October 30

Exercises: p. 400-401, problems 4, 7 and 8

0.0.18 Model

A - a finite set of alternatives for a group of \( N \in \mathbb{N} \) agents

\( \mathcal{L} \) - the set of strict linear orders (or rankings) of elements of \( A \)

an element \( (L_1, \ldots, L_i, \ldots, L_N) \) of \( \mathcal{L}^N \) is called a profile of rankings

\( f : \mathcal{L}^N \rightarrow A \) - a social choice function

Notice that the use of \( \mathcal{L} \) assumes that preferences are unrestricted, i.e., all possible orderings are worthy of consideration.

A social choice function \( f \) represents the process of aggregating individual preferences into a group choice. We might imagine \( f \) as the result of some voting procedure under some solution concept that determines how a person votes based upon his preferences. Obviously, social choice functions exist in reality; we want to explore what properties they should have and what properties they can and do have. A social welfare function fundamentally addresses the meaningfulness of group preferences (e.g., the preferences of a firm or society, or an academic department for that matter). Macroeconomics commonly uses the "representative agent" to represent a large number of people. Is this abstraction meaningful, i.e., is there a sense in which the preferences of the group can be represented as the preferences of an individual?

0.0.18.1 Properties of a social choice function:

**Unanimity:** If \( a \in A \) is at the top of every individual \( i \)'s ranking \( L_i \), then

\[ f(L_1, \ldots, L_i, \ldots, L_N) = a. \]

**Dictatorial:** \( f \) is dictatorial if there exists some agent \( i \) such that \( f(L_1, \ldots, L_i, \ldots, L_N) = a \) if and only if \( a \) is the top choice relative to \( L_i \).

This is not a desired property, but it will be part of the statement of Gibbard-Satterthwaite Theorem below.

**Strategy-Proof:** \( f \) is strategy-proof if for every individual \( i \), every \( (L_1, \ldots, L_i, \ldots, L_N) \), and every \( L_i' \),

\[ f(L_1, \ldots, L_i', \ldots, L_N) \]

is equal to or ranked below

\[ f(L_1, \ldots, L_i, \ldots, L_N) \]

when \( i \)'s ranking is \( L_i \).

If we regard \( L_i \) as \( i \)'s input into the social choice function that determines an alternative, "strategy-proof" means that \( i \) cannot profit by misreporting and submitting \( L_i' \) rather than his true ranking \( L_i \). This must hold regardless of the reports

\[ (L_1, \ldots, L_i-1, L_i+1, \ldots, L_N) \]

of the other agent. Honest reporting of one's ranking is thus a dominant strategy for each agent.

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5 This material is drawn from "Arrow’s Theorem and the Gibbard-Satterthwaite Theorem: A Unified Approach", by Phillip Reny (Economic Letters (70) (2001), 99-105).
Question: Is a dictatorship strategy-proof?

Exercise 1 (The Borda Count) Suppose that there are 3 individuals and 3 possible alternatives. As in our model, each individual strictly ranks the 3 alternatives. Suppose that the social choice function is determined by asking each individual to assign the numbers 1, 2, and 3 to the 3 alternatives. These are the "votes" of the 3 individuals. The alternative that receives the largest number of votes is the social choice.

Show that this social function is not strategy-proof. Express your answer as clearly as possible using the notation in the above model.

0.0.18.2 Back to the Gibbard-Satterthwaite Theorem

Theorem 12 (Gibbard-Satterthwaite Theorem) If A has at least 3 elements and f : \( \mathcal{L}^N \to A \) is onto and strategy-proof, then f is dictatorial.

Unanimity and unrestricted preferences \( \Rightarrow \) onto

The Gibbard-Satterthwaite Theorem is a discouraging result in the sense that it sharply limits what can be accomplished in a strategy-proof manner. The theorem reveals how little can be accomplished through dominant strategy implementation: in a general sense, dominant strategies are too much to ask for, in the sense there very few social choice functions can be implemented in this sense.

Most of the criticism of this result focuses on the assumption that preferences are unrestricted. This applies in many voting problems but not necessarily in economic problems (e.g., "more is better").

0.0.19 Digression: The Revelation Principle

The Revelation Principle originated in Gibbard’s proof of the Gibbard-Satterthwaite Theorem.

Notation for a game: agent i’s strategy set is \( S_i, \sigma_i : \mathcal{L} \to S_i \) denotes a strategy of player i (i.e., a choice based upon his preferences over the alternatives \( A \)), and

\[
\tau : \prod_{i=1}^{N} S_i \to A
\]

denotes the outcome function of the game (i.e., how the game determines an alternative based upon the strategic choices of the players). A game or mechanism is denoted

\[
\left( \prod_{i=1}^{N} S_i, \tau \right).
\]

Letting \( \sigma : \mathcal{L}^N \to \prod_{i=1}^{N} S_i \) be defined by

\[
\sigma = (\sigma_1, \ldots, \sigma_N),
\]

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the social choice function \( f \) is implemented by the strategy profile \((\sigma_i)_{1 \leq i \leq N}\) in the game \((\prod S_i, \tau)\) if

\[
\tau \circ \sigma = f, 
\]

i.e., \( f \) results when the agents employ the strategies \((\sigma_i)_{1 \leq i \leq N}\).

A revelation game or mechanism is a game in which each \( S_i = \mathcal{L} \) (i.e., each agent is given the opportunity to report his complete ranking of the alternatives in \( A \)).

**Theorem 13 (Revelation Principle for Dominant Strategies)** If \((\sigma_i)_{1 \leq i \leq N}\) is a dominant strategy equilibrium in the game \((\prod S_i, \tau)\), then honest revelation by each agent is a dominant strategy equilibrium in the game \((\mathcal{L}^N, \tau \circ \sigma)\).

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