $X$ to $Y$ than are in $C$ itself. Consequently, there is nothing that the members of the group $C$ can do to keep $X$ from defeating $Y$. 

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